RELATIVE FLATNESS AND FLATNESS OF IMPLICIT SYSTEMS*

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Abstract. In this work we define the concept of *relative flatness* of a system with respect to a subsystem. The subsystem associated to a set of outputs of a system is constructed, and called here *output subsystem*. It is shown that the relative flatness of a system with respect to the output subsystem implies the flatness of the corresponding implicit system obtained by setting these outputs to zero. A sufficient condition of relative flatness based on a *relative derived flag* is presented. Based on these results, a sufficient condition for the flatness of a class of nonlinear implicit systems is obtained.

Key words. Nonlinear systems, implicit systems, time-varying systems, flatness, relative flatness, feedback linearization.

1. Introduction and Motivation. The aim of this paper is to present the notion of *relative flatness* with respect to a subsystem. We show that this concept may be useful for control systems theory, in particular for studying the structure of nonlinear implicit systems. Our approach is based on the infinite dimensional geometric setting recently introduced in control theory [18, 42, 20] in combination with the ideas presented in [52, 50, 55]. Our sufficient conditions for flatness of implicit systems may be regarded as a generalization of the conditions obtained in [52] for explicit systems. Our setting has some connections with the ideas of [49], which has considered a different class of implicit systems.

Feedback linearization is an important problem in nonlinear control theory. This problem was completely solved in static-state feedback case [26, 24] but necessary and sufficient conditions for feedback linearizability by dynamic state feedback are not yet known (see [6, 50, 7, 21, 53, 55, 1, 46, 54, 23, 43, 56]).

The notion of differential flatness was introduced by Fliess et al [17, 19] and is strongly related to the problem of feedback linearization. This concept corresponds to a complete and finite parametrization of all solutions of a control system by a differentially independent family of functions, called flat output.

Linear *singular* (or implicit) systems are an important class of control systems and many papers and books on this subject are found in the literature [5, 33][1]. Solvability of nonlinear implicit differential equations is considered in [3, 45]. Other problems like controllability [30], stabilization [34, 8], canonical forms [47] and feedback control [9], have already been considered.

Feedback linearization of implicit systems has been studied for instance by [31, 32, 27]. These works consider the problem of finding a state transformation and a state feedback such that the closed loop system is a linear singular system. In this

*The first author was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) under grant 300492/95-2 and Fundação de Amparo à Pesquisa do Estado de São Paulo FAPESP, under grant 97 / 04668-1. This paper is an extended version of [41]. (To Appear in SIAM J. Control and Optimization)

[1]Note that the *module theoretic* approach of [16] is also valid for implicit systems.

work we tackle the problem of finding sufficient conditions for flatness of a class of time-varying implicit systems of the form[2]

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1.1a}$$

$$y(t) = h(t, x(t), u(t)) = 0 \tag{1.1b}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and all the components of $f(x)$ and $g(x)$ are analytical functions of x.

Given a set of p nonlinear differential equations of arbitrary order, one can always define an equivalent system of the form (1.1a)-(1.1b). In fact, consider the following differential equations[3]

$$\phi_i(w_1, \ldots, w_1^{(\alpha_{1i})}, \ldots, w_r, \ldots, w_r^{(\alpha_{ri})}) = 0,\ i \in \{1, \ldots, p\} \tag{1.2}$$

Let $\beta_j = \max_i\{\alpha_{ji}\}$, $i \in \{1, \ldots, p\}, j \in \{1, \ldots, r\}$. Consider the system S with state $x = (w_1, \ldots, w_1^{(\beta_1 - 1)}, \ldots, w_r, \ldots, w_r^{(\beta_r - 1)})$, input $u = (u_1, \ldots, u_r)$, where $u_j = w_j^{(\beta_j)}$, and output y defined by equations

$$\begin{aligned} &\left\{ \begin{array}{rcl} \dot{w}_j^{(0)} & = & w_j^{(1)} \\ & \vdots & \qquad\quad j \in \{1, \ldots, r\} \\ \dot{w}_j^{(\beta_j)} & = & u_j \end{array} \right. \\ &y_i = \phi_i(w_1, \ldots, w_1^{(\alpha_{1i})}, \ldots, w_r, \ldots, w_r^{(\alpha_{ri})}),\ i \in \{1, \ldots, p\}. \end{aligned} \tag{1.3}$$

It is clear that the system (1.2) is represented by the system (1.3) with the constraints $y_i = 0$, which is in the form (1.1a)-(1.1b). So, all the results developed here may be applied to a set of differential equations of arbitrary order.

We now present, without being precise, a summary of the ideas and the results of this paper. Roughly speaking, a subsystem S_1 of a system S is some part of S that may be considered as a system by itself. Note that S_1 may affect the "quotient system" S/S_1, but it is not affected by $S/S1$ as depicted in the Figure 1.1 :

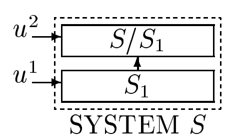

FIG. 1.1. *Structure of a system S with respect to a subsystem S_1.*

Remark 1.1. We stress that, in figure 1.1, S/S_1 is not a subsystem.

Recall that a system is *flat* if and only if there exists a differentially independent set of functions $y = (y_1, \ldots, y_m)$, called the *flat output*, such that every variable of the system is a function of the flat output and its derivatives. A system S is said to be *relatively flat with respect to a given subsystem* S_1 if, after a convenient *endogenous feedback*, S is decomposed into two independent subsystems S_1 and S_2 such that S_2 is a flat system[4] (see figure 1.2). We stress that the fact that the system is decomposed

[2]This class is more general than the one considered by [31, 27].

[3]As in the behavioral approach of Willems [59], we do not distinguish input, state and outputs among the variables $w_i, i = 1, \ldots r$ in the differential equations (1.2).

[4]See Definition 5.1 for a precise statement of *relative flatness*.

into two independent subsystems is not artificial since the same structure occurs for the algebraic counterpart of this definition (see Rem. 5.1).

In this paper, a sufficient condition for relative flatness is given (see Thm. 8.2). One can easily conclude that a system S that is relatively flat with respect to a flat subsystem is also flat[5] leading to a sufficient condition of flatness of system S.

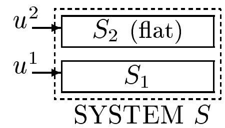

FIG. 1.2. *Structure of a system S that is relatively flat with respect to a subsystem S_1. Note that S_2 is flat.*

Now, let y be the output (not necessarily a flat output) of system S. We will show that one can construct a subsystem Y of S such that Y contains only the "information" of time and of y and its derivatives $y^{(k)}, k \in \mathbb{N}$ (see Theorem 4.3). Subsystem Y will be called *output subsystem*.

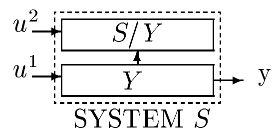

FIG. 1.3. *Structure of a System S with respect to the output subsystem Y.*

The structure of the implicit system, obtained from S by setting y to be equal to zero (see Figure 1.3) is directly related to the properties of S with respect to the output subsystem Y. Under some regularity assumptions, if S is relatively flat with respect to Y, then the implicit system obtained from S by including the constraint $y = 0$ is also flat[6].

The paper is organized as follows. In section 2 the notation and some mathematical background are presented. The infinite dimensional differential geometric approach of [20] is briefly summarized in section 3. The notion of *subsystem* is presented in section 4. The existence and some properties of local *output* subsystems are also discussed in § 4. The concept of *relative flatness* is discussed in section 5. In section 6 it is shown that, under regularity assumptions, an implicit system (1.1a)–(1.1b) may be considered as a system that is immersed in the explicit system (1.1a). In section 7, the results of the previous sections are used to derive a sufficient condition for flatness of implicit systems. A sufficient condition for relative flatness based on *relative derived flags* is developed in in section 8. Some examples are discussed in section 9. Finally, some auxiliary results and proofs are presented in appendices A, B and C.

2. Preliminaries and notation. The field of real numbers is denoted by $\mathbb{R}$ and $\mathbb{N}$ stands for the set natural numbers (including zero). The subset $\{1, \ldots, k\}$ of $\mathbb{N}$ is denoted by $\lfloor k \rceil$. Given a set W, then card W stands for the cardinality of W. We adopt the standard notations of differential geometry and exterior algebra in the finite and infinite dimensional case [57, 4, 60]. Let us briefly recall the main definitions of

[5] See Proposition 5.2 for a precise statement of this idea.

[6] See Theorem 7.2 for a precise statement of this sufficient condition of flatness.

the infinite dimension setting introduced in control systems theory [18, 42, 20]. This approach is mainly based on the differential geometry of jets and prolongations (see for instance [28, 60]) whereas the approach of [25] and [36] is based on finite dimensional differential geometry [57].

Let A be a countable set. Denote by $\mathbb{R}^A$ the set of functions from A to $\mathbb{R}$. One may define the coordinate function $x_i : \mathbb{R}^A \to \mathbb{R}$ by $x_i(\xi) = \xi(i), i \in A$. This set can be endowed with the Fréchet topology (*i.e.*, an *inverse limit* topology [2, 60]). A basis of this topology is given by the subsets of the form $\mathcal{B} = \{\xi \in \mathbb{R}^A \mid |x_i(\xi) - \delta_i| < \epsilon_i, i \in F\}$, where F is a finite subset of A, $\delta_i \in \mathbb{R}$ and ϵ_i is a positive real number for $i \in F$. A function $\phi : \mathbb{R}^A \to \mathbb{R}$ is smooth if $\phi = \psi(x_{i_1}, \ldots, x_{i_s})$, where $\psi : \mathbb{R}^s \to \mathbb{R}$ is a smooth function. Only the dependence on a finite number of coordinates is allowed.

From this notion of smoothness, one can easily state the notions of vector fields and differential forms[7] on $\mathbb{R}^A$ and smooth mappings from $\mathbb{R}^A$ to $\mathbb{R}^B$. The notion of $\mathbb{R}^A$-manifold can be also established easily as in the finitely dimensional case [60].

Given an $\mathbb{R}^A$-manifold $\mathcal{P}$, $C^\infty(\mathcal{P})$ denotes the set of smooth maps from $\mathcal{P}$ to $\mathbb{R}$. Let $\mathcal{Q}$ be an $\mathbb{R}^B$-manifold and let $\phi : \mathcal{P} \to \mathcal{Q}$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_p\mathcal{P} \to T_{\phi(p)}\mathcal{Q}$ and $\phi^* : T^*_{\phi(p)}\mathcal{Q} \to T^*_p\mathcal{P}$.

The map $\phi : \mathcal{P} \to \mathcal{Q}$ is called an *immersion* if, around every $\xi \in \mathcal{P}$ and $\phi(\xi) \in \mathcal{Q}$, there exist local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates $\phi(x) = (x, 0)$. The map ϕ is called a *submersion* if, around every $\xi \in \mathcal{P}$ and $\phi(\xi) \in \mathcal{Q}$, there exist local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates, $\phi(x, y) = x$.

In the finite dimensional case, immersion and submersions are locally characterized respectively by the injectivity and surjectivity of the tangent mappings. However, in the infinite dimensional case this is no longer true. Moreover, the inverse function Theorem and the classical Frobenius Theorem (for distributions) do not hold and a field does not admit a flow in general [60].

Given two forms η and ξ in $\Lambda(\mathcal{P})$, then $\eta \wedge \xi$ denotes their *wedge* multiplication. The *exterior derivative* of $\eta \in \Lambda(\mathcal{P})$ will be denoted by $d\eta$. Note that the graded algebra $\Lambda(\mathcal{P})$, as well as its homogeneous elements $\Lambda_k(\mathcal{P})$ of degree k, have a structure of $C^\infty(\mathcal{P})$-module. See [57, 4] for details. Given a family $\nu = (\nu_1, \ldots, \nu_k)$ of a $C^\infty(\mathcal{P})$-module, then span $\{\nu_1, \ldots, \nu_k\}$ stands for the span over $C^\infty(\mathcal{P})$.

Given a field f and a 1-form ω on $\mathcal{P}$, we denote $\omega(f)$ by $\langle \omega, f \rangle$. The set of smooth k-forms on $\mathcal{P}$ will be denoted by $\Lambda_k(\mathcal{P})$ and $\Lambda(\mathcal{P}) = \cup_{k \in \mathbb{N}} \Lambda_k(\mathcal{P})$.

The following useful result of finite dimensional differential geometry is known as "Cartan Lemma" ([57], p.80 ex. 16). Let $\{\omega_1, \ldots, \omega_r\} \subset \Lambda_1(\mathcal{P})$ be independent pointwise. Assume that there exist 1-forms $\eta_1, \ldots, \eta_r$ such that $\sum_{i=1}^r \eta_i \wedge \omega_i = 0$. Then there exist functions $a_{ij} \in C^\infty(\mathcal{P})$, with $a_{ij} = a_{ji}$, such that $\eta_i = \sum_{j=1}^r a_{ij}\omega_j \ \ (i = 1, \ldots, r)$. The same result is also valid pointwise, *i.e.*, $\sum_{i=1}^r \eta_i \wedge \omega_i\big|_p = 0$ implies that $\eta_i(p) = \sum_{j=1}^r a_{ij}\omega_j(p) \ \ (i = 1, \ldots, r)$ for convenient $a_{ij} = a_{ji} \in \mathbb{R}$.

A smooth codistribution J is a $C^\infty(\mathcal{P})$-submodule $J \subset T^*\mathcal{P}$. Given a submodule S of $\Lambda(\mathcal{P})$ and $p \in \mathcal{P}$, then $S(p)$ denotes the $\mathbb{R}$-linear subspace of $\Lambda_1(\mathcal{P})\big|_p$ given by $\mathrm{span}_{\mathbb{R}} \{\zeta(p) | \ \zeta \in S\}$. In particular, if J is a codistribution, then $J(p)$ denotes the subspace of $T^*_p\mathcal{P}$ given by $\mathrm{span}_{\mathbb{R}} \{\omega(p) | \ \omega \in J\}$[8].

[7] We stress that the forms are finite combinations of the form $\sum_i a_{I_i} dx_{I_i}$, where I_i is the multi-index $(j_{i,1}, \ldots, j_{i,r_i})$, the a_{I_i} are smooth functions, $dx_{I_i} = dx_{j_{i,1}} \wedge \ldots \wedge dx_{j_{i,r_i}}$. On the other hand, the fields are (possibly) infinite sums of the form $\sum_{i \in A} a_i \frac{\partial}{\partial x_i}$.

[8] One can also define a codistribution as a map $p \mapsto J(p)$ where $J(p)$ is a subspace of $T^*_p\mathcal{P}$.

Assume that a codistribution I is locally generated by $\eta_1, \dots, \eta_k$ and that $\Psi = \{x_i |\ i \in A\}$ is a local coordinate system around some open set $U \subset \mathcal{P}$. As $\eta_i = \sum_{\text{finite}} \alpha_{ij} dx_i$ for convenient smooth functions α_{ij}, then there must exist some finite subset $A_0 \subset A$ such that all the functions α_{ij} depend only on $\{x_i |\ i \in A_0\}$ and $\eta_i \in \operatorname{span}\{dx_j |\ j \in A_0\}$. Consider the finite dimensional vector space $\mathbb{R}^{A_0}$ and the canonical submersion $\pi : U \to \mathbb{R}^{A_0}$ such that $\pi \circ \Psi^{-1}(x_i |\ i \in A) = (x_i |\ i \in A_0)$. It is clear that the one-forms $\widetilde{\eta}_i = \sum_{\text{finite}} \alpha_{ij} dx_i$ on the open neighbourhood $\pi(U) \subset \mathbb{R}^{A_0}$ are such that $\widetilde{\eta}_i = \pi^* \eta_i, i \in \lfloor k \rceil$. Furthermore, if $\widetilde{I} = \operatorname{span}\{\widetilde{\eta}_i |\ i \in \lfloor k \rceil\}$, then $I = \pi^* \widetilde{I}$. In other words one may apply to (locally) finitely generated codistributions the standard techniques of differential geometry, for instance the Frobenius Theorem, by "pulling-back" the results that hold on the finite dimensional case [42].

3. Diffieties and Systems. In this section we recall the main concepts of the infinite dimensional geometric setting of [18, 42, 20]. We have chosen to present a simplified exposition. For a more complete and intrinsic presentation the reader may refer to the cited literature.

3.1. Diffieties. A *diffiety* M is a $\mathbb{R}^A$-manifold equipped with a distribution Δ of finite dimension r, called *Cartan distribution*. A section of the Cartan distribution is called a *Cartan field*. An *ordinary diffiety* is a diffiety for which $\dim \Delta = 1$ and a Cartan field ∂_M is distinguished and called *the Cartan field*. In this paper we will only consider ordinary diffieties, that will be called simply by *diffieties*.

A Lie-Bäcklund mapping $\phi : M \to N$ between diffieties is a smooth mapping that is compatible with the Cartan fields, *i.e.*, $\phi_* \partial_M = \partial_N \circ \phi$. A *Lie-Bäcklund immersion* (respectively, *submersion*) is a Lie-Bäcklund mapping that is an immersion (resp., submersion). A Lie-Bäcklund isomorphism between two diffieties is a diffeomorphism that is a Lie-Bäcklund mapping.

Context permitting, we will denote the Cartan field of an ordinary diffiety M simply by $\frac{d}{dt}$. Given a smooth object ϕ defined on M (a smooth function, field or form), then $L_{\frac{d}{dt}}(\phi)$ will be denoted by $\dot{\phi}$ and $L^n_{\frac{d}{dt}}(\phi)$ by $\phi^{(n)}, n \in \mathbb{N}$. In particular, if ω is a 1-form given by $\omega = \sum_{\text{finite}} \alpha_i dx_i$, then $\dot{\omega} = \sum_{\text{finite}} (\dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i)$.

3.2. Systems. The set of real numbers $\mathbb{R}$ has a trivial diffiety structure with the Cartan field defined by the operation of differentiation of smooth functions. A *system* is a triple $(S, \mathbb{R}, \tau)$ where S is a diffiety equipped with Cartan field $\frac{d}{dt}$, the mapping $\tau : S \to \mathbb{R}$ is a Lie-Bäcklund submersion and $\frac{d}{dt}(\tau) = 1$. The function τ represents *time*, that is chosen once and for all. Context permitting, the system $(S, \mathbb{R}, \tau)$ is denoted simply by S. A *Lie-Bäcklund mapping between two systems* $(S, \mathbb{R}, \tau)$ and $(S', \mathbb{R}, \tau')$ is a *time-respecting* Lie-Bäcklund mapping $\phi : S \to S'$, *i.e.*, $\tau' = \tau \circ \phi$. The previous condition means that the notion of time of both systems coincide. This notion of system is *time-varying* as it will be explained bellow.

3.3. State Representation. We present a simplified definition of state representation that introduces the state and the input and its derivatives as a local coordinate system (see [18, 20] for a more intrinsic presentation).

A local state representation of a system $(S, \mathbb{R}, \tau)$ is a local coordinate system $\psi = \{t, x, U\}$ where $x = \{x_i, i \in \lfloor n \rceil\}$, $U = \{u_j^{(k)} |\ j \in \lfloor m \rceil, k \in \mathbb{N}\}$, where $u_j^{(k)} = L^k_{\frac{d}{dt}} u_j$, and $\tau = t$. The set of functions $x = (x_1, \dots, x_n)$ is called state and $u = (u_1, \dots, u_m)$

is called input. In these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k\in\mathbb{N}} \sum_{j\in\lfloor m\rceil} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{3.1}$$

Note that f_i may depend on t, x and a finite number of elements of U. In this sense, the state representation defined here is said to be generalized, since one accepts that f_i may depend on the derivatives of the input. If the functions f_i depend only on $\{t, x, u\}$ for $i \in \lfloor n\rceil$, then the state representation is said to be *classical*. A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by (x, u). A state representation is said to be *analytic* if the f_i are all analytic[9].

3.4. Output. An output y of a system S is a set $y = (y_1, \ldots, y_p)$ of smooth functions defined on S. If (x, u) is a state representation of S, then it is clear that

$$y_j = y_j(t, x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p\rceil \tag{3.2}$$

If the y_j depend only on $\{t, x, u\}$ for $j \in \lfloor p\rceil$, then the output is said to be *classical* with respect to the state representation (x, u). A state representation (x, u) with output y is said to be *analytic* if the functions f_i and the y_j are all analytic with respect to its arguments x and $\{u^{(j)} \mid j \in \mathbb{N}\}$.

3.5. System associated to differential equations. Now assume that a control system is defined by a set of equations

$$\begin{array}{rcl} \dot{t} & = & 1 \\ \dot{x}_i & = & f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n\rceil \\ y_j & = & y_j(t, x, u, \ldots, u^{(\beta_j)}),\ j \in \lfloor p\rceil \end{array} \tag{3.3}$$

One can always associate to these equations a diffiety S of global coordinates $\psi = \{t, x, U\}$ and Cartan field given by (3.1).

3.6. Flatness. We present now a simple definition of flatness in terms of coordinates[10]. A system S equipped with Cartan field $\frac{d}{dt}$ and time function $t = \tau$ is locally flat around $\xi \in S$ if there exists a set of smooth functions $y = (y_1, \ldots, y_m)$, called *flat output*, such that the set $\{t, y_i^{(j)} \mid i \in \lfloor m\rceil, j \in \mathbb{N}\}$ is a (local) coordinate system of S around $\xi \in S$, where $y_i^{(j)} = L_{\frac{d}{dt}}^{j} y_i$. Note that the Cartan field is locally given by :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{j\in\mathbb{N}} \sum_{i\in\lfloor m\rceil} y_i^{(j+1)} \frac{\partial}{\partial y_i^{(j)}}$$

Let $\Psi : S \to T$ be a Lie-Bäcklund isomorphism between two systems. Then S is flat if and only if T is flat, also. If $y = (y_1, \ldots, y_m)$ is a flat output of T then $\{y_1 \circ \Psi, \ldots, y_m \circ \Psi\}$ is a flat output of S.

[9] This definition is coordinate dependent since only smooth atlases are considered on diffities [60].
[10] For more intrinsic definitions and some variations, see [18, 20].

3.7. Endogenous feedback and coordinate changes. Since a local state representation (x,u) is by definition a local coordinate system, a new local state representation (z,v) induces a coordinate change from $\{t,x,(u^{(i)}:i\in\mathbb{N})\}$ to $\{t,z,(v^{(j)}:i\in\mathbb{N})\}$. The coordinate changes of this kind are called *endogenous feedbacks*[11].

An example of endogenous feedback is static-state feedback. Two state representations (x,u) and (z,v) defined around $\xi\in S$ are said to be linked by (time-varying) static-state feedback if we locally have

$$\operatorname{span}\{dt,dx\}=\operatorname{span}\{dt,dz\} \tag{3.4a}$$

$$\operatorname{span}\{dt,dx,du\}=\operatorname{span}\{dt,dz,dv\}\,. \tag{3.4b}$$

Let (x,u) be a classical state-representation and let z and v be family of smooth functions such that card $x=$ card z and card $u=$ card v. Then it is easy to show that, if (3.4) locally holds, then (z,v) is a local state representation that is linked to (x,u) by static-state feedback [39, Prop. 3.2].

Another example of endogenous feedback is putting integrators in series with the first k inputs of the system (3.3). This procedure induces a local state representation (z,v) of the system S, where $z=(x_1,\ldots,x_n,u_1,\ldots,u_k)$ and $v=(\dot u_1,\ldots,\dot u_k,u_{k+1},\ldots,u_m)$, called *dynamic extension* of the state.

4. Subsystems. A (local) *subsystem* S_a of a given system S is a system S_a such that there exists a surjective[12] Lie-Bäcklund submersion $\pi:U\subset S\to S_a$, where U is an open subset of S. A (local) subsystem will be denoted by (S_a,π) or simply by S_a.

4.1. State equations adapted to subsystems. Assume that there exists a local classical state representation (x,u) of a system S of the form

$$\dot x_a=f_a(t,x_a,u_a) \tag{4.1a}$$

$$\dot x_b=f_b(t,x_a,x_b,u_a,u_b). \tag{4.1b}$$

where $x=(x_a,x_b)$ and $u=(u_a,u_b)$. Assume that (4.1a) represents the state equations of a subsystem S_a and $\pi:S\to S_a$ is such that $\pi(t,x,U)=(t,x_a,U_a)$, where U denotes the set $(u^{(j)}|\ j\in\mathbb{N})$ and U_a denotes the set $({u_a}^{(j)}|\ j\in\mathbb{N})$. A state representation of S the form (4.1a)-(4.1b) is said to be adapted to the subsystem S_a. In the end of this section we show that state equations adapted to a a subsystem can be generically constructed (see Proposition 4.4).

4.2. Relative static-state feedback. We will consider now a special case of endogenous feedback that will be called by *Relative Static-State Feedback.* Consider that $((x_a,x_b),(u_a,u_b))$ is a local state representation for system S such that the state equations are of the form (4.1a)–(4.1b). A relative state feedback is a new state representation $((x_a,z_b),(u_a,v_b))$ such that

$$\begin{array}{rcl} z_b &=& z_b(t,x_b,\,x_a,u_a,\ldots,{u_a}^{(r)})\\ v_b &=& v_b(t,x_b,u_b,x_a,u_a,\ldots,{u_a}^{(r+1)})\end{array} \tag{4.2}$$

where r is a convenient integer and similar equations do exist for x_b,u_b as functions of $x_a,z_b,\,u_a,v_b$ and the derivatives of u_a. In other words, this is an invertible time-varying feedback. The next definition renders this notion more intrinsic.

[11] See [18] for a definition of endogenous feedback that considers an equivalence relation between systems.

[12] Since submersions are open maps, one can always consider that $S_a=\pi(U)$ by restricting S_a to the image of π.

DEFINITION 4.1. *Let S be a system and let (π, S_a) be a (local) subsystem of S. Let (x,u) and (z,v) be two (local) state representations of S. Let Σ be the codistribution defined by the pull-back*[13] $\Sigma = \pi^*(T^*S_a)$. *Then (x,u) and (z,v) are linked by a* relative static-state feedback *with respect to the subsystem S_a if* $\mathrm{span}\,\{dx\}+\Sigma = \mathrm{span}\,\{dz\}+\Sigma$ *and* $\mathrm{span}\,\{dx, du\} + \Sigma = \mathrm{span}\,\{dz, dv\} + \Sigma$.

PROPOSITION 4.2. *Let S be a system with local state representation (x,u) defined on $V_\xi \subset S$, where $x = (x_a, x_b)$, and $u = (u_a, u_b)$ are such that the state equations are of the form* (4.1a)-(4.1b). *Let S_a be the (local) subsystem associated to equation* (4.1a). *Consider the set of smooth functions $z = (x_a, z_b)$ and $v = (u_a, v_b)$ defined on V_ξ, where* $\mathrm{card}\; x = \mathrm{card}\; z = n$ *and* $\mathrm{card}\; u = \mathrm{card}\; v = m$. *Then the following statements are equivalent :*

(i) *(z,v) is a local state representation around ξ and (x,u) and (z,v) are linked by relative static-state feedback.*

(ii) $\mathrm{span}\,\{dx\} + \Sigma = \mathrm{span}\,\{dz\} + \Sigma$ *and* $\mathrm{span}\,\{dx, du\} + \Sigma = \mathrm{span}\,\{dz, dv\} + \Sigma$.

Proof. Deferred to the Appendix B.1. □

Remark 4.1. The proof of the Prop. 4.2 shows that (i) implies that condition (4.2) is satisfied for a subsystem S_a defined by (4.1a). It will be shown (see Prop. 4.4) that all subsystems admit adapted state equations of the form (4.1a)–(4.1b), up to relative static-state feedbacks.

4.3. Output Subsystem. Given a system S with output y, a (local) *output subsystem* is a (local) subsystem $\pi : U \subset S \to Y$ such that $\pi^*\left(T^*_{\pi(\xi)}Y\right) = \mathrm{span}\left\{dt, dy^{(k)} : k \in \mathbb{N}\right\}|_\xi, \xi \in U$.

4.4. Existence of local output subsystems. Without loss of generality, assume that (x,u) is a classic state representation with output y. If it is not the case we can add integrators in series with the input until the required properties are fulfilled. The next Theorem shows that local output subsystems can be constructed generically and they admit adapted state equations up to relative static-state feedback. Furthermore, they are unique up to Lie-Bäcklund isomorphisms.

THEOREM 4.3. *(Existence and Uniqueness of Output Subsystems) Let S be a system and let (x,u) be a classical* analytical *state representation defined on a open neighbourhood $W \subset S$. Let y be a classical output of S. Let* $n = \mathrm{card}\; x$. *Let $U \subset W$ be the set of regular points of the codistributions $Y_k, \mathcal{Y}_k, k \in \lfloor n \rceil$, where* $Y_k = \mathrm{span}\left\{dt, dy, \ldots, dy^{(k)}\right\}$ *and* $\mathcal{Y}_k = \mathrm{span}\left\{dt, dx, dy, \ldots, dy^{(k)}\right\}$. *Then, around any $\xi \in U$, there exists an open neighbourhood V_ξ of ξ, a local* classical *state representation $(z,v) = ((z_a, z_b), (v_a, v_b))$ of the system S defined on V_ξ such that:*

(i) *The (local) state equations are :*

$$\dot{z}_a = f_a(t, z_a, v_a) \tag{4.3a}$$

$$\dot{z}_b = f_b(t, z_a, z_b, v_a, v_b). \tag{4.3b}$$

(ii) *Let Y be the local subsystem associated to* (4.3a) *and let $\pi : V_\xi \to Y$ be the corresponding Lie-Bäcklund submersion. We have* $\pi^*(T^*Y) = \mathrm{span}\;\{dt,\, dz_a,\, (dv_a^{(k)} : k \in \mathbb{N})\} = \mathrm{span}\left\{dt, dy^{(k)} : k \in \mathbb{N}\right\}$. *In particular, Y is an output subsystem of S. Let $\mathcal{Z} = \{z_a, (v_a^{(k)} : k \in \mathbb{N})\}$ and $\mathcal{Y} = \{y_j^{(k)} : j \in \lfloor p \rceil, k \in \mathbb{N}\}$. Then $\mathcal{Z} \subset \mathcal{Y}$.*

(iii) *The state representations (x,u) and (z,v) are linked by relative static-state feedback with respect to the subsystem Y associated to* (4.3a).

[13] Note that $\mathrm{span}\,\{dt\} \subset \Sigma$.

Furthermore, two local output subsystems around any $\xi \in S$ are (locally) Lie-Bäcklund isomorphic.

Proof. See Appendix B.2. □

We state now the result that assures that a subsystem can be generically represented by state equations of the form (4.1a)-(4.1b).

PROPOSITION 4.4. *Assume that S_a is a subsystem of S and that there exist local state representations for S_a and S around every point of S_a and S. Then, generically, there exists local state representations of S of the form* (4.1a)-(4.1b) *in a way that* (4.1a) *is a state representation of S_a.*

Proof. Let $\pi : S \to S_a$ be the corresponding Lie-Bäcklund submersion. Take a local state representation (z_a, e_a) of S_a around $\pi(\xi) \in S_a$. We abuse notation and denote $z_a \circ \pi$ and $e_a \circ \pi$ respectively by z_a and e_a. Now, consider system S with output $y = (z_a, e_a)$ and construct, possibly by extending the state with derivatives of the input, a classical state representation of S such that y is a classical output. The result follows easily from the application of Thm. 4.3 and the fact that $T^*S_a = \operatorname{span}\left\{dt, dz_a^{(k)}, de_a^{(k)}, k \in \mathbb{N}\right\}$. □

If the outputs are differentially independent, the next result shows that local output subsystems are generically flat.

PROPOSITION 4.5. *Let U be the open and dense subset of theorem 4.3. Assume that the (explicit) system* (1.1a) *with output $y = h(t, x, u)$ is right-invertible,* i. e.*, the the output rank ρ is equal to the number of output components*[14]*. Let $\pi : V_\xi \subset S \to Y$ be a local output subsystem with $V_\xi \subset U$. Then Y is (locally) flat with flat output y.*

Proof. The proof is deferred to Appendix B.3. □

5. Relative flatness. We now state the concept of *relative flatness.*

DEFINITION 5.1. *Let S be a system and (π_1, S_1) and (π_2, S_2) be two subsystems of S. The system S is said to be locally* decomposed by *S_1 and S_2 if, around $\xi \in S$, there exists local coordinates (t, x^1) for S_1, (t, x^2) for S_2 and (t, x^1, x^2) for*[15] *S such that $\pi_i(t, x^1, x^2) = (t, x^i), i = 1, 2$. A system S is said to be (locally)* relatively flat *with respect to a subsystem S_1 if there exists a flat subsystem S_2 such that S is (locally) decomposed by S_1 and S_2.*

PROPOSITION 5.2. *Let S_1 be a (locally) flat subsystem of a system S. Assume that S is relatively flat with respect to S_1. Then S is (locally) flat.*

Proof. The union of flat outputs of S_1 and S_2 is a flat output of S. □

Remark 5.1. In the differential algebraic approach of [15] (see also [22]) one can define a subsystem of a system K/k as a field extension L/k such that L is a subfield of K. Then a system K/k is relatively flat with respect to L if the system K/L is flat, considering L as the ground field (see [12] for a a result similar to Proposition 5.2.). However, these algebraic notions are not suitable for our purposes because integrability conditions are not available in this algebraic context.

It can be shown that, if K/k is relatively flat with respect to L, then K/k can be decomposed into two independent subsystems L/k and F/k, where F/k is flat. (see Appendix A and [58]). In this sense, the assumption that the system is decomposed into two independent subsystems in the definition of relative flatness is not restrictive with respect to the algebraic definition (see also [38] for similar facts that occur when L corresponds to the noncontrollable subsystem.)

[14] See Appendix C for the definition of the output rank ρ.

[15] We abuse notation and denote $x^i \circ \pi_i$ simply by x^i.

The following proposition is a necessary and sufficient condition for completing a given output y into a flat output (see [44] for related results).

PROPOSITION 5.3. *Let S be a system and let S_1 be a flat subsystem of S. Let y be a (local) flat output for S_1. Then there exists a set z of smooth functions such that S is locally flat with flat output (y, z) if and only if S is relatively flat with respect to the subsystem S_1.*

Proof. The necessity is obvious. The sufficiency follows from the proof of Prop. 5.2. ◻

6. Implicit systems regarded as Lie-Bäcklund immersions. Let S be the nonconstrained system defined by (1.1a). We show that, under some regularity assumptions, (1.1a)-(1.1b) may be regarded as a system that is immersed in S. We construct a system Γ and a Lie-Bäcklund immersion $\iota : \Gamma \to S$ such that every integral curve $\sigma(t)$ of the Cartan-field of S, respecting the constraints $y(t) \equiv 0$, is of the form $\sigma(t) = \iota \circ \gamma(t)$ for a suitable integral curve $\gamma(t)$ of the Cartan-field of Γ.

Consider the explicit (nonconstrained) system S defined by (1.1a) with output $y = h(t, x, u)$, global coordinates $\{t, x, (u_i^{(j)} : i \in \lfloor m \rceil; j \in \mathbb{N})\}$, and Cartan field (3.1). Consider now the following assumptions :

A1. Existence and Regularity Assumption. *Let $\Gamma = \{\xi \in S \mid y^{(k)}(\xi) = 0$ for all $k \in \mathbb{N}\}$. Assume that $\Gamma \neq \emptyset$ and furthermore, $\Gamma \subset U$, where U is the open and dense subset of the system S such that the statement of Theorem 4.3 holds*[16]*. In other words, around every point $\xi \in \Gamma$, we can construct a local output subsystem.*

A2. Time Interval Assumption. *For every $\xi \in \Gamma$ and every open neighbourhood $U \subset S$ of ξ, there exists some $\epsilon \geq 0$ such that $\tau(\Gamma \cap U)$ contains an open interval $(\tau(\xi) - \epsilon, \tau(\xi) + \epsilon)$.*

Remark 6.1. Note that assumption A2 means that Γ "does exist" during an interval of time. If the system is time-invariant it is easy to verify that assumption A2 is not needed. Note also that the set Γ may be empty, and in this case the implicit system has no solution. For instance, let $y_1 = x_1 + 1$ and $y_2 = x_1^2 + 2$. Then $y_1 = 0$ implies that $y_2 \neq 0$. A problem of this nature may occur with output derivatives.

When the assumptions A1, A2 holds, the set $\Gamma \subset S$ may be endowed with the structure of an immersed Fréchet manifold by choosing the subset topology, as shown by the following proposition.

PROPOSITION 6.1. *Suppose that assumptions A1 and A2 are satisfied for system S. Then the subset $\Gamma \subset S$ has a structure of immersed manifold in S. Let $\iota : \Gamma \to S$ be the canonical insertion. We can define a Cartan field ∂_Γ on Γ by the equation $\iota_* \partial_\Gamma(\gamma) = \frac{d}{dt} \circ \iota(\gamma), \gamma \in \Gamma$. Equipped with this Cartan field, Γ is a system such that ι is a Lie-Bäcklund immersion. Furthermore, all the solutions $\xi(t)$ of* (1.1a) *obeying the restriction* (1.1b) *are of the form $\xi(t) = \iota \circ \nu(t)$ where $\nu(t)$ is a solution of Γ.*

Proof. We show first that Γ is an immersed manifold. For this, consider the topological subspace $\Gamma \subset S$ with the subset topology. For each point $\xi \in \Gamma$, Thm. 4.3 gives local charts $\phi : \hat{U} \to \tilde{U} \subset \mathbb{R}^A$, where $\phi = \{t, z_a, V_a, z_b, V_b\}$, $V_a = \{v_a^{(k)} : k \in \mathbb{N}\}$ $V_b = \{v_b^{(k)} : k \in \mathbb{N}\}$, and we have span $\{dt, dz_a, dV_a\} =$ span $\{dt, dy^{(k)} : k \in \mathbb{N}\}$. This local chart is adapted to a local output subsystem $\pi : \hat{U} \to Y$, and is such that $\pi(t, z_a, V_a, z_b, V_b) = (t, z_a, V_a)$. Furthermore, by part (ii) of Thm. 4.3, the functions of the set $\mathcal{Z} = \{z_a, V_a\}$ are such that $\mathcal{Z} \subset \mathcal{Y}$, where $\mathcal{Y} = \{y^{(k)} : k \in \mathbb{N}\}$. By construction,

[16]Note that in this case the state representation (1.1a) is globally defined. According the proof of theorem 4.3 we have that U is the open and dense set of regular points of the codistributions $Y_k =$ span $\{dt, dy, \ldots, dy^{(k)}\}$ and $\mathcal{Y}_k =$ span $\{dt, dx, dy, \ldots, dy^{(k)}\}$ for $k \in \{0, 1, \ldots, n\}$.

if $\nu \in \hat{U}\cap\Gamma$ then $y^{(k)}(\nu) = 0$ for all $k \in \mathbb{N}$. This implies that all the components of $\mathcal{Z}$ are also null in ν. If we show that the functions in $\mathcal{W} = \mathcal{Y}-\mathcal{Z}$ are also null in $\nu \in \Gamma\cap\hat{U}$, we will show that a point ν is in $\Gamma\cap\hat{U}$ if and only if $z_a = 0$ and $V_a = 0$ in ν. In fact, note first that, since span $\{dt, d\mathcal{Z}\}$ = span $\{dt, d\mathcal{Y}\}$, all the functions θ in $\mathcal{Y}$ can be locally written in the form $\theta = \theta(t, z_a, V_a)$. By assumption A2, if we restrict $\tilde{U}$ to a basic open set of the form $I_{\tau(\xi)} \times W$ where $I_{\tau(\xi)} = (\tau(\xi)-\epsilon, \tau(\xi)+\epsilon)$, we may assume that, for every $\bar{t} \in I_{\tau(\xi)}$, then $\hat{U}\cap\Gamma$ contains a point $\xi_{\bar{t}} = (\bar{t}, z_a, V_a, z_b, V_b) = (\bar{t}, 0, 0, z_b, V_b)$. For any fixed $\bar{t} \in I_{\tau(\xi)}$, since $\xi_{\bar{t}} \in \Gamma\cap\hat{U}$, we have that $\theta(\xi_{\bar{t}}) = \theta(\bar{t}, 0, 0) = 0$. Since this is true for all $\bar{t} \in I_{\tau(\xi)}$, we have showed our claim.

Now consider the map $\mu : \Gamma\cap\hat{U} \to \mu(\Gamma\cap\hat{U}) \subset \mathbb{R}^B$ such that $\mu(t, 0, 0, z_b, V_b) = (t, z_b, V_b)$. We shall show that these maps form an smooth atlas of Γ. By construction it is clear that these maps are homeomorphisms. Hence it suffices to show that these charts are C^∞ compatible. By convenience denote the functions of the chart ϕ by $\{t, X, Z\}$ and the functions of the chart μ by $\{t, Z\}$, where $X = \{z_a, V_a\}$ and $Z = \{z_b, V_b\}$.

Now let $\mu_i : \Gamma\cap U_i \to \tilde{V}_i$, $i = 1, 2$, be two local charts constructed in that way, based respectively on the local charts of S given by $\phi_i = \{t, X_i, Z_i\}$, $i = 1, 2$. In particular, it follows that $\mu_i \circ \phi_i(t, 0, Z_i) = (t, Z_i)$, $i = 1, 2$. Without loss of generality, assume that $U_1 = U_2$. Consider the local coordinate change $(t, X_1, Z_1) = \phi_1 \circ \phi_2^{-1}(t, X_2, Z_2)$. Note that the map $\theta : \tilde{V}_2 \to \tilde{V}_1$ such that $(t, Z_1) \mapsto (t, Z_2)$ defined by $(t, 0, Z_1) = \phi_1 \circ \phi_2^{-1}(t, 0, Z_2)$ is a local diffeomorphism with inverse defined by $(t, 0, Z_2) = \phi_2 \circ \phi_1^{-1}(t, 0, Z_1)$. Since $\theta = \mu_1 \circ \mu_2^{-1}$, we conclude that such charts are C^∞ compatible.

Now let $\iota : \Gamma \to S$ be the insertion map. In the coordinates ϕ and μ previously constructed, we have $\iota(t, Z) = (t, 0, Z)$. In particular ι is an immersion between $\mathbb{R}^A$-manifolds and so $\iota_*(\zeta)$ is injective for all $\zeta \in \Gamma$. Remember that any function η of the set $X = \{z_a, V_a\} \subset \mathcal{Y}$ is such that $\dot{\eta}|_\nu = 0$ for every $\nu \in \Gamma\cap\hat{U}$. In particular, we have that the image of $\iota_*(\nu)$ contains $\frac{d}{dt}(\iota(\nu))$ for every $\nu \in \Gamma\cap\hat{U}$. So we can define ∂_Γ by the rule $\iota_*\partial_\Gamma = \frac{d}{dt} \circ \iota$. By definition, it follows that ι is a Lie-Bäcklund immersion.

The last affirmation of the statement is a consequence of the first one. □

Remark 6.2. Let $\phi = (t, x_a, V_a, x_b, V_b)$ and $\mu = (t, x_b, V_b)$ be respectively the coordinates of S and Γ constructed above. In this coordinates we have

$$\partial_\Gamma = \frac{\partial}{\partial t} + \sum_{i=1}^{n_b} f_{b_i}(t, 0, 0, x_b, V_b)\frac{\partial}{\partial x_{b_i}} + \sum_{i=1}^{m_b}\sum_{j\in\mathbb{N}} u_{b_i}^{(j+1)}\frac{\partial}{\partial u_{b_i}^{(j)}} \tag{6.1}$$

where $f_{b_i} = \frac{d}{dt}(x_{b_i}) = f_{b_i}(t, x_a, V_a, x_b, V_b)$, $i \in \lfloor n_b \rceil$. In other words, (x_b, u_b) is a state representation of Γ.

It is easy to show that the pull-back (by ι) of a relative static-state feedback for S w. r. t. a local output subsystem Y induces a static-state feedback for Γ, if one considers the state representation $((x_a, x_b), (u_a, u_b))$ for S and (x_b, u_b) for Γ.

7. Flatness of implicit systems. In this section we will derive a sufficient condition for flatness of implicit systems. Let us begin with an auxiliary result.

PROPOSITION 7.1. *Let Γ, S and Y be systems, where Γ is immersed in S and Y is a subsystem of S. Let $\iota : \Gamma \to S$ and $\pi : S \to Y$ be respectively the corresponding Lie-Bäcklund immersion and submersion. Assume that there exist local coordinates (t, γ) of Γ, (t, γ, y) of S and (t, y) of Y such that $\iota(t, \gamma) = (t, \gamma, 0)$ and*[17] *$\pi(t, \gamma, y) = (t, y)$.*

[17] We assume that (t, y) is inside the domain of our local chart of Y for $y = 0$.

Assume that S is relatively flat with respect to Y. Then Γ is (locally) flat.

Proof. Let S_2 be a flat subsystem of S such that S_2 and Y decomposes S (see Definition 5.1). Let $\pi_2 : S \to S_2$ be the corresponding Lie-Bäcklund submersion. Recall that there exists coordinates $(t, z, \tilde{y})$ of S, $(t, \tilde{y})$ of Y, and (t, z) of S_2 such that $\pi_2 : (t, z, \tilde{y}) = (t, z)$ and $\pi : (t, z, \tilde{y}) = (t, \tilde{y})$. Since the coordinate change map $(t, y) \to (t, \tilde{y})$ is a local diffeomorphism, we may assume without loss of generality that $\tilde{y} = y$. With a possible restriction of domains, we can consider the coordinate change mapping $\phi(t, \gamma, y) = (t, z, y)$. Note that the map $\phi_0(t, \gamma) = (t, z)$ such that $\phi(t, \gamma, 0) = (\phi_0(t, \gamma), 0) = (t, z, 0)$ is a local diffeomorphism. Let $\Psi : \Gamma \to S_2$ be such that $\Psi = \pi_2 \circ \iota$. By definition, Ψ is a Lie-Bäcklund mapping since it is a composition of Lie-Bäcklund mappings. In the coordinates (t, z) for S_2 and (t, γ) for Γ we have $\Psi(t, \gamma) = \phi_0(t, \gamma)$. Hence Ψ is a local Lie-Bäcklund isomorphism and so Γ is flat. In particular if θ is a flat output of S_2 then $\theta \circ \Psi$ is a flat output of Γ. $\square$

The following result is a sufficient condition for flatness of an implicit system.

THEOREM 7.2. *Let S be the explicit system defined by* (1.1a). *Let $y = h(t, x, u)$ be an output for system S and let Y be the corresponding output subsystem of S. Suppose that Assumptions A1-A2 of the previous section hold for the system* (1.1a) *with the constraints* (1.1b). *According to Proposition 6.1, equations* (1.1a)-(1.1b) *define a system Γ that is immersed in S. Assume that the explicit system* (1.1a) *is (locally) relatively flat w.r.t. the subsystem Y. Then the implicit system Γ is locally flat around all $\xi \in \Gamma$.*

Proof. Let $\iota : \Gamma \to S$ be the insertion map and let $\pi : U \subset S \to Y$ be the canonical submersion onto the local output subsystem Y. According to the proof of Prop. 6.1, we can define local charts $\phi = (t, X, Z)$ of S, $\mu = (t, Z)$ of Γ and $\Psi = (t, X)$ of Y such that $\iota(t, X) = (t, 0, Z)$ and $\pi(t, X, Z) = (t, X)$. Hence, by Prop. 7.1 (for $\gamma = Z$ and $y = X$) the result follows. $\square$

Let (1.1a) be a flat (explicit) system and assume that the output y of (1.1b) is part of the flat output of the explicit system (1.1a). Then next result shows that the implicit system (1.1a)-(1.1b) is flat.

COROLLARY 7.3. *Assume that S is locally flat with flat output $y = (y_1, \ldots, y_m)$. Assume that the local coordinate system $\{t, y_i^{(j)} : i \in \lfloor m \rceil, j \in \mathbb{N}\}$ is defined on open set V whose image is a basic open set $\tilde{V}$*[18]. *Let $\Gamma \subset V$ defined by $\{\xi \in V \, | y_i^{(j)}(\xi) = 0, i \in \lfloor r \rceil, j \in \mathbb{N}\}$. Assume that Γ is nonempty. Then Γ is an immersed system in $V \subset S$. Furthermore Γ is (locally) flat with flat output $y_{r+1}, \ldots, y_m$.*

Proof. Consider system S with output $\tilde{y} = (y_{r+1}, \ldots, y_m)$. Let $\tilde{x} = \emptyset$ and $\tilde{u} = (\tilde{y}_1, \ldots, \tilde{y}_m)$. Then $(\tilde{x}, \tilde{u})$ is a local state representation of S. Let $\tilde{Y}_r = \text{span}\,\{dt, dy, \ldots, dy^{(k)}\}$ and $\tilde{\mathcal{Y}}_r = \text{span}\,\{dt, d\tilde{x}, dy, \ldots, dy^{(k)}\}$. Then $\tilde{Y}_r = \tilde{\mathcal{Y}}_r$ are nonsingular codistributions on S for $r \in \mathbb{N}$ and hence the assumption A1 of § 6 holds. Since $\tilde{V}$ is a basic open set, it is also clear that assumption A2 holds. By Thm. 4.3, the output subsystem $\tilde{Y}$ is well defined, and by Prop. 4.5, it follows that $\tilde{Y}$ is locally flat. By Prop. 5.3, S is relatively flat w. r. t. $\tilde{Y}$. The desired result follows from Thm. 7.2. $\square$

8. A sufficient condition for relative flatness. Consider a system S and a subsystem S_1 of S given by equations (4.1a)-(4.1b) where (4.1a) represents S_1. Let $\dim x_a = n_a$, $\dim x_b = n_b$, $\dim u_a = m_a$, and $\dim u_b = m_b$. For this system one can

[18] Recall that a basic open set is of the form $\tilde{V} = \{\xi \in S \;\big|\; |y_i^{(j)}(\xi) - \bar{y}_i^{(j)}| < \epsilon_{ij} \;, (i,j) \in \Delta\}$, where Δ is a finite subset of $\lfloor m \rceil \times \mathbb{N}$, $\bar{y}_i^{(j)} \in \mathbb{R}$, and $\epsilon_{ij} \in \mathbb{R}^+$.

define the *Relative Derived Flag* as follows.

DEFINITION 8.1. *The Relative Derived Flag of the system* (4.1a)-(4.1b) *is the sequence of codistributions* $I^{(k)}$ *defined by* $I^{(-1)} = \text{span}\{(dx_b - f_b dt), (du_b - \dot{u}_b dt)\}$, *and* $I^{(k)}(p) = \text{span}\{\omega(p) \mid \ \omega \in I^{(k-1)},\ d\omega(p) \bmod \ (I^{(k-1)} + J)|_p \equiv 0\}$, $k \in \mathbb{N}$ *where*

$$J \quad = \quad \text{span}\left\{(dx_a - \dot{x}_a dt), (d{u_a}^{(j)} - {u_a}^{(j+1)} dt) |\ j \in \mathbb{N}\right\}. \tag{8.1}$$

Remark 8.1. In the proof of Prop. 8.3 it is shown that if $I^{(k)}$ is nonsingular then it is smooth (otherwise $I^{(k+1)}$ is not well defined). The 1-forms in span $\left\{\frac{d}{dt}\right\}^{\perp}$ are called *contact forms* [42]. Let $\pi : S \to S_a$ be the Lie-Bäcklund submersion of S onto subsystem S_a (see § 4.1). Then it is easy to show that J is the codistribution generated by the contact-forms of S_a, *i.e.*, $J = \pi^*(T^*S_a) \cap \text{span}\left\{\frac{d}{dt}\right\}^{\perp}$. It follows that J is invariant by coordinate changes, and in particular, it is invariant by endogenous feedback. In Appendix B.4 it is shown that :

$$I^{(0)} = \text{span}\{dx_b - \dot{x}_b dt\} \tag{8.2}$$

By construction we have $\dim I^{(-1)} = n_b + m_b$ and $\dim I^{(0)} = n_b$. Note also that $I^{(k)} + J \subset I^{(-1)} + J \subset \text{span}\left\{\frac{d}{dt}\right\}^{\perp}, k \in \mathbb{N}$. We will show that the relative derived flag carries an intrinsic structural information, at least if one restricts the class of transformations to *relative static-state feedback* (see Cor. 8.4).

THEOREM 8.2. *Assume that the codistributions* span $\left\{I^{(k)}, dt, J\right\}$ *are involutive, that* $I^{(k)}$ *are nonsingular for all* $k \in \mathbb{N}$ *and that* $I^{(N)} = 0$ *for* N *big enough. Then the system* S *is (locally) relatively flat w.r.t.* S_1.

Remark 8.2. It is easy to verify that J is involutive, *i.e.*, that $d\omega \bmod J \equiv 0$ for all 1-forms $\omega \in J$. Furthermore, the codistribution span $\left\{I^{(k)}, dt, J\right\}$ is involutive if and only if span $\left\{I^{(k)}, dt, J_{\rho_k}\right\}$ is involutive for ρ_k big enough, where

$$J_l = \text{span}\left\{(dx_a - f_a dt), (d{u_a}^{(j)} - {u_a}^{(j+1)} dt) |\ j \in \lfloor l \rceil\right\}. \tag{8.3}$$

To prove Theorem 8.2 we need the following auxiliary result whose proof is deferred to Appendix B.5.

PROPOSITION 8.3. *Assume that the conditions of the Theorem 8.2 are satisfied on an open neighbourhood* V_ξ *of* ξ *in* S. *Then, for every* $p \in V_\xi$ *and* $k \in \mathbb{N}$ *we have* $\dim(I^{(k)} + J)|_p / J(p) = \dim I^{(k)}(p)$. *Assume that* $I^{(k-1)} + J$ *has a local basis* $B = \bar{B} \cup B_J$, *where* B_J *is a local basis of* J *and* $\bar{B}$ *is of the form*

$$\bar{B} = \left\{\omega_i^{(j)} : i \in \lfloor s \rceil, j \in \{0, \ldots, r_i\}\right\} \tag{8.4}$$

where $\omega_i = d\theta_i - \dot{\theta}_i dt$, $\theta_i \in C^\infty(S)$, $i \in \lfloor s \rceil$ *(or* $\bar{B} = \emptyset$*). Assume that the subset* $\{\omega_i^{(r_i)} : i \in \lfloor s \rceil\}$ *is linearly independent* $\bmod \{I^{(k)} + J\}$. *Let* $\dot{B} = \{\omega_i^{(r_i+1)} : i \in \lfloor s \rceil\}$. *Then we may complete the set* $B \cup \dot{B}$ *with a set* $\hat{B} = \{\omega_i, i = s+1, \ldots, \sigma\}$, *where* $\omega_i = d\theta_i - \dot{\theta}_i dt$ *in a way that* $B \cup \dot{B} \cup \hat{B}$ *is a basis of* $I^{(k-2)} + J$ *such that* $\dot{B} \cup \hat{B}$ *is linearly independent* $\bmod \{I^{(k-1)} + J\}$.

Proof. (of Theorem 8.2[19]). Let $N \in \mathbb{N}$ be the smallest integer such that $I^{(k)} = I^{(k+1)} = 0$ for all $k \geq N$. Let $\mathcal{B}_N$ be a basis for $J = I^{(N)} + J$ given by $\mathcal{B}_N =$

[19] Most of the techniques that are necessary for the proof of our sufficient condition of relative flatness are very similar to the techniques of the proof of the main result of [38].

$\{\eta, \mu_l |\ l \in \mathbb{N}\}$ where $\eta = (dx_a - \dot{x}_a dt)$ and $\mu_l = (du_a{}^{(l)} - u_a{}^{(l+1)}dt), l \in \mathbb{N}$. Since span $\left\{I^{(N-1)}, J, dt\right\}$ is involutive and $I^{(N-1)}$ is nonsingular, by Prop. 8.3 with $\bar{B} = \emptyset$, we can construct a local basis $\mathcal{B}_{N-1}$ of $I^{(N-1)} + J$ of the form $\mathcal{B}_{N-1} = A_{N-1} \cup \mathcal{B}_N$ where $A_{N-1} = \{(d\theta_{1,i_1} - \frac{d}{dt}\theta_{1,i_1}dt) |\ i_1 \in \lfloor s_{N-1}\rceil\}$. Let $\dot{A}_{N-1} = \{(d\theta^{(1)}_{1,i_1} - \theta^{(2)}_{1,i_1}dt),\ i_1 \in \lfloor s_{N-1}\rceil\}$. By Prop. 8.3, we may construct a set $\hat{A}_{N-1} = \{(d\theta_{2,i_2} - \frac{d}{dt}\theta_{2,i_2}dt) |\ i_2 \in \lfloor s_{N-2}\rceil\}$ in a way that $\mathcal{B}_{N-2} = A_{N-2} \cup \mathcal{B}_N$ is a basis of $I^{(N-2)} + J$ where $A_{N-2} = \hat{A}_{N-1} \cup \dot{A}_{N-1} \cup A_{N-1} = \{(d\theta^{(j-1)}_{k,i_k} - \theta^{(j)}_{k,i_k}dt) |\ k \in \lfloor 2\rceil, i_k \in \lfloor s_{N-k}\rceil, j \in \lfloor 2-k+1\rceil\}$. Note also that, by Prop. 8.3, it follows that the set $\hat{A}_{N-1} \cup \dot{A}_{N-1} = \{(d\theta^{(2-k)}_{k,i_k} - \theta^{(2-k+1)}_{k,i_k}dt) |\ k \in \lfloor 2\rceil,\ i_k \in \lfloor s_{N-k}\rceil\}$ is linearly independent mod $I^{(N-1)} + J$.

Continuing in this way, using Prop. 8.3, we may construct in the rth step, a basis for $I^{(N-r)} + J$ of the form

$$\mathcal{B}_{N-r} = A_{N-r} \cup \mathcal{B}_N \tag{8.5}$$

where $A_{N-r} = \hat{A}_{N-r+1} \cup \dot{A}_{N-r+1} \cup A_{N-r+1}$ and

$$\begin{array}{c} A_{N-r+1} = \{(d\theta^{(j-1)}_{k,i_k} - \theta^{(j)}_{k,i_k}dt) |\ k \in \lfloor r-1\rceil, i_k \in \lfloor s_{N-k}\rceil, j \in \lfloor r-k\rceil\} \\ \hat{A}_{N-r+1} \cup \dot{A}_{N-r+1} = \{(d\theta^{(r-k)}_{k,i_k} - \theta^{(r-k+1)}_{k,i_k}dt) |\ k \in \lfloor r\rceil, i_k \in \lfloor s_{N-k}\rceil\} \end{array} \tag{8.6}$$

and where $\hat{A}_{N-r+1} \cup \dot{A}_{N-r+1}$ is linearly independent mod $\{I^{(N-r+1)} + J\}$ for $r \in \lfloor N+1\rceil$. From Prop. 8.3, note that that $\dim(I^{(k)}(p) + J(p))/J(p) = \dim I^{(k)}(p)$, $k \in \mathbb{N}$.

Taking $r = N+1$ in (8.6) we obtain a basis $\mathcal{B}_{-1} = A_{-1} \cup B_N$ where $A_{-1} = \hat{A}_0 \cup \dot{A}_0 \cup A_0$ and

$$\begin{array}{rcl} A_0 & = & \{(d\theta^{(j-1)}_{k,i_k} - \theta^{(j)}_{k,i_k}dt) \mid k \in \lfloor N\rceil, i_k \in \lfloor s_{N-k}\rceil, j \in \lfloor N-k+1\rceil\} \\ \hat{A}_0 \cup \dot{A}_0 & = & \{(d\theta^{(N-k+1)}_{k,i_k} - \theta^{(N-k+2)}_{k,i_k}dt) \mid k \in \lfloor N+1\rceil, i_k \in \lfloor s_{N-k}\rceil\}. \end{array} \tag{8.7}$$

where the set $\hat{A}_0 \cup \dot{A}_0$ is independent mod $I^{(0)} + J$. Since $\dim I^{(0)} = n_b$ and $\dim I^{(-1)} = n_b + m_b$ we have card $\hat{A}_0 \cup \dot{A}_0 = m_b$. Now define the set w of n_b (state) functions and the set v of m_b (input) functions given by

$$\begin{array}{rcl} w & = & \{w^l_{k,i_k} |\ w^l_{k,i_k} = \theta^{(l-1)}_{k,i_k} : k \in \lfloor N\rceil, i_k \in \lfloor s_{N-k}\rceil, l \in \lfloor N-k+1\rceil\} \\ v & = & \{v_{k,i_k} |\ v_{k,i_k} = \theta^{(N-k+1)}_{k,i_k} : k \in \lfloor N+1\rceil, i_k \in \lfloor s_{N-k}\rceil\} \end{array}$$

By construction of $\mathcal{B}_0$ and $\mathcal{B}_{-1}$ it is clear that $I^{(0)} + J + \text{span}\,\{dt\} = \text{span}\,\{dt,\ dx_a,\ dw\} + J = \text{span}\,\{dt,\ dx_a,\ dx_b\} + J$ and $I^{(-1)} + J + \text{span}\,\{dt\} = \text{span}\,\{dt,\ dx_a,\ dw,\ du_a,\ dv\} = \text{span}\,\{dt,\ dx_a,\ dx_b,\ du_a,\ du_b\} + J$. Since card x_b = card w and card v_b = card v then, by Prop. 4.2 we conclude that $((x_a, w), (u_a, v))$ is a state representation that is linked to $((x_a, x_b), (u_a,\ u_b))$ by relative static state feedback. Since $I^{(k)} \subset \text{span}\left\{\frac{d}{dt}\right\}^{\perp}$, the equations $\langle(d\theta^{(j)}_{k,i_k} - \theta^{(j+1)}_{k,i_k}dt), \frac{d}{dt}\rangle = 0,\ k \in \lfloor N\rceil,\ i_k \in \lfloor s_{N-k}\rceil,\ j \in \lfloor N-k+1\rceil$ implies the following closed loop state equations :

$$\begin{array}{l} \dot{t} = 1 \\ \dot{x}_a = f_a(x_a, u_a) \\ \left\{ \begin{array}{rcl} \dot{w}^1_{k,i_k} & = & w^2_{k,i_k} \\ \dot{w}^2_{k,i_k} & = & w^3_{k,i_k} \\ & \vdots & \\ \dot{w}^{N-k+1}_{k,i_k} & = & v_{k,i_k} \end{array} \right. \quad, k \in \lfloor N\rceil,\ i_k \in \lfloor s_{N-k}\rceil \end{array} \tag{8.8}$$

□

Remark 8.3. Note that , if $s_{-1} > 0$, then the inputs $\{v_{k,i_k} \mid k = N{+}1, i_k \in \lfloor s_{-1} \rceil\}$ are completely decoupled from the state of system (8.8), *i.e.*, (8.8) is not well-formed in this case [48]. Note also that, if one restricts the coordinate transformations to the class of *relative static-state feedback* (see Definition 4.1) then the conditions of Theorem 8.2 are necessary and sufficient. This follows from the invariance of the relative derived flag with respect to relative static-state feedback (see Cor. 8.4) and after (tedious) calculations of the relative derived flag of a system of the form (8.8).

COROLLARY 8.4. *Consider the system S of equations* (4.1a)-(4.1b). *Let $x = (x_a, x_b)$ and J be defined by equation* (8.1). *Let $\widehat{I}^{(-1)} = \text{span}\,\{dx - \dot{x}dt, du - \dot{u}dt\} + J$ and $\widehat{I}^{(k)}(p) = \text{span}\{\omega(p) \mid \ \omega \in \widehat{I}^{(k-1)},\ d\omega(p) \bmod \widehat{I}^{(k-1)}|_p \equiv 0\}$ for $k \in \mathbb{N}$. Assume that the codistributions* $\text{span}\{\widehat{I}^{(k)}, dt\}$ *are involutive,* $\dim \widehat{I}^{(k)}(q)/J(q)$ *is (locally) constant for $k \in \mathbb{N}$, and that $\widehat{I}^{(N)} = J$ for N big enough. Then the system S is (locally) relatively flat w. r. t. S_1. Furthermore, the codistributions $\widehat{I}^{(k)}$, $k \in \mathbb{N}$ are invariant by relative static-state feedback with respect to the subsystem defined by* (4.1a).

Proof. We show first that $\widehat{I}^{(k)} = I^{(k)} + J$ for $k \in \{-1\} \cup \mathbb{N}$. This is obviously true for $k = -1$. Assume that this is true for $k - 1$ and let $\hat{\omega} \in \widehat{I}^{(k-1)}$. Then $\hat{\omega} = \omega + \mu$, where $\omega \in I^{(k-1)}$ and $\mu \in J$. As J is involutive, then $d\hat{\omega} \bmod \widehat{I}^{(k-1)} \equiv 0$ if and only if $d\omega \bmod (I^{(k-1)} + J) \equiv 0$. In particular, $\hat{\omega} \in \widehat{I}^{(k)}$ if and only if $\omega \in I^{(k)}$. It follows that $\widehat{I}^{(k)} = I^{(k)} + J$, showing our claim. Hence, the first affirmation follows easily from Theorem 8.2. To show the invariance of the flag $\widehat{I}^{(k)}$, let $(\tilde{x}, \tilde{u})$ be a state representation of S that is linked to (x, u) by relative static-state feedback. Let $\widetilde{I}^{(-1)} = \text{span}\,\{d\tilde{x} - \dot{\tilde{x}}dt,\, d\tilde{u} - \dot{\tilde{u}}dt\} + J$. Since $J + \text{span}\,\{dt\} = \Sigma = \pi^*(T^*S_a)$, by Def. 4.1 it follows $\widetilde{I}^{(-1)} + \text{span}\,\{dt\} = \widehat{I}^{(-1)} + \text{span}\,\{dt\}$. Hence, $\tilde{\omega} \in \widetilde{I}^{(-1)}$ if and only if $\tilde{\omega} = \hat{\omega} + \beta dt$, where $\hat{\omega} \in \widehat{I}^{(-1)}$. Now note that $\widetilde{I}^{(-1)}$ and $\widetilde{I}^{(-1)}$ are both contained in span $\left\{\frac{d}{dt}\right\}^{\perp}$. In particular $\langle\tilde{\omega}, \frac{d}{dt}\rangle = \langle\hat{\omega}, \frac{d}{dt}\rangle = 0$ implies that $\beta = 0$. We conclude that $\widetilde{I}^{(-1)} = \widehat{I}^{(-1)}$. Since the computation of $\widetilde{I}^{(k)}$ follows the same rule than the computation $\widehat{I}^{(k)}$ and J is invariant by endogenous feedback (see Rem. 8.1), we conclude that $\widetilde{I}^{(k)} = \widehat{I}^{(k)}, k \in \mathbb{N}$. □

Remark 8.4. Let $\mathcal{U} = \text{span}\,\{dx\}^{\perp}$ and $H = J^{\perp}$. Let $G_0 = \mathcal{U} \cap H$ and let $G_{k+1} = G_k + [\frac{d}{dt}, G_k]$. It can be shown [10] that the conditions of Theorem 8.2 for time-invariant systems are equivalent to the involutivity of the distributions G_i and the existence of k such that $G_i = H$ for all $i \geq k$.

8.1. Flatness and local output subsystems. Theorem 8.5 is a sufficient condition for relative flatness w. r. t. a local output subsystem.

THEOREM 8.5. *Let S be the explicit system* (1.1a) *with state representation* (x, u) *and output $y = h(t, x, u)$. Let U be the the open and dense set where theorem 4.3 holds. Let $\widehat{I}^{(0)} = \text{span}\,\{dx - \dot{x}dt\} + J$, where $J = \text{span}\{dy^{(k-1)} - y^{(k)}dt : k \in \mathbb{N}\}$. Consider the relative derived flag $\widehat{I}^{(i)}(p) = \text{span}\{\omega(p) \mid \omega \in \widehat{I}^{(i-1)},\ d\omega(p) \bmod \widehat{I}^{(i-1)}(p) \equiv 0\}$. Assume that, in U, the codistributions* $\text{span}\{\widehat{I}^{(k)}, dt\}$ *are involutive, and that* $\dim \widehat{I}^{(k)}(q)/J(q)$ *is (locally) constant dimensional for $k \in \mathbb{N}$ and $\widehat{I}^{(N)} = J$ for N big enough. Then S is (locally) relatively flat* w.r.t. *the output subsystem Y around every $\xi \in U$.*

Proof. By Thm. 4.3 there exists a local output subsystem Y of S and new state representation $((z_a, z_b), (v_a, v_b))$ linked to (x, u) by a a relative static-state feedback, such that the closed loop state equations are given by (4.3a)–(4.3b), where $\text{span}\left\{dt, dz_a, dv_a^{(k)} : k \in \mathbb{N}\right\} = \text{span}\left\{dt, dy^{(k)} : k \in \mathbb{N}\right\} = J + \text{span}\,\{dt\}$. Let $\tilde{J} =$

span $\left\{(dz_a - \dot{z}_a), (dv_a^{(k)} - v_a^{(k+1)}) : k \in \mathbb{N}\right\}$. It follows easily that $\tilde{J} + \text{span}\,\{dt\} = J + \text{span}\,\{dt\}$. Using the fact that $J \subset \text{span}\left\{\frac{d}{dt}\right\}^{\perp}$ and $\tilde{J} \subset \text{span}\left\{\frac{d}{dt}\right\}^{\perp}$, (see the arguments of the proof of Cor. 8.4), it follows that $\tilde{J} = J$. Then the result follows from Cor. 8.4. □

9. Examples. We begin with an counterexample of the necessity of the condition of Prop. 5.2 for flatness of a system S.

9.1. A counterexample. Consider the system S defined by

$$\dot{x}_1 = u_1 \tag{9.1a}$$

$$\dot{x}_2 = x_1(x_2^2 + 1) \tag{9.1b}$$

$$\dot{x}_3 = u_2 \tag{9.1c}$$

Then this system is flat with flat output $y_1 = x_3$ and $y_2 = x_2$.

Consider the susbsystem S_1 defined by (9.1a). Then this system is not relatively flat with respect to S_1. An indirect way to see this is by noting that the "implicit" system that we get by making $y = x_1 = 0$ and $y^{(k)} = u_1^{(k-1)} = 0, k \in \mathbb{N}$ is given by

$$\begin{array}{rcl} \dot{x}_2 & = & 0 \\ \dot{x}_3 & = & u_2 \end{array}$$

It is easy to show that this system is not flat because it is not controllable [20].

Note that the explicit system S is time-invariant and the codistributions $Y_k = \text{span}\left\{dt, dy, \ldots, dy^{(k)}\right\}$ and $\mathcal{Y}_k = \text{span}\,\{dt, dx,\, dy,\, \ldots,\, dy^{(k)}\}$ are nonsingular for $k \in \mathbb{N}$. In particular the assumptions A1 and A2 of § 6 are satisfied. Notice also that S_1 is an output subsystem for system S. If S were relatively flat with respect to S_1, then Thm. 7.2 would imply that the implicit system is locally flat.

9.2. A second counterexample. The following example shows that the conditions of Thm. 7.2 are sufficient conditions for flatness of implicit systems, but they are not necessary conditions.

Consider the explicit system S with output y defined by :

$$\begin{array}{rcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & x_4^2 x_1 + x_3 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & u_2 \\ y & = & x_1 - \epsilon \end{array} \tag{9.2}$$

It is easy to see that the "implicit" system (that is already explicit in this case) obtained by making $y^{(k)} = 0, k \in \mathbb{N}$ is given by

$$\begin{array}{rcl} \dot{x}_2 & = & \epsilon x_4^2 + x_3 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & u_2 \end{array} \tag{9.3}$$

System (9.3) is linearizable by static state feedback if and only if $\epsilon = 0$. In other words the implicit system obtained by making $y^{(k)} = 0, k \in \mathbb{N}$ is flat if $\epsilon = 0$ (but is not flat if $\epsilon \neq 0$ because it is not linearizable by static-state feedback in this case).

However the explicit system is not relatively flat with respect to the subsystem defined by the first equation of (9.2). An indirect proof of this fact can be given by noting that, if system (9.2) were relatively flat with respect to the subsystem S_1 defined by the first equation, then by Thm. 7.2, taking any $\epsilon \in \mathbb{R}$, the implicit system would be locally flat (see the arguments of the end of the last example). However the implicit system is flat only for $\epsilon = 0$.

9.3. An academic example. Consider the implicit system

$$\dot{x}_1 = \frac{x_2^2}{(1+x_3^2)^2} + e^{x_3}u_1, \quad \dot{x}_2 = (1+x_3^2)u_1 + \frac{2x_2x_3}{(1+x_3^2)}u_2, \quad \dot{x}_3 = u_2 \tag{9.4a}$$

$$y = x_1 = 0 \tag{9.4b}$$

Let S be the (explicit) system (9.4a) with output $y = x_1$. It is easy to verify that the codistributions $\mathcal{Y}_k = \text{span}\left\{dt, dy, \ldots, dy^{(k)}\right\}$ and $Y_k = \text{span}\{dt, dx, dy, \ldots, dy^{(k)}\}$ of Lemma C.1 are nonsingular everywhere for $k \in \mathbb{N}$, and $\sigma_k = 1, k \geq 1$. Note also that $\Gamma = \{\xi \in S \mid y^{(k)}(\xi) = 0\}$ is nonempty because Γ contains the point $\xi \in S$ defined by $x_1(\xi) = x_2(\xi) = x_3(\xi) = u_1^{(k)}(\xi) = u_2^{(k)}(\xi) = 0$, $k \in \mathbb{N}$ (for any t). Since the system is time-invariant then the assumptions A1 and A2 of § 6 are satisfied. By Prop. 6.1, the implicit system is a immersed system in the nonconstrained system. Let $J = \text{span}\{dy^{(k)} - y^{(k+1)}dt : k \in \mathbb{N}\}$ and let $\widehat{I}^{(0)} = \text{span}\left\{dx - \dot{x}dt\right\} + J$. Using condition (B.4), some calculations show that[20] $\widehat{I}^{(1)} = \text{span}\{\eta - \langle\eta, \frac{d}{dt}\rangle dt\} + J$, where $\eta = dx_2 - \frac{2x_2x_3}{(1+x_3^2)}dx_3$, and $\widehat{I}^{(2)} = J$. Since $d\eta = \frac{2x_3}{(1+x_3^2)}(\eta \wedge dz_3)$. From Theorem 8.5, for every local output subsystem Y, the explicit system S is relatively flat w. r. t. Y. By Theorem 7.2, the implicit system Γ defined by (9.4a)–(9.4b) is locally flat around every point $\xi \in \Gamma$. By the proof of theorem 8.2 and the construction of Γ in § 6, a flat output of the implicit system can be constructed by finding a function ψ such that $d\phi = \alpha\eta$. A possible solution is $\psi = \frac{x_2}{(1+x_3^2)}$.

By Props. 4.5 and 5.3, one may complete the output y into a flat output (y, z) for system S. In this case one may take $z = \psi$.

9.4. Constrained robots. Constrained robots are robots whose movement is restricted by some physical contact surfaces. Such restrictions can be represented by adding r holonomic constraints $\phi_i(q) = 0 \ \ (i = 1, \ldots, r)$ to its original equations.

The following model can be obtained by taking into account the contact forces [29] :

$$M(q)\ddot{q} + H(q, \dot{q}) = (J\phi)^T(q)\lambda + \tau \tag{9.5a}$$

$$\phi_i(q) = 0 \ \ (i = 1, \ldots, r) \tag{9.5b}$$

where $q \in \mathbb{R}^n$, $J\phi(q) = \partial\phi/\partial q$, $\lambda = (\lambda_1, \ldots, \lambda_r)^T$ is a vector corresponding to the contact forces, $M(q)$ is the symmetric positive definite mass matrix, and $H(q, \dot{q})$ corresponds to Coriolis and gravity forces. We will assume that $\partial\phi/\partial q$ has rank r for all q in the operation region of the robot. A representation of the system (9.5a)–(9.5b) in the form (1.1a)–(1.1b) is given by

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ -M^{-1}H \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ M^{-1}(J\phi)^T & M^{-1} \end{pmatrix}\begin{pmatrix} \lambda \\ \tau \end{pmatrix} \tag{9.6a}$$

$$0 = \phi_i(q), \ i = 1, \ldots, r \tag{9.6b}$$

[20] The application of part (ii) of Prop. 8.3 is the easiest way for computing relative derived flags, and lead to linear equations with coefficients that are functions defined on S as shown in the proof of Prop. 8.3.

Let $\psi = (\psi_1, \dots, \psi_{n-r})$ be chosen in a way that map $q \mapsto (\phi, \psi)$ is a local diffeomorphism. Considering only the explicit system S defined by (9.6a), it is easy to show that (q, λ) is a flat output for S. In particular (ϕ, ψ, λ) is also a flat output for S. From Cor. 7.3, it follows that (ψ, λ) is a flat output for the constrained robot. Note now that ψ are local coordinates of the constraint surface. In particular, the simultaneous tracking of the position along the constraint surface and the contact forces are possible. The reader may refer to [37, 40] for details and the presentation of the design of a flatness based control, including the underactuated case. Another approach for the solution of this problem is considered for instance in [29].

10. Conclusions. In this paper we state the notion of (local) subsystem. The definition and construction of (local) output subsystems are also presented (Thm. 4.3). It is shown that subsystems admit (generically) adapted state equations (Prop. 4.4).

We show, under regularity assumptions, that an implicit system (1.1a)–(1.1b) defines a system Γ (in the sense of § 3.2) that admits state space representations and is immersed in the (explicit) system S defined by (1.1a) (Prop. 6.1). This immersion is in fact an embedding since the topology of the immersed system is the subset topology. This result may be regarded as a generalization of the fact that equations $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^p$, defines implicitly an embedded submanifold of $\mathbb{R}^n$ when the Jacobian matrix $Jf(x)$ has constant rank in the solutions of $f(x) = 0$.

The concept of relative flatness w. r. t. a subsystem is introduced (see §5). We show that a system S is relative flat with respect to a flat subsystem if and only if the flat output of the subsystem may be completed into a flat output of the system (see Prop. 5.3).

Given a system S with output y, we show that relative flatness w. r. t. a (local) output subsystem implies (local) flatness of the implicit system Γ obtained by restricting S to the constraint $y = 0$ (Thm. 7.2).

Sufficient conditions for relative flatness of a system w. r. t. a subsystem are developed (see Thm. 8.2 and Cor. 8.4). These condictions restricts the class of transformations to relative static-state feedbacks (Def. 4.1).

Sufficient conditions of relative flatness with respect to an output subsystem are obtained (see Thm. 8.5). This result can be combined with Thm. 7.2 in order to study flatness of implicit systems (1.1a)–(1.1b), as illustrated in the example of § 9.3.

Although it is assumed that system (1.1a)–(1.1b) is analytic, this hypothesis is only needed to assure that the output rank ρ of the explicit system (1.1a) with output $y = h(x, u, t)$ is a global invariant, at least in the subset $U \subset S$ of nonsingular points of the codistributions Y_k and $\mathcal{Y}_k$ for $k = 0, \dots, n$ (see Lemma C.1). Note that the differential dimension[21] of the implicit system Γ defined by (1.1a)–(1.1b) is $\tilde{m} = m - \rho$, where $m = \mathrm{card}\, u$. Hence the assumption of analycity implies that $\tilde{m}$ is an invariant. All the results of this paper could be rewritten in the smooth case (see [39, Lemma 6.2] for a smooth version of Lemma C.1), but in this case the differential dimension of Γ may depend on the working point. In the same way, it is easy to restrict our results to the time-invariant case (see [39, Lemma 8.1]).

All the definitions and results of this paper are local (note that the time-varying notions are also local in time). The only exception are the construction of the system Γ in § 6 and Prop. 6.1, that is a "global" construction.

[21] The local differential dimension is the cardinal of the input of a local state representation. Note that a differential dimension $\tilde{m}$ of a connected smooth system that admits a local state representation around every point is a global invariant [18], [39, Cor. 7.2].

Appendix A. A differential algebraic interpretation of relative flatness. The next proposition is a precise statement of the affirmation made in Rem. 5.1. The proof may be found in [58].

PROPOSITION A.1. *[58] Let K/k be a system. Let L/k be a subsystem of K, i.e., $k \subset L \subset K$. Then K/L is a flat system if and only if there exist a flat system F/k such that F is algebraic over K, K is algebraic over $k(L,F)$, and L and F are algebraically disjoint over k. In other words, $M = k(F,K)$ is decomposed into two independent subsystems L and F.*

Appendix B. Proof of Auxiliary Results.

B.1. Proof of Prop. 4.2. By definition, (i) implies (ii). To show that (ii) implies (i) it suffices to show that $\{t, z, (v^{(k)} : k \in \mathbb{N})\}$ is a local coordinate system. Consider the state representation $(\tilde{x}, \tilde{u})$ where $\tilde{x} = (x_b, x_a, u_a, \ldots, u_a^{(r)})$ and $\tilde{u} = (u_b, u_a^{(r+1)})$ for some r to be determined. It will be shown that $\tilde{z} = (z_b, x_a, u_a, \ldots, u_a^{(r)})$ and $\tilde{v} = (v_b, u_a^{(r+1)})$ are such that their differentials are linearly independent, that span $\{dt, d\tilde{x}\} =$ span $\{dt, d\tilde{z}\}$, and that span $\{dt, d\tilde{x}, d\tilde{u}\} =$ span $\{dt, d\tilde{z}, d\tilde{v}\}$. By § 3.7, it follows that $(\tilde{z}, \tilde{v})$ is a state representation and it is linked to $(\tilde{x}, \tilde{u})$ by static-state feedback, completing the proof. To show this, note that $\Sigma =$ span $\{dt, dx_a, ({du_a}^{(j)} : j \in \mathbb{N})\}$. From the particular form of x, z, u and v, it is clear that (ii) is equivalent to span $\{dx_b\} + \Sigma =$ span $\{dz_b\} + \Sigma$ and span $\{dx_b, du_b\} + \Sigma =$ span $\{dz_b, dv_b\} + \Sigma$. Since $\{t, x, (u^{(k)} : k \in \mathbb{N})\}$ is a coordinate system, it follows that $dz^b = \alpha_0 dt + \sum_i \alpha_i dx_{b_i} + \sum_l \beta_l dx_{a_l} + \sum_{j,k \le k^*} \gamma_{jk} du_{b_j}^{(k)}$. In particular one concludes that span $\{dx_b\} + \Sigma_r =$ span $\{dz_b\} + \Sigma_r$, where $\Sigma_r =$ span $\left\{dt, dx_a, ({du_a}^{(j)} : j \in \lfloor r \rceil)\right\}$, for any $r \ge k^*$. One shows similarly, possibly by taking a bigger r, that span $\{dt, dx_b, du_b\} + \Sigma_{r+1} =$ span $\{dt, dz_b, dv_b\} + \Sigma_{r+1}$. By § 3.7 it follows that $(\tilde{z}, \tilde{v})$ and $(\tilde{x}, \tilde{u})$ are linked by static-state feedback. □

B.2. Proof of Thm. 4.3. In this proof we use the results and the notations of Lemma C.1. Let $n = \dim x$. By that Lemma, around $\xi \in U$, there exists a local state representation (x_n, u_n) defined in V_ξ such that

(B.1a) $$\text{span}\,\{dt, dx_n\} = \text{span}\left\{dt, dx, dy, \ldots, dy^{(n)}\right\}$$

(B.1b) $$\text{span}\,\{dt, dx_n, du_n\} = \text{span}\left\{dt, dx, du, dy, \ldots, dy^{(n+1)}\right\}$$

and where $u_n = (\bar{y}_n^{(n+1)}, \hat{u}_n)$. Now choose a subset z_a of $\{y, \ldots, y^{(n)}\}$ in a way that $\{dt, dz_a\}$ is a local basis of span $\left\{dt, dy, \ldots, dy^{(n)}\right\}$ and choose z_b in a way that $\{dt, dz_a, dz_b\}$ is a local basis of span $\left\{dt, dx, dy, \ldots, dy^{(n)}\right\}$ around ξ. Let $u_a = \bar{y}_n^{(n+1)}$ and $u_b = \hat{u}_n$. By construction, $((z_a, z_b), (u_a, u_b))$ is a local state-representation of S around ξ, since it is linked to (x_n, u_n) by local static-state feedback (see (3.4)).

By Lemma C.1 part 8, it follows that span $\{d\dot{z}_a\} \subset$ span $\{t, z_a, u_a\}$ and that (i) and (ii) holds. Now note that (iii) follows easily from Def. 4.1 and conditions (B.1).

To show that two output subsystems are Lie-Bäcklund isomorphic, let $\pi_i : V_\xi^i \to Y_i$ be local output subsystems for $i = 1, 2$. Assume that $V_\xi^1 \cap V_\xi^2 \ne \emptyset$. We will show that there exist a local Lie-Bäcklund isomorphism $\delta : W_1 \to W_2$ where H is some open neighbourhood of ξ for which $H \subset V_\xi^1 \cap V_\xi^2$ and $W_i = \pi_i(H), i = 1, 2$.

Since the π_i are Lie-Bäcklund submersions for $i = 1, 2$, there exists local charts of $\phi_i = (t, X_i, Z_i)$, $i = 1, 2$, defined in some $H \subset S$ and local charts $\psi_i = (t, X_i)$, of Y_i, $i = 1, 2$, defined on $W_i = \pi_i(H)$ such that, in these coordinates $\phi_i \circ \pi_i^{-1} \circ$

$\psi_i(t, X_i, Z_i) = (t, X_i)$, $i = 1, 2$. Since Y_1 and Y_2 are both local subsystems we have $\text{span}\{dt, dX_i\} = \text{span}\left\{dt, dy^{(k)} : k \in \lfloor \mathbb{N} \rceil\right\}$, for $i = 1, 2$. In particular, it follows that the local coordinate change $(t, X_1, Z_1) = \phi_1 \circ \phi_2^{-1}(t, X_2, Z_2)$ is such that $X_1 = \theta(t, X_2)$ and $X_2 = \tilde{\theta}(t, X_1)$. So the map μ defined by $(t, X_2) \mapsto (t, \theta(t, X_2))$ is a local diffeomorphism[22]. Let $\delta : W_2 \subset Y_2 \to W_1 \subset Y_1$ be the local diffeomorphism defined by $\delta = \psi_1^{-1} \circ \mu \circ \psi_2$. To complete the proof it suffices to show that δ is Lie-Bäcklund. For this, we show first that $\delta \circ \pi_2|_H = \pi_1|_H$. In fact, note that

$$\begin{array}{rcl} \psi_1 \circ (\delta \circ \pi_2) \circ \phi_1^{-1}(t, X_1, Z_1) & = & \psi_1 \circ (\delta \circ \pi_2 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})(t, X_1, Z_1) = \\ (\psi_1 \circ \delta) \circ \pi_2 \circ \phi_2^{-1}(t, X_2, Z_2) & = & (\mu \circ \psi_2) \circ \pi_2 \circ \phi_2^{-1}(t, X_2, Z_2) = \\ \mu \circ (\psi_2 \circ \pi_2 \circ \phi_2^{-1})(t, X_2, Z_2) & = & \mu(t, X_2) = \\ (t, X_1) & = & \psi_1 \circ \pi_1 \circ \phi_1^{-1}(t, X_1, Z_1) \end{array}$$

From the first and the last terms above, we have that $\delta \circ \pi_2|_H = \pi_1|_H$. Denote by ∂_i the Cartan fields respectively of Y_i, for $i = 1, 2$. By definition $\pi_i^* \frac{d}{dt} = \partial_i \circ \pi_i$. In particular $\partial_1 \circ \delta \circ \pi_2 = \partial_1 \circ \pi_1 = (\pi_1) * \frac{d}{dt} = (\delta \circ \pi_2)_* \frac{d}{dt} = \delta_*(\pi_2)_* \frac{d}{dt} = \delta_* \partial_2 \circ \pi_2$. As π_2 is surjective it follows that $\partial_1 \circ \delta = \delta_* \partial_2$, showing that δ is Lie-Bäcklund. □

B.3. Proof of Prop. 4.5. Let $\rho = p$ be the output rank and let $\{\sigma_1, \ldots, \sigma_n\}$ be the structure at infinity of the system. By Lemma C.1, the sequence σ_i is nonincreasing. Since σ_i converges to ρ, we must have $\sigma_i = \rho, i \in \lfloor n \rceil$. Thus, the differentials $\{dt, dy^{(j)} : j \in \mathbb{N}\}$ must be independent in U. In particular, in the proof of Theorem 4.3, a possible choice of z_a and u_a can be $z_a = (y, \ldots, y^{(n)})$ and $u_a = y^{(n+1)}$. It is then clear that the subsystem Y is (locally) flat with flat output y. □

B.4. Proof of equation (8.2). From (B.5) it follows that $\omega \in I^{(0)}$ if and only if $\omega \in I^{(-1)}$ and $\dot{\omega} \in I^{(-1)} + J + \text{span}\{dt\} = \text{span}\left\{dx_b, du_b, dx_a, (du_a^{(j)} : j \in \mathbb{N}), dt\right\}$. Let $\omega \in I^{(-1)}$. Then $\omega = \sum_i \alpha_i(dx_{b\,i} - \dot{x}_{b_i} dt) + \sum_j \beta_j(du_{b\,j} - \dot{u}_{b_j} dt)$ for convenient functions α_i, β_j defined on S. Hence $\dot{\omega} = \sum_i \dot{\alpha}_i(dx_{b\,i} - \dot{x}_{b_i} dt) + \sum_j \dot{\beta}_j(du_{b\,j} - \dot{u}_{b_j} dt) + \sum_i \alpha_i(d\dot{x}_{b_i} - \ddot{x}_{b_i} dt) + \sum_j \beta_j(d\dot{u}_{b_j} - \ddot{u}_{b_j} dt) = \sum_l \gamma_l dx_{a\,l} + \sum_l \delta_r du_{a\,r} + \sum_i \epsilon_i dx_{b\,i} + \zeta_j du_{b\,j} + \theta dt + \sum_j \beta_j d\dot{u}_{b_j}$ where $\gamma_l, \delta_r, \epsilon_i, \zeta_j$ and θ are convenient functions. Notice that $\{t, x, (u^{(k)} : k \in \mathbb{N})\}$ is a local chart, where $x = (x_a, x_b)$ and $u = (u_a, u_b)$. Hence $\omega \in I^{(0)}$ if and only if $\beta_j = 0$ for all $j \in \lfloor m_b \rceil$. □

B.5. Proof of Proposition 8.3. In order to prove Prop. 8.3 we need the following lemmas :

LEMMA B.1. *For all integers $k \geq 0$, $r \geq 0$ and for every point $p \in S$, we have :*

(i) $\left(I^{(k)} + J_r + \text{span}\{dt\}\right)\big|_p \cap \text{span}\left\{\frac{d}{dt}\right\}^\perp \subset I^{(k)}(p) + J_r$. *The same result also holds when replacing J_r by J.*

(ii) $\text{span}\left\{I^{(k)}, J, dt\right\}|_p = I^{(k)}(p) \oplus J(p) \oplus \text{span}\{dt\}(p)$

Proof. **(i)** Assume that $\omega(p) \in \left(I^{(k)} + J_r + \text{span}\{dt\}\right)\big|_p \cap \text{span}\left\{\frac{d}{dt}\right\}^\perp$. Then $\omega(p) = \varpi(p) + \beta dt|_p$ where $\varpi \in I^{(k)} + J_r$ and $\beta \in C^\infty(S)$. Then $\langle \omega, \frac{d}{dt} \rangle|_p = \langle \varpi, \frac{d}{dt} \rangle|_p + \beta \langle dt, \frac{d}{dt} \rangle|_p$. Since $\langle dt, \frac{d}{dt} \rangle = 1$ and $I^{(k)} + J \subset \text{span}\left\{\frac{d}{dt}\right\}^\perp$, it follows that $\beta(p) = 0$ and hence $\omega(p) = \varpi(p) \in I^{(k)}(p) + J_r(p)$.

(ii) Let $\omega(p) \in \left\{(I^{(k)} + J) \cap \text{span}\{dt\}\right\}\big|_p$. We have $\omega(p) = \beta(p) dt$. So $\langle \omega, \frac{d}{dt} \rangle|_p = \beta(p) = 0$ and so $\beta(p) = 0$.

[22] We stress that we are not using the Inverse Function Theorem, but only the existence of the inverse of the coordinate change map.

Let $\omega(p) \in \left\{(I^{(k)} + \mathrm{span}\,\{dt\}) \cap J\right\}\big|_p$. We have $\omega(p) = \varpi(p) + \beta(p)dt$, where $\varpi(p) \in I^{(k)} \subset \mathrm{span}\left\{\frac{d}{dt}\right\}^{\perp}$. Since $J \subset \mathrm{span}\left\{\frac{d}{dt}\right\}^{\perp}$, then $\langle \omega(p), \frac{d}{dt}\rangle = \langle \varpi(p), \frac{d}{dt}\rangle + \beta(p)\langle dt, \frac{d}{dt}\rangle = 0$. Hence $\omega(p) \in I^{(k)}$. Since $I^{(k)} \cap J = 0$, it follows that $\omega(p) = 0$.

Let $\omega(p) \in \left\{(J + \mathrm{span}\,\{dt\}) \cap I^{(k)}\right\}\big|_p$. Using the same reasoning above, one verify easily that $\omega(p) = 0$. ∎

Lemma B.2. *Assume that the conditions of the Theorem 8.2 are satisfied on an open neighbourhood V_ξ of ξ in S. Then $I^{(k)}, k \in \mathbb{N}$ is a smooth codistribution and for every $p \in V_\xi$ and $k \in \mathbb{N}$ we have :*

(i) *For all $k \in \mathbb{N}$ there exists a set of covector fields $\Omega = \{\omega_1, \ldots, \omega_{r_k}\} \subset I^{(k)} + J$, where $r_k = \dim I^{(k)}$, $\omega_i = (d\theta_i - \dot{\theta}_i dt)$, with $\theta_i \in C^\infty(S)$, and an open neighbourhood V of ξ such that the canonical projections of the elements of $\Omega(\nu)$ form a basis for $(I^{(k)}(\nu) + J(\nu)) \bmod J(\nu)$ for all ν in V.*

(ii) *If ω is of the form $\left(d\theta - \dot{\theta}dt\right)$ for a function $\theta \in C^\infty(S)$, then $\omega \in I^{(k+1)} + J$ if and only if $\dot{\omega} \in I^{(k)} + J$. In particular $I^{(k)} + J \supset I^{(k+1)} + \frac{d}{dt}I^{(k+1)} + J$.*

(iii) *Let $\{\omega_1, \ldots, \omega_r\} \subset I^{(k-1)} + J$ be a set of 1-forms such that $\omega_i = (d\theta_i - \dot{\theta}_i dt)$, where $\theta_i \in C^\infty(S)$. Assume that the set $\{\omega_1(p), \ldots, \omega_r(p)\}$ is linearly independent*[23] *$\bmod I^{(k)}(p) + J(p)$. Then $\{\dot{\omega}_1(p), \ldots, \dot{\omega}_r(p)\} \subset (I^{(k-2)}(p) + J(p))$ is linearly independent $\bmod (I^{(k-1)} + J + \mathrm{span}\,\{dt\})|_p$.*

Proof. Assume by induction that $I^{(j)}, j = -1, 0, \ldots, k$ is smooth. We will show first that (i) and (ii) holds. **(i)** We show now that, for an integer l_k big enough then $\mathrm{span}\left\{I^{(k)}, J_{l_k}, dt\right\}$ is involutive (see eq. (8.3)). In fact, since $I^{(k)}$ is nonsingular and finite dimensional, there exist a local basis $\{\tilde{\omega}_i : i \in \lfloor r_k \rceil\}$ of $I^{(k)}$. By part (ii) of Lemma B.1, it follows that the set $\{(\tilde{\omega}_i : i \in \lfloor r_k \rceil), dx_a, du_a, \ldots, du_a^{(l_k)}, dt\}$ is a local basis of $I^{(k)} + J_{l_k} + \mathrm{span}\,\{dt\}$. Since the codistribution $\mathrm{span}\left\{I^{(k)}, J, dt\right\}$ is involutive, then $d\tilde{\omega}_i = \sum_{j=1}^{r_k} \eta_{ij} \wedge \nu_{ij}$ for convenient 1-forms η_{ij}, ν_{ij} with $\nu_{ij} \in \mathrm{span}\left\{I^{(k)}, J, dt\right\}$. Hence $\nu_{ij} \in \mathrm{span}\left\{(\tilde{\omega}_i : i \in \lfloor r_k \rceil), dx_a, du_a, \ldots, du_a^{(s_{ij})}, dt\right\}$. Let $l_k^* = \max_{i,j}\{s_{ij}\}$. Then $\mathrm{span}\left\{I^{(k)}, J_{l_k}, dt\right\}$ is involutive for every $l_k \geq l_k^*$. By the Frobenius theorem and part (ii) of Lemma B.1, we see that $\mathrm{span}\left\{I^{(k)}, J_{l_k}, dt\right\}$ is spanned by linearly independent 1-forms $\{d\theta_1, \ldots d\theta_{r_k}, dx_a, du_a, \ldots, du_a{}^{(l_k)}, dt\}$, where $\dim I^{(k)} = r_k$. Now note that $\omega_i = (d\theta_i - \dot{\theta}_i dt) \in \mathrm{span}\left\{\frac{d}{dt}\right\}^{\perp}$. Since $\omega_i \in \mathrm{span}\left\{I^{(k)}, J_{l_k}, dt\right\}$, by Lemma B.1 part (i) it follows that $\omega_i \in I^{(k)} + J_{l_k}$. Let $K = \mathrm{span}\,\{J_{l_k}, dt\}\,|_p$ and $L = \mathrm{span}\,\{J, dt\}_p$. By construction the canonical projection of the set $\{\omega_i, i \in \lfloor r_k \rceil\}$ on $(I^{(k)}(p) + K)/K$ form a basis of $(I^{(k)}(p) + K)/K$. By part (ii) of lemma B.1, it is easy to see that the map $\Psi : (I^{(k)}(p) + K)/K \to (I^{(k)}(p) + L)/L$ such that $\omega(p) \bmod K \mapsto \omega(p) \bmod L$ is an isomorphism. In particular the canonical projections of the ω_i on $(I^{(k)}(p) + L)/L$ also form a basis.

(ii) We show first that we have

$$d\omega(p) \bmod (I^{(k)}(p) + J(p)) \equiv -\,\dot{\omega} \wedge dt\big|_p \bmod (I^{(k)}(p) + J(p)) \tag{B.2}$$

for all $p \in S$. For, by (i), note that $\omega = \sum_{i=1}^{r_k} \alpha_i\omega_i + \beta\eta + \sum_{j=0}^{l_k} \gamma_j\mu_j$ for $\omega_i = d\theta_i - \dot{\theta}_i dt$, $\eta = (dx_a - \dot{x}_a dt)$, and $\mu_j = (du_a{}^{(j)} - u_a{}^{(j+1)}dt)$ for convenient smooth functions α_i, β and γ_j. Hence, $d\omega \bmod (I^{(k)} + J) = [\sum_{i=1}^{r_k}(\alpha_i d\omega_i + d\alpha_i \wedge \omega_i) +\ (\beta d\eta + d\beta \wedge \eta) +$

[23] The linear independence of the set $\{\omega_1(p), \ldots, \omega_r(p)\} \bmod (I^{(k)}(p) + J)$ for some $p \in S$ means that $\left(\sum_{i=1}^{r} \alpha_i\omega_i(p) + \omega(p)\right)\big|_p = 0$ for $\omega \in I^{(k)} + J$ and $\alpha_i \in \mathbb{R}$ implies that $\omega(p) = 0$ and $\alpha_i = 0$.

$\sum_{j=1}^{l_k}(\gamma_j d\mu_j + d\gamma_j \wedge \mu_j)]$ mod $(I^{(k)} + J)$. Since $d(d\theta - \dot\theta dt) = -d\dot\theta \wedge dt$ for all $\theta \in C^\infty(S)$, and $[\sum_i d\alpha_i \wedge \omega_i + d\beta \wedge \eta + \sum_j d\gamma_j \wedge \eta_j]$ mod $(I^{(k)} + J) \equiv 0$, we see that

$$\text{(B.3)} \quad d\omega \bmod (I^{(k)} + J) = -\sum_i \alpha_i d\dot\theta_i \wedge dt - \beta d\dot x_a \wedge dt - \sum_j \gamma_j d{u_a}^{(j+1)} \wedge dt$$

Now note that $\dot\omega \wedge dt = [\sum_{i=1}^{r_k}(\alpha_i\dot\omega_i + \dot\alpha_i\omega_i) + (\beta\dot\eta + \dot\beta\eta) + \sum_{j=1}^{l_k}(\gamma_j\dot\mu_j + \dot\gamma_j\mu_j)] \wedge dt$. Since $\{[\sum_{i=1}^{r_k}\dot\alpha_i\omega_i + \dot\beta\eta + \sum_{j=1}^{l_k}\dot\gamma_j\mu_j] \wedge dt\}$ mod $(I^{(k)} + J) \equiv 0$, $\frac{d}{dt}(d\theta - \dot\theta dt) = d\dot\theta - \ddot\theta dt$, it follows by direct computation that $\dot\omega \wedge dt$ is also given by the right hand side of equation (B.3) and so (B.2) holds.

Now we will show that, for all $p \in S$ and $\omega \in I^{(k)}$, we have

$$\text{(B.4)} \qquad \omega(p) \in I^{(k+1)}(p) \;\Leftrightarrow\; \dot\omega(p) \in \operatorname{span}\left\{I^{(k)}, J, dt\right\}(p)$$

Let $\{dt, (\omega_i : i \in \lfloor r_k\rceil), \eta, (\mu_j : j \in \lfloor l_k\rceil)\}$ be a basis for span $\left\{I^{(k)}, J_{l_k}, dt\right\}$. Notice that, $\dot\omega \wedge dt|_p$ mod $(I^{(k)}(p) + J(p)) \equiv 0$ means that $\dot\omega \wedge dt|_p + \sum_{i=1}^{r_k}\zeta_i \wedge \omega_i|_p + \xi \wedge \eta|_p + \sum_{j=i}^{l_k}\rho_j \wedge \mu_j\Big|_p = 0$ for convenient 1-forms ζ_i, ξ, ρ_j. From the Cartan Lemma (see section 2), we conclude that $\dot\omega(p) \in \operatorname{span}\left\{I^{(k)}, J_{l_k}, dt\right\}(p)$. Then, (B.4) follows from (B.2) and the Definition 8.1. It is easy to show that the same arguments and the fact that J is involutive imply that

$$\text{(B.5)} \qquad \omega(p) \in I^{(k+1)}(p) + J(p) \;\Leftrightarrow\; \dot\omega(p) \in \operatorname{span}\left\{I^{(k)}, J, dt\right\}(p)$$

If $\omega = d\theta - \dot\theta dt$ then $\dot\omega \in \operatorname{span}\left\{\frac{d}{dt}\right\}^\perp$. By (B.5) and from Lemma B.1 part (i), it follows that $\dot\omega \in I^{(k)} + J$. Now note that, by (i), $I^{(k+1)} + J$ has a basis for this particular form. This completes the proof of (ii). We show now that our induction hypothesis (*i.e.*, that $I^{(j)}$ is smooth for $j = -1, 0, \ldots, k$) implies that $I^{(k+1)}$ is smooth. In fact, by the proof of (i), given a local basis $\{\tilde\omega_i : i \in \lfloor r_k\rceil\}$ of $I^{(k)}$, there exists a local basis $\{(\tilde\omega_i : i \in \lfloor r_k\rceil), dx_a, (du_a^{(k)} : k \in \mathbb{N}), dt\}$ of $W_k = \operatorname{span}\left\{I^{(k)}, J, dt\right\}$. Note that $W_k \subset W_0 = \operatorname{span}\{dx_b, J, dt\}$. In particular we have $\tilde\omega_i = \hat\omega_i + \gamma_i$, where $\hat\omega_i \in \operatorname{span}\{dx_b\}$ and $\gamma_i \in \operatorname{span}\{J, dt\} = T^*S_a$. Note that $\mu_i = \dot\gamma_i \in \operatorname{span}\{J, dt\}$ and we may replace $\tilde\omega_i$ by $\hat\omega_i$ in the basis of W_k , obtaining another basis of W_k. Note also that there exist a subset $\hat x_b$ of x_b such that $\{d\hat x_b, du_b, (\tilde\omega_i : i \in \lfloor r_k\rceil), dx_a, (du_a^{(k)} : k \in \mathbb{N}), dt\}$ is a basis of W_{-1}. Let $z = (\hat x_b, u_b)$. Let $\dot{\tilde\omega}_i = \dot{\hat\omega}_i + \dot\gamma_i = \sum_j a_{ij}dz_j + \mu_i$, where $\mu_i \in \operatorname{span}\{J, dt\}$. Denote the matrix formed by the functions a_{ij} by A. By (B.4), $\omega(p) = \sum_j \alpha_i\tilde\omega_i \in I^{(k+1)}(p)$ if and only if $\sum_j(\dot\alpha_i\tilde\omega_i + \alpha_i\dot{\tilde\omega}_i)|_p \in W_k(p)$. Denoting by α the column vector with components α_i, then $\omega(p) \in I^{(k+1)}(p)$ if and only if $A(p)\alpha(p) = 0$. Then if $I^{(k+1)}$ is nonsingular if and only if $A(p)$ has (locally) constant rank and in this case it is clear that $I^{(k)}$ is smooth[24].

(iii) To prove (iii), assume that there exists ω in $I^{(k-1)} + J$ and functions $\alpha_i \in C^\infty(S)$ such that, for $p \in S$ then $(\omega + \alpha_0 dt + \sum_{i=1}^r \alpha_i\dot\omega_i)|_p = 0$. Hence, $\{[\omega - \sum_{i=1}^r(\dot\alpha_i)\omega_i] + \alpha_0 dt + \frac{d}{dt}(\sum_{i=1}^r \alpha_i\omega_i)\}\big|_p = 0$. Since $[\omega - \sum_{i=1}^r(\dot\alpha_i)\omega_i](p) \in I^{(k-1)}(p) + J(p)$, it follows that $\frac{d}{dt}(\sum_{i=1}^r \alpha_i\omega_i)\big|_p \in \operatorname{span}\left\{I^{(k-1)}, J, dt\right\}(p)$. It follows from (B.5) that $(\sum_{i=1}^r \alpha_i\omega_i)(p) \in I^{(k)}(p) + J(p)$ and hence the set $\{\omega_1, \ldots \omega_r\}$ is not linearly independent mod $I^{(k)} + J(p)$ in $p \in S$. □

[24] This proof shows also that one may compute the relative derived flag by solving linear equations.

Proof (of Prop. 8.3). Since $\omega_i = d\theta_i - \dot{\theta}_i dt$, $i \in \lfloor s \rceil$, by part (ii) of Lemma B.1, the set $\mathcal{B} = \{(d\theta_i^{(j)},\, j \in \{0,\ldots,r_i\},\, i \in \lfloor s \rceil),\, dx_a,\, du_a,\, \ldots,\, du_a^{(l_{k-1})},\, dt\}$ is a local basis of $I^{(k-1)} + J_{l_{k-1}} + \mathrm{span}\,\{dt\}$, for any $l_{k-1} > l_{k-1}^*$ for some l_{k-1}^*.

By part (iii) of Lemma B.2, $\dot{B}$ is linearly independent $\mod \{I^{(k-1)} + J + \mathrm{span}\,\{dt\}\}$. Hence $\mathcal{B} \cup \dot{\mathcal{B}} = \{(d\theta_i^{(j)},\, j \in \{0,\ldots,r_i+1\},\, i \in \lfloor s \rceil),\, dx_a,\, du_a,\, \ldots,\, du_a^{(l_{k-1})},\, dt\}$ is linearly independent for all l_{k-1}.

From the proof of part (i) of Lemmma B.2, we also have that there exists a local basis $\{(\tilde{\theta}_i : i \in \lfloor r \rceil),\, dx_a,\, du_a,\, \ldots,\, du_a^{(l_{k-2})},\, dt\}$ of $I^{(k-2)} + J_{l_{k-2}} + \mathrm{span}\,\{dt\}$ for every $l_{k-2} \geq l_{k-2}^*$. Let $l_{k-1} = l_{k-2} = \max\{l_{k-1}^*, l_{k-2}^*\}$.

As $I^{(k-1)} \subset I^{(k-2)}$, we may complete $\mathcal{B} \cup \dot{\mathcal{B}}$ with a subset $\hat{\mathcal{B}} = \{\theta_i, i = s+1, \ldots, \sigma\}$ of $\{\tilde{\theta}_i : i \in \lfloor r \rceil\}$ in order to form a basis of $I^{(k-2)} + J_{l_{k-2}} + \mathrm{span}\,\{dt\}$. By the same reasoning of the end of the proof of part (i) of Lemmma B.2, it follows that $B \cup \dot{B} \cup \hat{B}$ is a basis of $I^{(k-2)} + J$. The fact that $\dot{\mathcal{B}} \cup \hat{\mathcal{B}}$ is linearly independent $\mod (I^{(k-1)} + J + \mathrm{span}\,\{dt\})$ implies that $\dot{B} \cup \hat{B}$ is also linearly independent $\mod (I^{(k-1)} + J)$. ∎

Appendix C. Geometric Interpretation of the Dynamic Extension Algorithm.

In [14] it was shown, using an algebraic approach, that the output rank (the number of differentially independent outputs [15]) can be computed by the *structure algorithm* [51] and the *dynamic extension algorithm* [13, 35]. This interpretation was developed further in [11] in order to study control synthesis problems by quasi-static state feedback. In [39], the algebraic results of [14, 11] are translated to the differential geometric approach of [20], giving the following Lemma :

Lemma C.1. *[39, Lemma 8.2] Consider the analytic (explicit) system S defined by* (1.1a) *with analytic output $y = h(t,x,u)$. Let S_k be the open and dense set of regular points of the codistributions $Y_i = \mathrm{span}\,\{dt, dy, \ldots, dy^{(i)}\}$ and $\mathcal{Y}_i = span\,\{dt,\, dx,\, dy,\, \ldots,\, dy^{(i)}\}$. In the* kth *step of the dynamic extension algorithm, one may construct a partition*[25] *$y = (\bar{y}_k, \hat{y}_k)$ and a new local classical state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$ and input $u_k = (\overline{y}_k^{(k+1)}, \hat{u}_k)$, defined in an open neighbourhood V_ξ of $\xi \in S_k$, such that*

1. $\mathrm{span}\,\{dt, dx_k\} = \mathrm{span}\,\{dt, dx, dy, \ldots, dy^{(k)}\}$.
2. $\mathrm{span}\,\{dt, dx_k, du_k\} = \mathrm{span}\,\{dt, dx, dy, \ldots, dy^{(k+1)}, du\}$.
3. *It is always possible to choose $\bar{y}_{k+1}^{(k+1)}$ in a way that $\bar{y}_k^{(k+1)} \subset \bar{y}_{k+1}^{(k+1)}$.*
4. *It is always possible to choose $\hat{u}_{k+1} \subset \hat{u}_k$.*
5. *Let $\mathcal{D}(\mathcal{C})$ denote the generic dimension of a codistribution $\mathcal{C}$ generated by the differentials of a finite set of analytic functions. The sequence $\sigma_k = \mathcal{D}(\mathcal{Y}_k) - \mathcal{D}(\mathcal{Y}_{k-1})$ is nondecreasing, the sequence $\rho_k = \mathcal{D}(Y_k) - \mathcal{D}(Y_{k-1})$ is nonincreasing, and both sequences converge to the same integer ρ, called the* output rank, *for some $k^* \leq n =$* $\dim x$.
6. *$S_k = S_{k^*}$ for $k \geq k^*$.*
7. *$Y_k \cap \mathrm{span}\,\{dx\}|_\nu = Y_{k^*-1} \cap \mathrm{span}\,\{dx\}|_\nu$ for every $\nu \in S_{k^*}$ and $k \geq k^*$.*
8. *Around $\xi \in U_k$, one may choose, $\bar{y}_k = \bar{y}_{k^*}$ for $k \geq k^*$. Furthermore, $Y_{k+1} = Y_k + \mathrm{span}\,\{\bar{y}_k^{(k+1)}\}$ for $k \geq k^*$.*

Proof. A complete proof of this result can be found in [39]. (see [14, Thm. 2.5] and [11, Lemma 4.1.6] for similar results in algebraic contexts). ∎

[25] Including a possible reordering of the outputs.

Acknowledgments. The authors are indebted to Michel Fliess, Joachim Rudolph, Felipe M. Pait and Emmanuel Delaleau for their suggestions concerning this paper.

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