RELATIVE FLATNESS AND FLATNESS OF IMPLICIT SYSTEMS*

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Abstract. In this work we define the concept of relative flatness of a system with respect to a subsystem. The subsystem associated to a set of outputs of a system is constructed, and called here output subsystem. It is shown that the relative flatness of a system with respect to the output subsystem implies the flatness of the corresponding implicit system obtained by setting these outputs to zero. A sufficient condition of relative flatness based on a relative derived flag is presented. Based on these results, a sufficient condition for the flatness of a class of nonlinear implicit systems is obtained.

Key words. Nonlinear systems, implicit systems, time-varying systems, flatness, relative flatness, feedback linearization.

1. Introduction and Motivation. The aim of this paper is to present the notion of relative flatness with respect to a subsystem. We show that this concept may be useful for control systems theory, in particular for studying the structure of nonlinear implicit systems. Our approach is based on the infinite dimensional geometric setting recently introduced in control theory [18, 42, 20] in combination with the ideas presented in [52, 50, 55]. Our sufficient conditions for flatness of implicit systems may be regarded as a generalization of the conditions obtained in [52] for explicit systems. Our setting has some connections with the ideas of [49], which has considered a different class of implicit systems.

Feedback linearization is an important problem in nonlinear control theory. This problem was completely solved in static-state feedback case [26, 24] but necessary and sufficient conditions for feedback linearizability by dynamic state feedback are not yet known (see [6, 50, 7, 21, 53, 55, 1, 46, 54, 23, 43, 56]).

The notion of differential flatness was introduced by Fliess et al [17, 19] and is strongly related to the problem of feedback linearization. This concept corresponds to a complete and finite parametrization of all solutions of a control system by a differentially independent family of functions, called flat output.

Linear singular (or implicit) systems are an important class of control systems and many papers and books on this subject are found in the literature [5, 33]. Solvability of nonlinear implicit differential equations is considered in [3, 45]. Other problems like controllability [30], stabilization [34, 8], canonical forms [47] and feedback control [9], have already been considered.

Feedback linearization of implicit systems has been studied for instance by [31, 32, 27]. These works consider the problem of finding a state transformation and a state feedback such that the closed loop system is a linear singular system. In this

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†Note that the module theoretic approach of [16] is also valid for implicit systems.
work we tackle the problem of finding sufficient conditions for flatness of a class of time-varying implicit systems of the form\(^2\)

\[
\begin{align*}
(1.1a) & \quad \dot{x}(t) = f(t, x(t), u(t)) \\
(1.1b) & \quad y(t) = h(t, x(t), u(t)) = 0
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p, u(t) \in \mathbb{R}^m\) and all the components of \(f(x)\) and \(g(x)\) are analytical functions of \(x\).

Given a set of \(p\) nonlinear differential equations of arbitrary order, one can always define an equivalent system of the form (1.1a)-(1.1b). In fact, consider the following differential equations\(^3\)

\[
\phi_i(w_1, \ldots, w_1^{(\alpha_1)}, \ldots, w_r, \ldots, w_r^{(\alpha_r)}) = 0, \quad i \in \{1, \ldots, p\}
\]

Let \(\beta_j = \max_j \{\alpha_j\}, \quad i \in \{1, \ldots, p\}, j \in \{1, \ldots, r\}\). Consider the system \(S\) with state \(x = (w_1, \ldots, w_1^{(\beta_1-1)}, \ldots, w_r, \ldots, w_r^{(\beta_r-1)})\), input \(u = (u_1, \ldots, u_r)\), where \(u_j = w_j^{(\beta_j)}\), and output \(y\) defined by equations

\[
\begin{align*}
\begin{cases}
\dot{w}_j^{(0)} = w_j^{(1)} \\
\vdots \\
\dot{w}_j^{(\beta_j)} = u_j \\
y_i = \phi_i(w_1, \ldots, w_1^{(\alpha_1)}, \ldots, w_r, \ldots, w_r^{(\alpha_r)}), \quad i \in \{1, \ldots, p\}.
\end{cases}
\end{align*}
\]

It is clear that the system (1.2) is represented by the system (1.3) with the constraints \(y_i = 0\), which is in the form (1.1a)-(1.1b). So, all the results developed here may be applied to a set of differential equations of arbitrary order.

We now present, without being precise, a summary of the ideas and the results of this paper. Roughly speaking, a subsystem \(S_1\) of a system \(S\) is some part of \(S\) that may be considered as a system by itself. Note that \(S_1\) may affect the “quotient system” \(S/S_1\), but it is not affected by \(S/S_1\) as depicted in the Figure 1.1:

![Figure 1.1. Structure of a system S with respect to a subsystem S_1.](image)

Remark 1.1. We stress that, in figure 1.1, \(S/S_1\) is not a subsystem.

Recall that a system is flat if and only if there exists a differentially independent set of functions \(y = (y_1, \ldots, y_m)\), called the flat output, such that every variable of the system is a function of the flat output and its derivatives. A system \(S\) is said to be relatively flat with respect to a given subsystem \(S_1\) if, after a convenient endogenous feedback, \(S\) is decomposed into two independent subsystems \(S_1\) and \(S_2\) such that \(S_2\) is a flat system\(^4\) (see figure 1.2). We stress that the fact that the system is decomposed

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\(^2\)This class is more general than the one considered by [31, 27].

\(^3\)As in the behavioral approach of Willems [59], we do not distinguish input, state and outputs among the variables \(w_j, i = 1, \ldots r\) in the differential equations (1.2).

\(^4\)See Definition 5.1 for a precise statement of relative flatness.
into two independent subsystems is not artificial since the same structure occurs for the algebraic counterpart of this definition (see Rem. 5.1).

In this paper, a sufficient condition for relative flatness is given (see Thm. 8.2). One can easily conclude that a system \( S \) that is relatively flat with respect to a flat subsystem is also flat\(^5\) leading to a sufficient condition of flatness of system \( S \).

\[
\begin{array}{c}
\text{SYSTEM } S \\
\text{ } \\
\begin{array}{c}
S_1 \\
S_2 \text{ (flat)}
\end{array}
\end{array}
\]

\text{Fig. 1.2. Structure of a system } S \text{ that is relatively flat with respect to a subsystem } S_1. \text{ Note that } S_2 \text{ is flat.}

Now, let \( y \) be the output (not necessarily a flat output) of system \( S \). We will show that one can construct a subsystem \( Y \) of \( S \) such that \( Y \) contains only the “information” of time and of \( y \) and its derivatives \( y^{(k)}, k \in \mathbb{N} \) (see Theorem 4.3). Subsystem \( Y \) will be called \textit{output subsystem}.

\[
\begin{array}{c}
\text{SYSTEM } S \\
\begin{array}{c}
S/Y \\
Y
\end{array}
\end{array}
\]

\text{Fig. 1.3. Structure of a System } S \text{ with respect to the output subsystem } Y.

The structure of the implicit system, obtained from \( S \) by setting \( y \) to be equal to zero (see Figure 1.3) is directly related to the properties of \( S \) with respect to the output subsystem \( Y \). Under some regularity assumptions, if \( S \) is relatively flat with respect to \( Y \), then the implicit system obtained from \( S \) by including the constraint \( y = 0 \) is also flat\(^6\).

The paper is organized as follows. In section 2 the notation and some mathematical background are presented. The infinite dimensional differential geometric approach of [20] is briefly summarized in section 3. The notion of subsystem is presented in section 4. The existence and some properties of local output subsystems are also discussed in § 4. The concept of relative flatness is discussed in section 5. In section 6 it is shown that, under regularity assumptions, an implicit system (1.1a)–(1.1b) may be considered as a system that is immersed in the explicit system (1.1a). In section 7, the results of the previous sections are used to derive a sufficient condition for flatness of implicit systems. A sufficient condition for relative flatness based on relative derived flags is developed in in section 8. Some examples are discussed in section 9. Finally, some auxiliary results and proofs are presented in appendices A, B and C.

2. Preliminaries and notation. The field of real numbers is denoted by \( \mathbb{R} \) and \( \mathbb{N} \) stands for the set natural numbers (including zero). The subset \( \{1, \ldots, k\} \) of \( \mathbb{N} \) is denoted by \( [k] \). Given a set \( W \), then \( \text{card} \ W \) stands for the cardinality of \( W \). We adopt the standard notations of differential geometry and exterior algebra in the finite and infinite dimensional case [57, 4, 60]. Let us briefly recall the main definitions of

\(^5\)See Proposition 5.2 for a precise statement of this idea.

\(^6\)See Theorem 7.2 for a precise statement of this sufficient condition of flatness.
the infinite dimension setting introduced in control systems theory [18, 42, 20]. This approach is mainly based on the differential geometry of jets and prolongations (see for instance [28, 60]) whereas the approach of [25] and [36] is based on finite dimensional differential geometry [57].

Let $A$ be a countable set. Denote by $\mathbb{R}^A$ the set of functions from $A$ to $\mathbb{R}$. One may define the coordinate function $x_i : \mathbb{R}^A \rightarrow \mathbb{R}$ by $x_i(\xi) = \xi(i), i \in A$. This set can be endowed with the Fréchet topology (i.e., an inverse limit topology [2, 60]). A basis of this topology is given by the subsets of the form $B = \{ \xi \in \mathbb{R}^A \mid |x_i(\xi) - \delta_i| < \epsilon_i, i \in F \}$, where $F$ is a finite subset of $A$, $\delta_i \in \mathbb{R}$ and $\epsilon_i$ is a positive real number for $i \in F$. A function $\phi : \mathbb{R}^A \rightarrow \mathbb{R}$ is smooth if $\phi = \psi(x_1, \ldots, x_n)$, where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function. Only the dependence on a finite number of coordinates is allowed.

From this notion of smoothness, one can easily state the notions of vector fields and differential forms$^7$ on $\mathbb{R}^A$ and smooth mappings from $\mathbb{R}^A$ to $\mathbb{R}^B$. The notion of $\mathbb{R}^A$-manifold can be also established easily as in the finitely dimensional case [60].

Given an $\mathbb{R}^A$-manifold $P$, $C^\infty(P)$ denotes the set of smooth maps from $P$ to $\mathbb{R}$. Let $Q$ be an $\mathbb{R}^B$-manifold and let $\phi : P \rightarrow Q$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_pP \rightarrow T_{\phi(p)}Q$ and $\phi^* : T^*_{\phi(p)}Q \rightarrow T^*_pP$.

The map $\phi : P \rightarrow Q$ is called an immersion if, around every $\xi \in P$ and $\phi(\xi) \in Q$, there exist local charts of $P$ and $Q$ such that, in these coordinates $\phi(x) = (x, 0)$. The map $\phi$ is called a submersion if, around every $\xi \in P$ and $\phi(\xi) \in Q$, there exist local charts of $P$ and $Q$ such that, in these coordinates, $\phi(x, y) = x$.

In the finite dimensional case, immersion and submersions are locally characterized respectively by the injectivity and surjectivity of the tangent mappings. However, in the infinite dimensional case this is no longer true. Moreover, the inverse function Theorem and the classical Frobenius Theorem (for distributions) do not hold and a field does not admit a flow in general [60].

Given two forms $\eta$ and $\xi$ in $\Lambda(P)$, then $\eta \wedge \xi$ denotes their wedge multiplication. The exterior derivative of $\eta \in \Lambda(P)$ will be denoted by $d\eta$. Note that the graded algebra $\Lambda(P)$, as well as its homogeneous elements $\Lambda_k(P)$ of degree $k$, have a structure of $C^\infty(P)$-module. See [57, 4] for details. Given a family $\nu = (\nu_1, \ldots, \nu_r)$ of a $C^\infty(P)$-module, then span $\{\nu_1, \ldots, \nu_r\}$ stands for the span over $C^\infty(P)$.

Given a field $f$ and a 1-form $\omega$ on $P$, we denote $\omega(f)$ by $\langle \omega, f \rangle$. The set of smooth $k$-forms on $P$ will be denoted by $\Lambda_k(P)$ and $\Lambda(P) = \cup_{k \in \mathbb{N}} \Lambda_k(P)$.

The following useful result of finite dimensional differential geometry is known as “Cartan Lemma” ([57], p.80 ex. 16). Let $\{\omega_1, \ldots, \omega_r\} \subset \Lambda_1(P)$ be independent pointwise. Assume that there exist 1-forms $\eta_1, \ldots, \eta_r$ such that $\sum_{i=1}^r \eta_i \wedge \omega_i = 0$. Then there exist functions $a_{ij} \in C^\infty(P)$, with $a_{ij} = a_{ji}$, such that $\eta_i = \sum_{j=1}^r a_{ij} \omega_j$ ($i = 1, \ldots, r$). The same result is also valid pointwise, i.e., $\sum_{i=1}^r \eta_i \wedge \omega_i|_p = 0$ implies that $\eta_i(p) = \sum_{j=1}^r a_{ij}(p) \omega_j(p)$ ($i = 1, \ldots, r$) for convenient $a_{ij} = a_{ji} \in \mathbb{R}$.

A smooth codistribution $J$ is a $C^\infty(P)$-submodule $J \subset T^*P$. Given a submodule $S$ of $\Lambda_1(P)$ and $p \in P$, then $S(p)$ denotes the $\mathbb{R}$-linear subspace of $\Lambda_1(P)|_p$ given by span$_\mathbb{R}$ $\{\xi(p) \mid \xi \in S\}$. In particular, if $J$ is a codistribution, then $J(p)$ denotes the subspace of $T^*_pP$ given by span$_\mathbb{R}$ $\{\omega(p) \mid \omega \in J\}$.\footnote{We stress that the forms are finite combinations of the form $\sum_{i} a_i dx_{i_1} \wedge \cdots \wedge dx_{i_n}$, where $I_i$ is the multi-index $\{i_1, \ldots, i_n\}$, the $a_i$ are smooth functions, $dx_{i_1} = dx_{i_1,j_1} \wedge \cdots \wedge dx_{i_1,j_n}$. On the other hand, the fields are (possibly) infinite sums of the form $\sum_{i \in A} a_i \frac{\partial}{\partial x_i}$.}

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Assume that a codistribution $I$ is locally generated by $\eta_1, \ldots, \eta_k$ and that $\Psi = \{x_i | i \in A\}$ is a local coordinate system around some open set $U \subset \mathcal{P}$. As $\eta_i = \sum_{\alpha \in \mu_i} \alpha_{ij} dx_i$ for convenient smooth functions $\alpha_{ij}$, then there must exist some finite subset $A_0 \subset A$ such that all the functions $\alpha_{ij}$ depend only on $\{x_i | i \in A_0\}$ and $\eta_i \in \text{span} \{dx_j | j \in A_0\}$. Consider the finite dimensional vector space $\mathbb{R}^{A_0}$ and the canonical submersion $\pi : U \rightarrow \mathbb{R}^{A_0}$ such that $\pi \circ \Psi^{-1}(x_i | i \in A) = (x_i | i \in A_0)$. It is clear that the one-forms $\tilde{\eta}_i = \sum_{\alpha \in \mu_i} \alpha_{ij} dx_i$ on the open neighbourhood $\pi(U) \subset \mathbb{R}^{A_0}$ are such that $\tilde{\eta}_i = \pi^* \eta_i, i \in [k]$. Furthermore, if $I = \text{span} \{\tilde{\eta}_i | i \in [k]\}$, then $I = \pi^* I$. In other words one may apply to (locally) finitely generated codistributions the standard techniques of differential geometry, for instance the Frobenius Theorem, by “pulling-back” the results that hold on the finite dimensional case [42].

3. Diffieties and Systems. In this section we recall the main concepts of the infinite dimensional geometric setting of [18, 42, 20]. We have chosen to present a simplified exposition. For a more complete and intrinsic presentation the reader may refer to the cited literature.

3.1. Diffieties. A diffiety $M$ is a $\mathbb{R}^4$-manifold equipped with a distribution $\Delta$ of finite dimension $r$, called Cartan distribution. A section of the Cartan distribution is called a Cartan field. An ordinary diffiety is a diffiety for which $\dim \Delta = 1$ and a Cartan field $\partial_M$ is distinguished and called the Cartan field. In this paper we will only consider ordinary diffieties, that will be called simply by diffieties.

A Lie-B"{a}cklund mapping $\phi : M \rightarrow N$ between diffieties is a smooth mapping that is compatible with the Cartan fields, i.e., $\phi, \partial_M = \partial_N \circ \phi$. A Lie-B"{a}cklund immersion (respectively, submersion) is a Lie-B"{a}cklund mapping that is an immersion (resp., submersion). A Lie-B"{a}cklund isomorphism between two diffieties is a diffeomorphism that is a Lie-B"{a}cklund mapping.

Context permitting, we will denote the Cartan field of an ordinary diffiety $M$ simply by $\frac{d}{d\tau}$. Given a smooth object $\phi$ defined on $M$ (a smooth function, field or form), then $L_{\frac{d}{d\tau}}(\phi)$ will be denoted by $\dot{\phi}$ and $L^n_{\frac{d}{d\tau}}(\phi)$ by $\phi^{(n)}$, $n \in \mathbb{N}$. In particular, if $\omega$ is a 1-form given by $\omega = \sum_{\alpha \in \mu_i} \alpha_i dx_i$, then $\tilde{\omega} = \sum_{\alpha \in \mu_i} (\dot{\alpha}_i dx_i + \alpha_i d\tau_i)$.

3.2. Systems. The set of real numbers $\mathbb{R}$ has a trivial diffiety structure with the Cartan field defined by the operation of differentiation of smooth functions. A system is a triple $(S, \mathbb{R}, \tau)$ where $S$ is a diffiety equipped with Cartan field $\frac{d}{d\tau}$, the mapping $\tau : S \rightarrow \mathbb{R}$ is a Lie-B"{a}cklund submersion and $\frac{d}{d\tau}(\tau) = 1$. The function $\tau$ represents time, that is chosen once and for all. Context permitting, the system $(S, \mathbb{R}, \tau)$ is denoted simply by $S$. A Lie-B"{a}cklund mapping between two systems $(S, \mathbb{R}, \tau)$ and $(S', \mathbb{R}, \tau')$ is a time-respecting Lie-B"{a}cklund mapping $\phi : S \rightarrow S'$, i.e., $\tau' = \tau \circ \phi$. The previous condition means that the notion of time of both systems coincide. This notion of system is time-varying as it will be explained below.

3.3. State Representation. We present a simplified definition of state representation that introduces the state and the input and its derivatives as a local coordinate system (see [18, 20] for a more intrinsic presentation).

A local state representation of a system $(S, \mathbb{R}, \tau)$ is a local coordinate system $\psi = \{t, x, U\}$ where $x = \{x_i, i \in [n]\}$, $U = \{u_j^{(k)} | j \in [m], k \in \mathbb{N}\}$, where $u_j^{(k)} = L^k u_j$, and $\tau = t$. The set of functions $x = (x_1, \ldots, x_n)$ is called state and $u = (u_1, \ldots, u_m)$
is called input. In these coordinates the Cartan field is locally written by

\[(3.1) \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k \in \mathbb{N}} \sum_{j \in [m]} u^{(k+1)}_j \frac{\partial}{\partial u_j^{(k)}} \]

Note that \(f_i\) may depend on \(t, x\) and a finite number of elements of \(U\). In this sense, the state representation defined here is said to be generalized, since one accepts that \(f_i\) may depend on the derivatives of the input. If the functions \(f_i\) depend only on \(\{t, x, u\}\) for \(i \in [n]\), then the state representation is said to be classical. A state representation of a system \(S\) is completely determined by the choice of the state \(x\) and the input \(u\) and will be denoted by \((x, u)\). A state representation is said to be analytic\(^9\).

3.4. Output. An output \(y\) of a system \(S\) is a set \(y = (y_1, \ldots, y_p)\) of smooth functions defined on \(S\). If \((x, u)\) is a state representation of \(S\), then it is clear that

\[(3.2) \quad y_j = y_j(t, x, u, \ldots, u^{(\alpha_j)}), \quad j \in [p] \]

If the \(y_j\) depend only on \(\{t, x, u\}\) for \(j \in [p]\), then the output is said to be classical with respect to the state representation \((x, u)\). A state representation \((x, u)\) with output \(y\) is said to be analytic if the functions \(f_i\) and the \(y_j\) are all analytic with respect to its arguments \(x\) and \(\{u^{(j)} \mid j \in \mathbb{N}\}\).

3.5. System associated to differential equations. Now assume that a control system is defined by a set of equations

\[(3.3) \quad \begin{align*}
\dot{t} &= 1 \\
\dot{x}_i &= f_i(t, x, u, \ldots, u^{(\alpha_i)}), \quad i \in [n] \\
y_j &= y_j(t, x, u, \ldots, u^{(\beta_j)}), \quad j \in [p]
\end{align*} \]

One can always associate to these equations a diffeity \(S\) of global coordinates \(\psi = \{t, x, U\}\) and Cartan field given by (3.1).

3.6. Flatness. We present now a simple definition of flatness in terms of coordinates\(^{10}\). A system \(S\) equipped with Cartan field \(\frac{\partial}{\partial x}\) and time function \(t = \tau\) is locally flat around \(\xi \in S\) if there exists a set of smooth functions \(y = (y_1, \ldots, y_m)\), called flat output, such that the set \(\{t, y^{(i)}_i \mid i \in [m], j \in \mathbb{N}\}\) is a (local) coordinate system of \(S\) around \(\xi \in S\), where \(y^{(i)}_i = L^i_j y_i\). Note that the Cartan field is locally given by:

\[\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i \in [m]} \sum_{j \in [m]} y^{(j+1)}_i \frac{\partial}{\partial y^{(j)}_i} \]

Let \(\Psi : S \to T\) be a Lie-Bäcklund isomorphism between two systems. Then \(S\) is flat if and only if \(T\) is flat, also. If \(y = (y_1, \ldots, y_m)\) is a flat output of \(T\) then \(\{y_1 \circ \Psi, \ldots, y_m \circ \Psi\}\) is a flat output of \(S\).

\(^9\)This definition is coordinate dependent since only smooth atlases are considered on diffeities [60].

\(^{10}\)For more intrinsic definitions and some variations, see [18, 20].
3.7. Endogenous feedback and coordinate changes. Since a local state representation \((x, u)\) is by definition a local coordinate system, a new local state representation \((z, v)\) induces a coordinate change from \(\{t, x, (u^i) : i \in \mathbb{N}\}\) to \(\{t, z, (v^j) : j \in \mathbb{N}\}\). The coordinate changes of this kind are called endogenous feedbacks\(^{11}\).

An example of endogenous feedback is static-state feedback. Two state representations \((x, u)\) and \((z, v)\) defined around \(\xi \in S\) are said to be linked by (time-varying) static-state feedback if we locally have

\[
(3.4a) \quad \text{span} \{dt, dx\} = \text{span} \{dt, dz\}
\]
\[
(3.4b) \quad \text{span} \{dt, dx, du\} = \text{span} \{dt, dz, dv\}.
\]

Let \((x, u)\) be a classical state-representation and let \(z\) and \(v\) be families of smooth functions such that \(x = \text{card } z\) and \(u = \text{card } v\). Then it is easy to show that, if (3.4) locally holds, then \((z, v)\) is a local state representation that is linked to \((x, u)\) by static-state feedback [39, Prop. 3.2].

Another example of endogenous feedback is putting integrators in series with the first \(k\) inputs of the system (3.3). This procedure induces a local state representation \((z, v)\) of the system \(S\), where \(z = (x_1, \ldots, x_n, u_1, \ldots, u_k)\) and \(v = (u_1, \ldots, u_k, u_{k+1}, \ldots, u_m)\), called dynamic extension of the state.

4. Subsystems. A (local) subsystem \(S_a\) of a given system \(S\) is a system \(S_a\) such that there exists a surjective\(^{12}\) Lie-Bäcklund submersion \(\pi : U \subset S \rightarrow S_a\), where \(U\) is an open subset of \(S\). A (local) subsystem will be denoted by \((S_a, \pi)\) or simply by \(S_a\).

4.1. State equations adapted to subsystems. Assume that there exists a local classical state representation \((x, u)\) of a system \(S\) of the form

\[
(4.1a) \quad \dot{x}_a = f_a(t, x_a, u_a)
\]
\[
(4.1b) \quad \dot{x}_b = f_b(t, x_a, x_b, u_a, u_b),
\]

where \(x = (x_a, x_b)\) and \(u = (u_a, u_b)\). Assume that (4.1a) represents the state equations of a subsystem \(S_a\) and \(\pi : S \rightarrow S_a\) is such that \(\pi(t, x, U) = (t, x_a, U_a)\), where \(U\) denotes the set \((u^i| j \in \mathbb{N})\) and \(U_a\) denotes the set \((u_a^j| j \in \mathbb{N})\). A state representation of \(S\) the form (4.1a)-(4.1b) is said to be adapted to the subsystem \(S_a\). In the end of this section we show that state equations adapted to a subsystem can be generically constructed (see Proposition 4.4).

4.2. Relative static-state feedback. We will consider now a special case of endogenous feedback that will be called by Relative Static-State Feedback. Consider that \(((x_a, x_b), (u_a, u_b))\) is a local state representation for system \(S\) such that the state equations are of the form (4.1a)-(4.1b). A relative state feedback is a new state representation \(((x_a, z_b), (u_a, v_b))\) such that

\[
(4.2) \quad z_b = z_b(t, x_b, x_a, u_a, \ldots, u_a^{(r)}),
\]
\[
(4.2) \quad v_b = v_b(t, x_b, u_b, x_a, u_a, \ldots, u_a^{(r+1)}).
\]

where \(r\) is a convenient integer and similar equations do exist for \(x_b, u_b\) as functions of \(x_a, z_b, u_a, v_b\) and the derivatives of \(u_a\). In other words, this is an invertible time-varying feedback. The next definition renders this notion more intrinsic.

\(^{11}\) See [18] for a definition of endogenous feedback that considers an equivalence relation between systems.

\(^{12}\) Since submersions are open maps, one can always consider that \(S_a = \pi(U)\) by restricting \(S_a\) to the image of \(\pi\).
Definition 4.1. Let S be a system and let \((\pi, S_\pi)\) be a (local) subsystem of S. Let \((x,u)\) and \((z,v)\) be two (local) state representations of S. Let \(\Sigma\) be the codistribution defined by the pull-back\(^{13}\) \(\Sigma = \pi'(T^*S_\pi)\). Then \((x,u)\) and \((z,v)\) are linked by a relative static-state feedback with respect to the subsystem \(S_\pi\) if \(\text{span}\ \{dx\} + \Sigma = \text{span}\ \{dz\} + \Sigma\) and \(\text{span}\ \{dx, du\} + \Sigma = \text{span}\ \{dz, dv\} + \Sigma\).

Proposition 4.2. Let S be a system with local state representation \((x,u)\) defined on \(V_\xi \subset S\), where \(x = (x_a, x_b)\), and \(u = (u_a, u_b)\) are such that the state equations are of the form \((4.1a)-(4.1b)\). Let \(S_\pi\) be the (local) subsystem associated to equation \((4.1a)\). Consider the set of smooth functions \(z = (x_a, z_b)\) and \(v = (u_a, v_b)\) defined on \(V_\xi\), where \(\text{card } x = \text{card } z = n\) and \(\text{card } u = \text{card } v = m\). Then the following statements are equivalent:

(i) \((z,v)\) is a local state representation around \(\xi\) and \((x,u)\) and \((z,v)\) are linked by relative static-state feedback.

(ii) \(\text{span}\ \{dx\} + \Sigma = \text{span}\ \{dz\} + \Sigma\) and \(\text{span}\ \{dx, du\} + \Sigma = \text{span}\ \{dz, dv\} + \Sigma\).

Proof. Deferred to the Appendix B.1. \(\square\)

Remark 4.1. The proof of the Prop. 4.2 shows that (i) implies that condition \((4.2)\) is satisfied for a subsystem \(S_\pi\) defined by \((4.1a)\). It will be shown (see Prop. 4.4) that all subsystems admit adapted state equations of the form \((4.1a)-(4.1b)\), up to relative static-state feedbacks.

4.3. Output Subsystem. Given a system S with output \(y\), a (local) output subsystem is a (local) subsystem \(\pi : U \subset S \rightarrow Y\) such that \(\pi'(T^*_{\pi(\xi)}Y) = \text{span}\ \{dt, dy^{(k)} : k \in \mathbb{N}\}\|_{\xi, \xi \in U}\).

4.4. Existence of local output subsystems. Without loss of generality, assume that \((x,u)\) is a classic state representation with output \(y\). If it is not the case we can add integrators in series with the input until the required properties are fulfilled. The next Theorem shows that local output subsystems can be constructed generically and they admit adapted state equations up to relative static-state feedback. Furthermore, they are unique up to Lie-Backlund isomorphisms.

Theorem 4.3. (Existence and Uniqueness of Output Subsystems) Let S be a system and let \((x,u)\) be a classical analytical state representation defined on an open neighbourhood \(W \subset S\). Let \(y\) be a classical output of S. Let \(n = \text{card } x\). Let \(U \subset W\) be the set of regular points of the codistributions \(Y_k, Y_k, k \in [n]\), where \(Y_k = \text{span}\ \{dt, dy, \ldots, dy^{(k)}\}\) and \(Y_k = \text{span}\ \{dt, dx, dy, \ldots, dy^{(k)}\}\). Then, around any \(\xi \in U\), there exists an open neighbourhood \(V_\xi\) of \(\xi\), a local classical state representation \((z,v) = ((z_a, z_b), (v_a, v_b))\) of the system S defined on \(V_\xi\) such that:

(i) The (local) state equations are:

\[
\begin{align*}
(4.3a) & \quad \dot{z}_a = f_a(t, z_a, v_a) \\
(4.3b) & \quad \dot{z}_b = f_b(t, z_a, z_b, v_a, v_b).
\end{align*}
\]

(ii) Let \(Y\) be the local subsystem associated to \((4.3a)\) and let \(\pi : V_\xi \rightarrow Y\) be the corresponding Lie-Backlund submersion. We have \(\pi'(T^*Y) = \text{span}\ \{dt, dz_a, (dv_a^{(k)} : k \in \mathbb{N}\}\) = \(\text{span}\ \{dt, dy^{(k)} : k \in \mathbb{N}\}\). In particular, \(Y\) is an output subsystem of S. Let \(Z = \{z_a, (v_a^{(k)} : k \in \mathbb{N}\}\) and \(Y = \{y^{(k)} : j \in [p], k \in \mathbb{N}\}\). Then \(Z \subset Y\).

(iii) The state representations \((x,u)\) and \((z,v)\) are linked by relative static-state feedback with respect to the subsystem \(Y\) associated to \((4.3a)\).

\[^{13}\text{Note that } \text{span}\ \{dt\} \subset \Sigma.\]
Furthermore, two local output subsystems around any \( \xi \in S \) are (locally) Lie-Bäcklund isomorphic.

Proof. See Appendix B.2. \( \square \)

We state now the result that assures that a subsystem can be generically represented by state equations of the form (4.1a)-(4.1b).

**Proposition 4.4.** Assume that \( S_a \) is a subsystem of \( S \) and that there exist local state representations for \( S_a \) and \( S \) around every point of \( S_a \) and \( S \). Then, generically, there exists local state representations of \( S \) of the form (4.1a)-(4.1b) in a way that (4.1a) is a state representation of \( S_a \).

Proof. Let \( \pi : S \to S_a \) be the corresponding Lie-Bäcklund submersion. Take a local state representation \( (z_a, e_a) \) of \( S_a \) around \( \pi(\xi) \in S_a \). We abuse notation and denote \( z_a \circ \pi \) and \( e_a \circ \pi \) respectively by \( z_a \) and \( e_a \). Now, consider system \( S \) with output \( y = (z_a, e_a) \) and construct, possibly by extending the state with derivatives of the input, a classical state representation of \( S \) such that \( y \) is a classical output. The result follows easily from the application of Thm. 4.3 and the fact that \( T^*S_a = \text{span} \{ dt, dz_a^{(k)}, de_a^{(k)}, k \in \mathbb{N} \} \).

If the outputs are differentially independent, the next result shows that local output subsystems are generically flat.

**Proposition 4.5.** Let \( U \) be the open and dense subset of theorem 4.3. Assume that the (explicit) system (1.1a) with output \( y = h(t, x, u) \) is right-invertible, i.e., the output rank \( \rho \) is equal to the number of output components\(^{14}\). Let \( \pi : V \subset S \to Y \) be a local output subsystem with \( V \subset U \). Then \( Y \) is (locally) flat with flat output \( y \).

Proof. The proof is deferred to Appendix B.3. \( \square \)

5. **Relative flatness.** We now state the concept of relative flatness.

**Definition 5.1.** Let \( S \) be a system and \( (\pi_1, S_1) \) and \( (\pi_2, S_2) \) be two subsystems of \( S \). The system \( S \) is said to be locally decomposed by \( S_1 \) and \( S_2 \) if, around \( \xi \in S \), there exists local coordinates \( (t, x^1) \) for \( S_1 \), \( (t, x^2) \) for \( S_2 \) and \( (t, x^1, x^2) \) for \( S \) such that \( \pi_i(t, x^1, x^2) = (t, x^i), i = 1, 2 \). A system \( S \) is said to be (locally) relatively flat with respect to a subsystem \( S_1 \) if there exists a flat subsystem \( S_2 \) such that \( S \) is (locally) decomposed by \( S_1 \) and \( S_2 \).

**Proposition 5.2.** Let \( S_1 \) be a (locally) flat subsystem of a system \( S \). Assume that \( S \) is relatively flat with respect to \( S_1 \). Then \( S \) is (locally) flat.

Proof. The union of flat outputs of \( S_1 \) and \( S_2 \) is a flat output of \( S \). \( \square \)

**Remark 5.1.** In the differential algebraic approach of [15] (see also [22]) one can define a subsystem of a system \( K/k \) as a field extension \( L/k \) such that \( L \) is a subfield of \( K \). Then a system \( K/k \) is relatively flat with respect to \( L \) if the system \( K/L \) is flat, considering \( L \) as the ground field (see [12] for a result similar to Proposition 5.2). However, these algebraic notions are not suitable for our purposes because integrability conditions are not available in this algebraic context.

It can be shown that, if \( K/k \) is relatively flat with respect to \( L \), then \( K/k \) can be decomposed into two independent subsystems \( L/k \) and \( F/k \), where \( F/k \) is flat. (see Appendix A and [38]). In this sense, the assumption that the system is decomposed into two independent subsystems in the definition of relative flatness is not restrictive with respect to the algebraic definition (see also [38] for similar facts that occur when \( L \) corresponds to the noncontrollable subsystem.)

\(^{14}\) See Appendix C for the definition of the output rank \( \rho \).

\(^{15}\) We abuse notation and denote \( x^i \circ \pi_i \) simply by \( x^i \).
The following proposition is a necessary and sufficient condition for completing a given output \( y \) into a flat output (see [44] for related results).

**Proposition 5.3.** Let \( S \) be a system and let \( S_1 \) be a flat subsystem of \( S \). Let \( y \) be a (local) flat output for \( S_1 \). Then there exists a set \( z \) of smooth functions such that \( S \) is locally flat with flat output \( (y, z) \) if and only if \( S \) is relatively flat with respect to the subsystem \( S_1 \).

Proof. The necessity is obvious. The sufficiency follows from the proof of Prop. 5.2. \( \square \)

6. Implicit systems regarded as Lie-Bäcklund immersions. Let \( S \) be the nonconstrained system defined by (1.1a). We show that, under some regularity assumptions, (1.1a)-(1.1b) may be regarded as a system that is immersed in \( S \). We construct a system \( \Gamma \) and a Lie-Bäcklund immersion \( \iota : \Gamma \to S \) such that every integral curve \( \sigma(t) \) of the Cartan-field of \( S \), respecting the constraints \( y(t) \equiv 0 \), is of the form \( \sigma(t) = \iota \circ \gamma(t) \) for a suitable integral curve \( \gamma(t) \) of the Cartan-field of \( \Gamma \).

Consider the explicit (nonconstrained) system \( S \) defined by (1.1a) with output \( y = h(t, x, u) \), global coordinates \( \{ t, x, (u_i^{(j)} : i \in \mathbb{M}; j \in \mathbb{N}) \} \), and Cartan field (3.1). Consider now the following assumptions:

**A1. Existence and Regularity Assumption.** Let \( \Gamma = \{ \xi \in S : y^{(k)}(\xi) = 0 \text{ for all } k \in \mathbb{N} \} \). Assume that \( \Gamma \neq \emptyset \) and furthermore, \( \Gamma \subset U \), where \( U \) is the open and dense subset of the system \( S \) such that the statement of Theorem 4.3 holds\(^1\). In other words, around every point \( \xi \in \Gamma \), we can construct a local output subsystem.

**A2. Time Interval Assumption.** For every \( \xi \in \Gamma \) and every open neighborhood \( U \subset S \) of \( \xi \), there exists some \( \epsilon \geq 0 \) such that \( \tau(\Gamma \cap U) \) contains an open interval \( (\tau(\xi) - \epsilon, \tau(\xi) + \epsilon) \).

Remark 6.1. Note that assumption A2 means that \( \Gamma \) “does exist” during an interval of time. If the system is time-invariant it is easy to verify that assumption A2 is not needed. Note also that the set \( \Gamma \) may be empty, and in this case the implicit system has no solution. For instance, let \( y_1 = x_1 + 1 \) and \( y_2 = x_2^2 \). Then \( y_1 = 0 \) implies that \( y_2 \neq 0 \). A problem of this nature may occur with output derivatives.

When the assumptions A1, A2 holds, the set \( \Gamma \subset S \) may be endowed with the structure of an immersed Fréchet manifold by choosing the subset topology, as shown by the following proposition.

**Proposition 6.1.** Suppose that assumptions A1 and A2 are satisfied for system \( S \). Then the subset \( \Gamma \subset S \) has a structure of an immersed manifold in \( S \). Let \( \iota : \Gamma \to S \) be the canonical insertion. We can define a Cartan field \( \partial_\tau \) on \( \Gamma \) by the equation \( \iota_\ast \partial_\tau(\gamma) = \frac{\partial}{\partial t} \circ \iota(\gamma), \gamma \in \Gamma \). Equipped with this Cartan field, \( \Gamma \) is a system such that \( \iota \) is a Lie-Bäcklund immersion. Furthermore, all the solutions \( \xi(t) \) of (1.1a) obeying the restriction (1.1b) are of the form \( \xi(t) = \iota \circ \nu(t) \) where \( \nu(t) \) is a solution of \( \Gamma \).

Proof. We show first that \( \Gamma \) is an immersed manifold. For this, consider the topological subspace \( \Gamma \subset S \) with the subset topology. For each point \( \xi \in \Gamma \), Thm. 4.3 gives local charts \( \phi : \tilde{U} \to \tilde{U} \subset \mathbb{R}^A \), where \( \phi = \{ t, z_a, V_a, z_b, V_b \} \), \( V_a = \{ u_a^{(k)} : k \in \mathbb{N} \} \), \( V_b = \{ u_b^{(k)} : k \in \mathbb{N} \} \), and we have span \( \{ dt, dz_a, dV_a \} = \text{span} \{ dt, dy^{(k)} : k \in \mathbb{N} \} \). This local chart is adapted to a local output subsystem \( \pi : \tilde{U} \to Y \), and is such that \( \pi(t, z_a, V_a, z_b, V_b) = (t, z_a, V_a) \). Furthermore, by part (ii) of Thm. 4.3, the functions of the set \( \mathcal{Z} = \{ z_a, V_a \} \) are such that \( \mathcal{Z} \subset \mathcal{Y} \), where \( \mathcal{Y} = \{ y^{(k)} : k \in \mathbb{N} \} \). By construction,

\(^1\)Note that in this case the state representation (1.1a) is globally defined. According the proof of theorem 4.3 we have that \( U \) is the open and dense set of regular points of the codistributions \( Y_k = \text{span} \{ dt, dy, \ldots, dy^{(k)} \} \) and \( \mathcal{Y}_k = \text{span} \{ dt, dx, dy, \ldots, dy^{(k)} \} \) for \( k \in \{ 0, 1, \ldots, n \} \).
if \( \nu \in \hat{U} \cap \Gamma \) then \( y^{(k)}(\nu) = 0 \) for all \( k \in \mathbb{N} \). This implies that all the components of \( Z \) are also null in \( \nu \). If we show that the functions in \( W = Y - Z \) are also null in \( \nu \in \Gamma \cap \hat{U} \), we will show that a point \( \nu \) is in \( \Gamma \cap \hat{U} \) if and only if \( z_a = 0 = V_a \) in \( \nu \). In fact, note first that, since \( \text{span} \{ dt, dZ \} = \text{span} \{ dt, dY \} \), all the functions \( \theta \) in \( Y \) can be locally written in the form \( \theta = \theta(t, z_a, V_a) \). By assumption A2, if we restrict \( \hat{U} \) to a basic open set of the form \( I_{t(\xi)} \times W \) where \( I_{t(\xi)} = (\tau(\xi) - \epsilon, \tau(\xi) + \epsilon) \), we may assume that, for every \( \tilde{t} \in I_{t(\xi)} \), then \( \hat{U} \cap \Gamma \) contains a point \( \xi_t = (\tilde{t}, z_a, V_a) = (\tilde{t}, 0, 0, z_b, V_b) \). For any fixed \( \tilde{t} \in I_{t(\xi)} \), since \( \xi_t \in \Gamma \cap \hat{U} \), we have that \( \theta(\xi_t) = \theta(\tilde{t}, 0, 0) = 0 \). Since this is true for all \( \tilde{t} \in I_{t(\xi)} \), we have showed our claim.

Now consider the map \( \mu : \Gamma \cap \hat{U} \rightarrow \mu(\Gamma \cap \hat{U}) \subset \mathbb{R}^B \) such that \( \mu(t, 0, 0, z_b, V_b) = (t, z_b, V_b) \). We shall show that these maps form an smooth atlas of \( \Gamma \). By construction it is clear that these maps are homeomorphisms. Hence it suffices to show that these charts are \( C^\infty \) compatible. By convenience denote the functions of the chart \( \phi \) by \( \{t, X, Z\} \) and the functions of the chart \( \mu \) by \( \{t, Z, W\} \), where \( X = \{z_a, V_a\} \) and \( Z = \{z_b, V_b\} \).

Now let \( \mu_i : \Gamma \cap U_i \rightarrow \hat{V}_i \), \( i = 1, 2 \), be two local charts constructed in that way, based respectively on the local charts of \( S \) given by \( \phi_i = \{t, X_i, Z_i\} \), \( i = 1, 2 \). In particular, it follows that \( \mu_i \circ \phi_i(t, 0, Z_i) = (t, Z_i) \), \( i = 1, 2 \). Without loss of generality, assume that \( U_1 = U_2 \). Consider the local coordinate change \( \mu_1 \circ \phi_2 = \phi_1 \circ \phi_1^{-1} \). Note that the map \( \theta : \hat{V}_2 \rightarrow \hat{V}_1 \) such that \( \theta(t, Z_1) = (t, Z_2) \) defined by \( \theta(t, 0, Z_1) = \phi_1 \circ \phi_2^{-1}(t, 0, Z_2) \) is a local diffeomorphism with inverse defined by \( \theta(t, 0, Z_2) = \phi_2 \circ \phi_1^{-1}(t, 0, Z_1) \). Since \( \theta = \mu_1 \circ \mu_2^{-1} \), we conclude that such charts are \( C^\infty \) compatible.

Now let \( \iota : \Gamma \rightarrow S \) be the insertion map. In the coordinates \( \phi \) and \( \mu \) previously constructed, we have \( \iota(t, Z) = (t, 0, Z) \). In particular \( \iota \) is an immersion between \( \mathbb{R}^A \)-manifolds and so \( \iota_*(\xi) \) is injective for all \( \xi \in \Gamma \). Remember that any function \( \eta \) of the set \( X = \{z_a, V_a\} \subset Y \) is such that \( \eta|_U = 0 \) for every \( \nu \in \Gamma \cap \hat{U} \). In particular, we have that the image of \( \iota_*(\nu) \) contains \( \frac{\partial}{\partial t}(\iota(\nu)) \) for every \( \nu \in \Gamma \cap \hat{U} \). So we can define \( \partial_t \) by the rule \( \iota_* \partial_t = \frac{\partial}{\partial t} \circ \iota \). By definition, it follows that \( \iota \) is a Lie-Bäcklund immersion.

The last affirmation of the statement is a consequence of the first one. \( \square \)

Remark 6.2. Let \( \phi = (t, x_a, V_a, x_b, V_b) \) and \( \mu = (t, x_b, V_b) \) be respectively the coordinates of \( S \) and \( \Gamma \) constructed above. In this coordinates we have

\[
\partial_{t(\xi)} = \frac{\partial}{\partial t} + \sum_{i=1}^{m} f_b(t, 0, 0, x_b, V_b) \frac{\partial}{\partial x_b} + \sum_{i=1}^{m} \sum_{j \in \mathbb{N}} u_{b(i)}^{(j+1)} \frac{\partial}{\partial u_{b(i)}^{(j)}}
\]

where \( f_b = \frac{\partial}{\partial t}(x_b) = f_b(t, x_a, V_a, x_b, V_b) \), \( i \in \left[ u_b \right] \). In other words, \( (x_b, u_b) \) is a state representation of \( \Gamma \).

It is easy to show that the pull-back (by \( \iota \)) of a relative static-state feedback for \( S \) w.r.t. a local output subsystem \( Y \) induces a static-state feedback for \( \Gamma \), if one considers the state representation \( (x_a, x_b), (u_a, u_b) \) for \( S \) and \( (x_b, u_b) \) for \( \Gamma \).

7. Flatness of implicit systems. In this section we will derive a sufficient condition for flatness of implicit systems. Let us begin with an auxiliary result.

Proposition 7.1. Let \( \Gamma, S \) and \( Y \) be systems, where \( \Gamma \) is immersed in \( S \) and \( Y \) is a subsystem of \( S \). Let \( \iota : \Gamma \rightarrow S \) and \( \pi : S \rightarrow Y \) be respectively the corresponding Lie-Bäcklund immersion and submersion. Assume that there exist local coordinates \( (t, \gamma) \) of \( \Gamma, (t, \gamma, y) \) of \( S \) and \( (t, y) \) of \( Y \) such that \( \iota(t, \gamma) = (t, \gamma, 0) \) and \( \pi(t, \gamma, y) = (t, y) \).

---

\(^{17}\)We assume that \( (t, y) \) is inside the domain of our local chart of \( Y \) for \( y = 0 \).
Assume that $S$ is relatively flat with respect to $Y$. Then $\Gamma$ is (locally) flat.

Proof. Let $S_2$ be a flat subsystem of $S$ such that $S_2$ and $Y$ decomposes $S$ (see Definition 5.1). Let $\pi_2 : S \rightarrow S_2$ be the corresponding Lie-Bäcklund submersion. Recall that there exists coordinates $(t, z, \tilde{y})$ of $S$, $(t, \tilde{y})$ of $Y$, and $(t, z)$ of $S_2$ such that $\pi_2 : (t, z, \tilde{y}) = (t, z)$ and $\pi : (t, z, \tilde{y}) = (t, \tilde{y})$. Since the coordinate change map $(t, y) \rightarrow (t, \tilde{y})$ is a local diffeomorphism, we may assume without loss of generality that $\tilde{y} = y$. With a possible restriction of domains, we can consider the coordinate change mapping $\phi(t, \gamma, y) = (t, z, y)$. Note that the map $\phi_0(t, \gamma) = (t, z)$ such that $\phi(t, \gamma, 0) = (\phi_0(t, \gamma), 0) = (t, z, 0)$ is a local diffeomorphism. Let $\Psi : \Gamma \rightarrow S_2$ be such that $\Psi = \pi_2 \circ \iota$. By definition, $\Psi$ is a Lie-Bäcklund mapping since it is a composition of Lie-Bäcklund mappings. In the coordinates $(t, z)$ for $S_2$ and $(t, \gamma)$ for $\Gamma$ we have $\Psi(t, \gamma) = \phi_0(t, \gamma)$. Hence $\Psi$ is a local Lie-Bäcklund isomorphism and so $\Gamma$ is flat. In particular if $\theta$ is a flat output of $S_2$ then $\theta \circ \Psi$ is a flat output of $\Gamma$.

The following result is a sufficient condition for flatness of an implicit system.

**Theorem 7.2.** Let $S$ be the explicit system defined by (1.1a). Let $y = h(t, x, u)$ be an output for system $S$ and let $Y$ be the corresponding output subsystem of $S$. Suppose that Assumptions $A1$-$A2$ of the previous section hold for the system (1.1a) with the constraints (1.1b). According to Proposition 6.1, equations (1.1a)-(1.1b) define a system $\Gamma$ that is immersed in $S$. Assume that the explicit system (1.1a) is (locally) relatively flat w.r.t. the subsystem $Y$. Then the implicit system $\Gamma$ is locally flat around all $\xi \in \Gamma$.

Proof. Let $\iota : \Gamma \rightarrow S$ be the insertion map and let $\pi : U \subset S \rightarrow Y$ be the canonical submersion onto the local output subsystem $Y$. According to the proof of Prop. 6.1, we can define local charts $\phi = (t, X, Z)$ of $S$, $\mu = (t, Z)$ of $\Gamma$ and $\Psi = (t, X)$ of $Y$ such that $\iota(t, X) = (t, 0, Z)$ and $\pi(t, X, Z) = (t, X)$. Hence, by Prop. 7.1 (for $\gamma = Z$ and $y = X$) the result follows.

Let (1.1a) be a flat (explicit) system and assume that the output $y$ of (1.1b) is part of the flat output of the explicit system (1.1a). Then next result shows that the implicit system (1.1a)-(1.1b) is flat.

**Corollary 7.3.** Assume that $S$ is locally flat with flat output $y = (y_1, \ldots, y_m)$. Assume that the local coordinate system $\{t, y_i^{(j)} : i \in \mathbb{N}, j \in \mathbb{N}\}$ is defined on open set $V$ whose image is a basic open set $\tilde{V}$. Let $\tilde{V} \subset V$ defined by $\{\xi \in V \mid y_i^{(j)}(\xi) = 0, i \in [m], j \in \mathbb{N}\}$. Assume that $\Gamma$ is nonempty. Then $\Gamma$ is an immersed system in $V \subset S$. Furthermore $\Gamma$ is (locally) flat with flat output $y_{r+1}, \ldots, y_n$.

Proof. Consider system $S$ with output $\tilde{y} = (y_{r+1}, \ldots, y_m)$. Let $\tilde{x} = \emptyset$ and $\tilde{u} = (\tilde{y}_1, \ldots, \tilde{y}_m)$. Then $(\tilde{x}, \tilde{u})$ is a local state representation of $S$. Let $\tilde{Y}_r = \text{span} \{dt, dy_1, \ldots, dy_m\}$ and $\tilde{Y}_r = \text{span} \{dt, d\tilde{x}, dy_1, \ldots, dy_m\}$. Then $\tilde{Y}_r = \tilde{Y}_r$ are nonsingular codistributions on $S$ for $r \in \mathbb{N}$ and hence the assumption $A1$ of §6 holds. Since $\tilde{V}$ is a basic open set, it is also clear that assumption $A2$ holds. By Thm. 4.3, the output subsystem $\tilde{Y}$ is well defined, and by Prop. 4.5, it follows that $\tilde{Y}$ is locally flat. By Prop. 5.3, is relatively flat w.r.t. $\tilde{Y}$. The desired result follows from Thm. 7.2.

8. A sufficient condition for relative flatness. Consider a system $S$ and a subsystem $S_1$ of $S$ given by equations (4.1a)-(4.1b) where (4.1a) represents $S_1$. Let $\dim x_a = n_a$, $\dim x_b = n_b$, $\dim u_a = m_a$, and $\dim u_b = m_b$. For this system one can

\[\text{Recall that a basic open set is of the form } \tilde{V} = \{\xi \in S \mid |y_i^{(j)}(\xi) - g_i^{(j)}(\xi)| < \epsilon_{ij} \mid (i, j) \in \Delta\}, \text{ where } \Delta \text{ is a finite subset of } [m] \times \mathbb{N}, g_i^{(j)} \in \mathbb{R}, \text{ and } \epsilon_{ij} \in \mathbb{R}^+.\]
define the *Relative Derived Flag* as follows.

**Definition 8.1.** The Relative Derived Flag of the system (4.1a)-(4.1b) is the sequence of codistributions $I^{(k)}$ defined by $I^{(-)} = \text{span} \{(dx_b - f_idt), (du_a - u_idt)\}$, and $I^{(k)}(p) = \text{span} \{\omega(p) \mid \omega \in I^{(k-1)}, d\omega(p) \mod (I^{(k-1)} + J)_p \equiv 0\}, k \in \mathbb{N}$ where

$$(8.1) \quad J = \text{span} \{(dx_a - \dot{x}_adt), (du_a^{(j)} - u_a^{(j+1)}dt) \mid j \in \mathbb{N}\}.$$  

**Remark 8.1.** In the proof of Prop. 8.3 it is shown that if $I^{(k)}$ is nonsingular then it is smooth (otherwise $I^{(k+1)}$ is not well defined). The 1-forms in $\text{span} \{\frac{dx}{dt}\}^\perp$ are called *contact forms* [42]. Let $\pi : S \to S_a$ be the Lie-Bäcklund submersion of $S$ onto subsystem $S_a$ (see § 4.1). Then it is easy to show that $J$ is the codistribution generated by the contact-forms of $S_a$, i.e., $J = \pi^*(T^* S_a) \cap \text{span} \{\frac{dx}{dt}\}^\perp$. It follows that $J$ is invariant by coordinate changes, and in particular, it is invariant by endogenous feedback. In Appendix B.4 it is shown that:

$$(8.2) \quad I^{(0)} = \text{span} \{dx_b - \dot{x}_bdt\}$$

By construction we have $\dim I^{(-1)} = n_b + m_b$ and $\dim I^{(0)} = n_b$. Note also that $I^{(k)} + J \subset I^{(-1)} + J \subset \text{span} \{\frac{dx}{dt}\}^\perp, k \in \mathbb{N}$. We will show that the relative derived flag carries an intrinsic structural information, at least if one restricts the class of transformations to relative static-state feedback (see Cor. 8.4).

**Theorem 8.2.** Assume that the codistributions $\text{span} \{I^{(k)} dt, J\}$ are involutive, that $I^{(k)}$ are nonsingular for all $k \in \mathbb{N}$ and that $I^{(N)} = 0$ for $N$ big enough. Then the system $S$ is (locally) relatively flat w.r.t. $S_1$.

**Remark 8.2.** It is easy to verify that $J$ is involutive, i.e., that $d\omega \mod J \equiv 0$ for all 1-forms $\omega \in J$. Furthermore, the codistribution $\text{span} \{I^{(k)} dt, J\}$ is involutive if and only if $\text{span} \{I^{(k)} dt, J_{\rho_k}\}$ is involutive for $\rho_k$ big enough, where

$$(8.3) \quad J_{\rho_k} = \text{span} \{(dx_a - f_i dt), (du_a^{(j)} - u_a^{(j+1)}dt) \mid j \in [I]\}.$$  

To prove Theorem 8.2 we need the following auxiliary result whose proof is deferred to Appendix B.5.

**Proposition 8.3.** Assume that the conditions of the Theorem 8.2 are satisfied on an open neighbourhood $V_\xi$ of $\xi$ in $S$. Then, for every $p \in V_\xi$ and $k \in \mathbb{N}$ we have $\dim(I^{(k)} + J)_p / J(p) = \dim I^{(k)}(p)$. Assume that $I^{(k-1)} + J$ has a local basis $B = \tilde{B} \cup B_J$, where $B_J$ is a local basis of $J$ and $\tilde{B}$ is of the form

$$(8.4) \quad \tilde{B} = \{\omega^{(j)}_i : i \in [s], j \in \{0, \ldots, r_i\}\}$$

where $\omega_i = d\theta_i - \tilde{\theta}_idt, \theta_i \in C^\infty(S), i \in [s]$ (or $\tilde{B} = \emptyset$). Assume that the subset $\{\omega^{(r_i)}_i : i \in [s]\}$ is linearly independent mod $\{I^{(k)} + J\}$. Let $\check{B} = \{\omega^{(r_i+1)}_i : i \in [s]\}$. Then we may complete the set $B \cup \check{B}$ with a set $\hat{B} = \{\omega_i, i = s + 1, \ldots, \sigma\},$ where $\omega_i = d\theta_i - \theta_idt$ in a way that $B \cup \check{B} \cup \hat{B}$ is a basis of $I^{(k-2)} + J$ such that $B \cup \hat{B}$ is linearly independent mod $\{I^{(k-1)} + J\}$.

**Proof.** (of Theorem 8.210). Let $N \in \mathbb{N}$ be the smallest integer such that $I^{(k)} = I^{(k+1)} = 0$ for all $k \geq N$. Let $B_N$ be a basis for $J = I^{(N)} + J$ given by $B_N =$

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10 Most of the techniques that are necessary for the proof of our sufficient condition of relative flatness are very similar to the techniques of the proof of the main result of [38].
where $\eta = (dx_a - \hat{x}_a \, dt)$ and $\mu_l = (du_a - u_a \, dt), l \in \mathbb{N}$. Since $\text{span}\, \{I^{(N-1)}, J, dt\}$ is involutive and $I^{(N-1)}$ is nonsingular, by Prop. 8.3 with $\mathcal{B} = \emptyset$, we can construct a local basis $B_{N-1}$ of $I^{(N-1)}+J$ of the form $B_{N-1} = A_{N-1} \cup \mathcal{B}_N$ where

$$A_{N-1} = \{(\theta_{j_k, i_{l}} - \frac{\partial}{\partial \theta_{j_k, i_{l}}} dt), i_{l} \in [s_{N-1}]\}$.

Let $\hat{A}_{N-1} = \{(\theta_{j_k, i_{l}} - \frac{\partial}{\partial \theta_{j_k, i_{l}}} dt), i_{l} \in [s_{N-1}]\}$. By Prop. 8.3, we may construct a set $\hat{A}_{N-1} = \{(\theta_{j_k, i_{l}} - \frac{\partial}{\partial \theta_{j_k, i_{l}}} dt), i_{l} \in [s_{N-2}]\}$ in a way that $B_{N-2} = A_{N-2} \cup \mathcal{B}_N$ is a basis of $I^{(N-2)} + J$ where $A_{N-2} = \hat{A}_{N-1} \cup \hat{A}_{N-1} \cup A_{N-1} = \{(\theta_{j_k, i_{l}} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [2], i_{l} \in [s_{N-1}], j \in [2 + k + 1]\}$.

Note also that, by Prop. 8.3, it follows that the set $\hat{A}_{N-1} \cup \hat{A}_{N-1} = \{(\theta_{j_k, i_{l}} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [2], i_{l} \in [s_{N-1}]\}$ is linearly independent mod $I^{(N-1)} + J$.

Continuing in this way, using Prop. 8.3, we may construct in the $r$th step, a basis for $I^{(N-r)} + J$ of the form

$$B_{N-r} = A_{N-r} \cup \mathcal{B}_N$$

where $A_{N-r} = \hat{A}_{N-r+1} \cup \hat{A}_{N-r+1} \cup A_{N-r+1}$ and

$$A_{N-r+1} = \{(\theta_{j_k, i_{l}} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [r-1], i_{l} \in [s_{N-r}], j \in [r-1]\}
\hat{A}_{N-r+1} \cup \hat{A}_{N-r+1} = \{(\theta_{j_k, i_{l}} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [r], i_{l} \in [s_{N-r}]\}$$

and where $\hat{A}_{N-r+1} \cup \hat{A}_{N-r+1}$ is linearly independent mod $I^{(N-r+1)} + J$ for $r \in [N+1]$. From Prop. 8.3, note that the dimension $I^{(k)}(p) + J(p))/J(p) = \dim I^{(k)}(p)$, $k \in \mathbb{N}$.

Taking $r = N+1$ in (8.6) we obtain a basis $B_{N-1} = A_{N-1} \cup \mathcal{B}_N$ where $A_{N-1} = \hat{A}_0 \cup \hat{A}_0 \cup \hat{A}_0$ and

$$A_0 = \{(\theta_{j_k, i_{l}} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [N], i_{l} \in [s_{N-r}], j \in [N-r+1]\}
\hat{A}_0 \cup \hat{A}_0 = \{(\theta_{j_k, i_{l}} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [N+1], i_{l} \in [s_{N-r}]\}$$

where the set $\hat{A}_0 \cup \hat{A}_0$ is independent mod $I^{(0)} + J$. Since $\dim I^{(0)} = n_b$ and $\dim I^{(-1)} = n_b + m_b$, we have $\text{card} \hat{A}_0 \cup \hat{A}_0 = m_b$. Now define the set $w$ of $n_b$ (state) functions and the set $v$ of $m_b$ (input) functions given by

$$w = \{w_{j_k, i_{l}}^{(i_1)} | \theta_{j_k, i_{l}}^{(i_1)} = \theta_{j_k, i_{l}}^{(i_1)}, k \in [N+1], i_{l} \in [s_{N-r}]\}
\hat{w}_{j_k, i_{l}} = \theta_{j_k, i_{l}}^{(i_1)} = \theta_{j_k, i_{l}}^{(i_1)}, k \in [N+1], i_{l} \in [s_{N-r}]\}$$

By construction of $\mathcal{B}_0$ and $B_{N-1}$ it is clear that $I^{(0)} + J + \text{span}\{dt\} = \text{span}\{dt, dx_a, dw, dx_a, dx_b, dx_b, dx_b, dx_b\} + J$ and $I^{(-1)} + J + \text{span}\{dt\} = \text{span}\{dt, dx_a, dw, dx_a, dx_b, dx_b, dw, dw, dw\} + J$. Since card $x_a = 3$ and card $v_a$ = card $v$ then, by Prop. 4.2 we conclude that $((x_a, w), (u_a, v))$ is a state representation that is linked to $((x_a, x_b), (u_a, u_b))$ by relative static state feedback. Since $I^{(k)} \subseteq \text{span}\{\frac{d}{dt}\}$, the equations $((\theta_{j_k, i_{l}}^{(j_1)} - \theta_{j_k, i_{l}}^{(j_1)}), k \in [N], i_{l} \in [s_{N-r}], j \in [N-r+1])$ implies the following closed loop state equations:

$$\dot{t} = 1$$
$$\dot{x}_a = f_a(x_a, u_a)$$
$$\dot{w}_{j_k, i_{l}}^{(i_1)} = u_{j_k, i_{l}}^{(i_1)}$$
$$\dot{w}_{j_k, i_{l}}^{(i_2)} = \theta_{j_k, i_{l}}^{(i_1)}$$
$$\dot{w}_{j_k, i_{l}}^{(i_3)} = \theta_{j_k, i_{l}}^{(i_1)}$$
$$\vdots$$
$$\dot{w}_{j_k, i_{l}}^{(i_{N-k+1})} = v_{j_k, i_{l}}^{(i_1)}$$

(8.8)
Remark 8.3. Note that, if $s_{-1} > 0$, then the inputs $\{v_{k,i_k} \mid k = N+1, i_k \in [s_{-1}]\}$ are completely decoupled from the state of system (8.8), i.e., (8.8) is not well-formed in this case [48]. Note also that, if one restricts the coordinate transformations to the class of relative static-state feedback (see Definition 4.1) then the conditions of Theorem 8.2 are necessary and sufficient. This follows from the invariance of the relative derived flag with respect to relative static-state feedback (see Cor. 8.4) and after (tedious) calculations of the relative derived flag of a system of the form (8.8).

Corollary 8.4. Consider the system $S$ of equations (4.1a)-(4.1b). Let $x = (x, x, u)$ and $J$ be defined by equation (8.1). Let $\hat{I}^{-1} = \text{span} \{dx - xdt, du - udt\} + J$ and $\hat{F}^k(p) = \text{span} \{\omega(p) \mid \omega \in \hat{I}^{k-1}, \omega \equiv 0\}$ for $k \in \mathbb{N}$. Assume that the codistributions $\text{span} \{\hat{F}^k, dt\}$ are involutive, $\dim \hat{F}^k/J(q)$ is (locally) constant for $k \in \mathbb{N}$, and that $\hat{I}^{(N)} = J$ for $N$ big enough. Then the system $S$ is (locally) relatively flat w. r. t. $S_1$. Furthermore, the codistributions $\hat{F}^k$, $k \in \mathbb{N}$ are invariant by relative static-state feedback with respect to the subsystem defined by (4.1a).

Proof. We show first that this is true for $k = -1$. Assume that this is true for $k = 1$ and let $\hat{\omega} \in \hat{I}^{(k-1)}$. Then $\hat{\omega} = \omega + \mu$, where $\omega \in \hat{I}^{(k-1)}$ and $\mu \in J$. As $J$ is involutive, then $d\hat{\omega}$ mod $\hat{I}^{(k-1)} = 0$ if and only if $d\omega \text{ mod } (I^{(k-1)} + J) = 0$ in particular, $\hat{\omega} \in \hat{I}^{(k)}$ if and only if $\omega \in I^{(k)}$. It follows that $\hat{F}^k = I^{(k)} + J$, showing our claim. Hence, the first affirmation follows easily from Theorem 8.2. To show the invariance of the flag $\hat{F}^k$, let $(\hat{x}, \hat{u})$ be a state representation of $S$ that is linked to $(x, u)$ by relative static-state feedback. Let $\hat{I}^{-1} = \text{span} \{d\hat{x} - xdt, d\hat{u} - udt\} + J$. Since $J$ is span $\{dt\} = \Sigma = \pi^*(T^*S_u)$, by Def. 4.1 it follows $\hat{I}^{-1} = \hat{I}^{-1} + \text{span} \{dt\}$. Hence, $\hat{\omega} \in \hat{I}^{-1}$ if and only if $\hat{\omega} = \omega + \beta dt$, where $\omega \in \hat{I}^{-1}$. Now note that $\hat{I}^{-1}$ and $\hat{I}^{-1}$ are both contained in $\text{span} \{dx, dt\}$. In particular $\langle \omega, \frac{dt}{\omega} \rangle \in \hat{I}^{-1}$ if and only that $\beta = 0$. We conclude that $\hat{I}^{-1} = \hat{I}^{-1}$. Since the computation of $\hat{F}^k$ follows the same rule than the computation of $\hat{F}^k$ and $J$ is invariant by endogenous feedback (see Rem. 8.1), we conclude that $\hat{F}^k = \hat{F}^k$, $k \in \mathbb{N}$.

Remark 8.4. Let $U = \text{span} \{dx\} + H = J^\perp$. Let $G_0 = U \cap H$ and let $G_{k+1} = G_k + \{\frac{dt}{\omega}|G_k\}$. It can be shown [10] that the conditions of Theorem 8.2 for time-invariant systems are equivalent to the involutivity of the distributions $G_i$ and the existence of $k$ such that $G_i = H$ for all $i \geq k$.

8.1. Flatness and local output subsystems. Theorem 8.5 is a sufficient condition for relative flatness w. r. t. a local output subsystem.

Theorem 8.5. Let $S$ be the explicit system (1.1a) with state representation $(x, u)$ and output $y = h(t, x, u)$. Let $U$ be the the open and dense set where theorem 4.3 holds. Let $\hat{F}^0 = \text{span} \{dx - xdt\} + J$, where $J = \text{span} \{dy^{k-1} - y^{k}dt : k \in \mathbb{N}\}$. Consider the relative derived flag $\hat{F}^k(p) = \text{span} \{\omega(p) \mid \omega \in \hat{I}^{k-1}, \omega \equiv 0\}$ for $k \in \mathbb{N}$. Assume that, in $U$, the codistributions $\text{span} \{\hat{F}^k, dt\}$ are involutive, and that $\dim \hat{F}^k/J(q)$ is (locally) constant dimensional for $k \in \mathbb{N}$ and $\hat{I}^{(N)} = J$ for $N$ big enough. Then $S$ is (locally) relatively flat w.r.t. the output subsystem $Y$ around every $\xi \in U$.

Proof. By Thm. 4.3 there exists a local output subsystem $Y$ of $S$ and new state representation $((z, z), (v, v))$ linked to $(x, u)$ by a relative static-state feedback, such that the closed loop state equations are given by (4.3a)-(4.3b), where $\text{span} \{dt, dz, dv^k \mid k \in \mathbb{N}\} = \text{span} \{dt, dy^k \mid k \in \mathbb{N}\} = J + \text{span} \{dt\}$. Let $\hat{J} =$
span \( \{ (dx_a - \hat{z}_a), (dv_a^{(k)} - u^{(k+1)}_a) : k \in \mathbb{N} \} \). It follows easily that \( \hat{J} + \text{span} \{ dt \} = J + \text{span} \{ dt \} \). Using the fact that \( J \subset \text{span} \{ \frac{dJ}{dt} \} \) and \( \hat{J} \subset \text{span} \{ \frac{d\hat{J}}{dt} \} \), (see the arguments of the proof of Cor. 8.4), it follows that \( \hat{J} = J \). Then the result follows from Cor. 8.4.

9. Examples. We begin with an counterexample of the necessity of the condition of Prop. 5.2 for flatness of a system \( S \).

9.1. A counterexample. Consider the system \( S \) defined by

\begin{align*}
(9.1a) & \quad \dot{x}_1 = u_1 \\
(9.1b) & \quad \dot{x}_2 = x_1 x_2^2 + 1 \\
(9.1c) & \quad \dot{x}_3 = u_2
\end{align*}

Then this system is flat with flat output \( y_1 = x_3 \) and \( y_2 = x_2 \).

Consider the subsystem \( S_1 \) defined by (9.1a). Then this system is not relatively flat with respect to \( S_1 \). An indirect way to see this is by noting that the “implicit” system that we get by making \( y = x_1 = 0 \) and \( y^{(k)} = u^{(k-1)}_1 = 0, k \in \mathbb{N} \) is given by

\begin{align*}
\dot{x}_2 &= 0 \\
\dot{x}_3 &= u_2
\end{align*}

It is easy to show that this system is not flat because it is not controllable \[20\].

Note that the explicit system \( S \) is time-invariant and the codistributions \( Y_k = \text{span} \{ dt, dy, \ldots, dy^{(k)} \} \) and \( \bar{Y}_k = \text{span} \{ dt, dx, dy, \ldots, dy^{(k)} \} \) are nonsingular for \( k \in \mathbb{N} \). In particular the assumptions A1 and A2 of \S 6 are satisfied. Notice also that \( S_1 \) is an output subsystem for system \( S \). If \( S \) were relatively flat with respect to \( S_1 \), then Thm. 7.2 would imply that the implicit system is locally flat.

9.2. A second counterexample. The following example shows that the conditions of Thm. 7.2 are sufficient conditions for flatness of implicit systems, but they are not necessary conditions.

Consider the explicit system \( S \) with output \( y \) defined by:

\begin{align*}
(9.2) & \quad \dot{x}_1 = u_1 \\
& \quad \dot{x}_2 = x_3 x_1 + x_3 \\
& \quad \dot{x}_3 = x_4 \\
& \quad \dot{x}_4 = u_2 \\
& \quad y = x_1 - \epsilon
\end{align*}

It is easy to see that the “implicit” system (that is already explicit in this case) obtained by making \( y^{(k)} = 0, k \in \mathbb{N} \) is given by

\begin{align*}
(9.3) & \quad \dot{x}_2 = \epsilon x_3^2 + x_3 \\
& \quad \dot{x}_3 = x_4 \\
& \quad \dot{x}_4 = u_2
\end{align*}

System (9.3) is linearizable by static state feedback if and only if \( \epsilon = 0 \). In other words the implicit system obtained by making \( y^{(k)} = 0, k \in \mathbb{N} \) is flat if \( \epsilon = 0 \) (but is not flat if \( \epsilon \neq 0 \) because it is not linearizable by static-state feedback in this case).

However the explicit system is not relatively flat with respect to the subsystem defined by the first equation of (9.2). An indirect proof of this fact can be given
by noting that, if system (9.2) were relatively flat with respect to the subsystem $S_1$ defined by the first equation, then by Thm. 7.2, taking any $\epsilon \in \mathbb{R}$, the implicit system would be locally flat (see the arguments of the end of the last example). However the implicit system is flat only for $\epsilon = 0$.

9.3. An academic example. Consider the implicit system

\begin{align}
(9.4a) \quad \dot{x}_1 &= \frac{x_2^2}{(1 + x_2^2)^2} + e^{x_3}u_1, \quad \dot{x}_2 = (1 + x_2^2)u_1 + \frac{2x_2x_3}{(1 + x_2^2)^2}u_2, \quad \dot{x}_3 = u_2 \\
(9.4b) \quad y &= x_1 = 0
\end{align}

Let $S$ be the (explicit) system (9.4a) with output $y = x_1$. It is easy to verify that the codistributions $\mathcal{Y}_k = \text{span}\{dt, dy, \ldots, dy^{(k)}\}$ and $\mathcal{Z}_k = \text{span}\{dt, dx, dy, \ldots, dy^{(k)}\}$ of Lemma C.1 are non-singular everywhere for $k \in \mathbb{N}$, and $\sigma_k = 1, k \geq 1$. Note also that $\Gamma = \{ \xi \in S \mid y^{(k)}(\xi) = 0\}$ is nonempty because $\Gamma$ contains the point $\xi \in S$ defined by $x_1(\xi) = x_2(\xi) = x_3(\xi) = u_1^{(k)}(\xi) = u_2^{(k)}(\xi) = 0, k \in \mathbb{N}$ (for any $t$). Since the system is time-invariant then the assumptions A1 and A2 of § 6 are satisfied. By Prop. 6.1, the implicit system is a immersed system in the non-constrained system. Let $J = \text{span}\{dy^{(k)} = y^{(k+1)}dt : k \in \mathbb{N}\}$ and $\tilde{\Gamma}(0) = \text{span}\{dx - \dot{x}dt\} + J$. Using condition (B.4), some calculations show that$^{20} \tilde{\Gamma}(1) = \text{span}\{\eta - \langle \eta, \frac{\partial}{\partial t}\rangle dt\} + J$, where $\eta = dx_2 - \frac{2x_2x_3}{1 + x_2^2}dx_3$, and $\tilde{\Gamma}(2) = J$. Since $d\eta = \frac{2x_2}{1 + x_2^2}(\eta \wedge dz_3)$. From Theorem 8.5, for every local output subsystem $Y$, the explicit system $S$ is relatively flat w. r. t. $Y$. By Theorem 7.2, the implicit system $\Gamma$ defined by (9.4a)-(9.4b) is locally flat around every point $\xi \in \Gamma$. By the proof of theorem 8.2 and the construction of $\Gamma$ in § 6, a flat output of the implicit system can be constructed by finding a function $\psi$ such that $d\psi = a\eta$. A possible solution is $\psi = \frac{x_2}{1 + x_2^2}$. By Props. 4.5 and 5.3, one may completely the output $y$ into a flat output $(y, z)$ for system $S$. In this case one may take $z = \psi$.

9.4. Constrained robots. Constrained robots are robots whose movement is restricted by some physical contact surfaces. Such restrictions can be represented by adding $r$ holonomic constraints $\phi_i(q) = 0 \ (i = 1, \ldots, r)$ to its original equations.

The following model can be obtained by taking into account the contact forces

\begin{align}
(9.5a) \quad M(q)\ddot{q} + H(q, \dot{q}) &= (J\phi)^T(q)\lambda + \tau \\
(9.5b) \quad \phi_i(q) &= 0 \ (i = 1, \ldots, r)
\end{align}

where $q \in \mathbb{R}^n, J\phi(q) = \partial\phi/\partial q, \lambda = (\lambda_1, \ldots, \lambda_r)^T$ is a vector corresponding to the contact forces, $M(q)$ is the symmetric positive definite mass matrix, and $H(q, \dot{q})$ corresponds to Coriolis and gravity forces. We will assume that $\partial\phi/\partial q$ has rank $r$ for all $q$ in the operation region of the robot. A representation of the system (9.5a)-(9.5b) in the form (1.1a)-(1.1b) is given by

\begin{align}
(9.6a) \quad \frac{d}{dt} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} &= \begin{pmatrix} \dot{q} \\ -M^{-1}H \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ M^{-1}(J\phi)^T & M^{-1} \end{pmatrix} \begin{pmatrix} \lambda \\ \tau \end{pmatrix} \\
(9.6b) \quad 0 &= \phi_i(q), \ i = 1, \ldots, r
\end{align}

$^{20}$The application of part (ii) of Prop. 8.3 is the easiest way for computing relative derived flags, and lead to linear equations with coefficients that are functions defined on $S$ as shown in the proof of Prop. 8.3.
Let \( \psi = (\psi_1, \ldots, \psi_{n-m}) \) be chosen in a way that map \( q \mapsto (\phi, \psi) \) is a local diffeomorphism. Considering only the explicit system \( S \) defined by (9.6a), it is easy to show that \((q,\lambda)\) is a flat output for \( S \). In particular \((\phi,\psi,\lambda)\) is also a flat output for \( S \). From Cor. 7.3, it follows that \((\psi,\lambda)\) is a flat output for the constrained robot. Note now that \( \psi \) are local coordinates of the constraint surface. In particular, the simultaneous tracking of the position along the constraint surface and the contact forces are possible. The reader may refer to [37, 40] for details and the presentation of the design of a flatness based control, including the underactuated case. Another approach for the solution of this problem is considered for instance in [29].

10. Conclusions. In this paper we state the notion of (local) subsystem. The definition and construction of (local) output subsystems are also presented (Thm. 4.3). It is shown that subsystems admit (generically) adapted state equations (Prop. 4.4).

We show, under regularity assumptions, that an implicit system (1.1a)–(1.1b) defines a system \( \Gamma \) (in the sense of §3.2) that admits state space representations and is immersed in the (explicit) system \( S \) defined by (1.1a) (Prop. 6.1). This immersion is in fact an embedding since the topology of the immersed system is the subset topology. This result may be regarded as a generalization of the fact that equations \( f(x) = 0 \), where \( f : \mathbb{R}^n \to \mathbb{R}^p \), defines implicitly an embedded submanifold of \( \mathbb{R}^n \) when the Jacobian matrix \( Jf(x) \) has constant rank in the solutions of \( f(x) = 0 \).

The concept of relative flatness w. r. t. a subsystem is introduced (see §5). We show that a system \( S \) is relative flat with respect to a flat subsystem if and only if the flat output of the subsystem may be completed into a flat output of the system (see Prop. 5.3).

Given a system \( S \) with output \( y \), we show that relative flatness w. r. t. a (local) output subsystem implies (local) flatness of the implicit system \( \Gamma \) obtained by restricting \( S \) to the constraint \( y = 0 \) (Thm. 7.2).

Sufficient conditions for relative flatness of a system w. r. t. a subsystem are developed (see Thm. 8.2 and Cor. 8.4). These conditions restricts the class of transformations to relative static-state feedbacks (Def. 4.1).

Sufficient conditions of relative flatness with respect to an output subsystem are obtained (see Thm. 8.5). This result can be combined with Thm. 7.2 in order to study flatness of implicit systems (1.1a)–(1.1b), as illustrated in the example of §9.3.

Although it is assumed that system (1.1a)–(1.1b) is analytic, this hypothesis is only needed to assure that the output rank \( \rho \) of the explicit system (1.1a) with output \( y = h(x, u, t) \) is a global invariant, at least in the subset \( U \subset S \) of nonsingular points of the codistributions \( Y_k \) and \( Y_k^* \) for \( k = 0, \ldots, n \) (see Lemma C.1). Note that the differential dimension\(^{21} \) of the implicit system \( \Gamma \) defined by (1.1a)–(1.1b) is \( \tilde{m} = m - \rho \), where \( m = \text{card} \, u \). Hence the assumption of analyticity implies that \( \tilde{m} \) is an invariant. All the results of this paper could be rewritten in the smooth case (see [39, Lemma 6.2] for a smooth version of Lemma C.1), but in this case the differential dimension of \( \Gamma \) may depend on the working point. In the same way, it is easy to restrict our results to the time-invariant case (see [39, Lemma 8.1]).

All the definitions and results of this paper are local (note that the time-varying notions are also local in time). The only exception are the construction of the system \( \Gamma \) in §6 and Prop. 6.1, that is a “global” construction.

\(^{21}\)The local differential dimension is the cardinal of the input of a local state representation. Note that a differential dimension \( \tilde{m} \) of a connected smooth system that admits a local state representation around every point is a global invariant [18], [39, Cor. 7.2].
Appendix A. A differential algebraic interpretation of relative flatness.
The next proposition is a precise statement of the affirmation made in Rem. 5.1. The proof may be found in [58].

**Proposition A.1.** [58] Let $K/k$ be a system. Let $L/k$ be a subsystem of $K$, i.e., $k \subseteq L \subseteq K$. Then $K/L$ is a flat system if and only if there exist a flat system $F/k$ such that $F$ is algebraic over $K$, $K$ is algebraic over $k(L,F)$, and $L$ and $F$ are algebraically disjoint over $k$. In other words, $M = k(F,K)$ is decomposed into two independent subsystems $L$ and $F$.

Appendix B. Proof of Auxiliary Results.

**B.1. Proof of Prop. 4.2.** By definition, (i) implies (ii). To show that (ii) implies (i) it suffices to show that \( \{t, z, (u^{(k)} : k \in \mathbb{N})\} \) is a local coordinate system. Consider the state representation \( (\hat{x}, \hat{u}) \) where \( \hat{x} = (x_b, x_a, u_q, \ldots, u_r^{(r^0)}) \) and \( \hat{u} = (u_b, u_r^{(r^1)}) \) for some $r$ to be determined. It will be shown that \( \hat{z} = (z_b, x_a, u_q, \ldots, u_r^{(r^1)}) \) are such that their differentials are linearly independent, that span \( \{dt, dx_b\} \) = span \( \{dt, dz_b\} \), and that span \( \{dt, dz_b, du\} \) = span \( \{dt, dz_b, du^{(j)} : j \in \mathbb{N}\} \). From the particular form of $x, z, u$ and $v$, it is clear that (ii) is equivalent to span \( \{dx_b\} + \Sigma_r = \{dz_b\} + \Sigma_r \) and span \( \{dx_b, du\} + \Sigma_r \) = span \( \{dz_b, du^{(j)} : j \in \{r\}\} \), for any $r \geq k^*$. One shows similarly, possibly by taking a bigger $r$, that span \( \{dt, dx_b, du\} + \Sigma_{r+1} = \{dz_b, du^{(j)} + \Sigma_{r+1} \). By § 3.7 it follows that \( (\hat{z}, \hat{v}) \) and \( (\hat{x}, \hat{u}) \) are linked by static-state feedback. \( \square \)

**B.2. Proof of Thm. 4.3.** In this proof we use the results and the notations of Lemma C.1. Let $n = \dim x$. By that Lemma, around $\xi \in U$, there exists a local state representation \( (x_a, u_a) \) defined in $V_\xi$ such that

\[
\begin{align}
\text{(B.1a)} & \quad \text{span} \{dt, dx_a\} = \text{span} \{dt, dx, dy, \ldots, dy^{(n)}\} \\
\text{(B.1b)} & \quad \text{span} \{dt, dx_a, du_a\} = \text{span} \{dt, dx, du, dy, \ldots, dy^{(n+1)}\}
\end{align}
\]

and where $u_a = (\hat{y}_a^{(n+1)}, \hat{u}_a)$. Now choose a subset $z_a$ of $\{y, \ldots, y^{(n)}\}$ in a way that $\{dt, dz_a\}$ is a local basis of span $\{dt, dy, \ldots, dy^{(n)}\}$ and choose $z_b$ in a way that $\{dt, dx_a, dz_b\}$ is a local basis of span $\{dt, dx_a, dy, \ldots, dy^{(n)}\}$ around $\xi$. Let $u_a = \hat{y}_a^{(n+1)}$ and $u_b = \hat{u}_a$. By construction, \( ((z_a, z_b), (u_a, u_b)) \) is a local state-representation of $S$ around $\xi$, since it is linked to \( (x_a, u_a) \) by local static-state feedback (see (3.4)).

By Lemma C.1 part 8, it follows that span $\{dz_b\} \subseteq \{t, z_a, u_a\}$ and that (i) and (ii) holds. Now note that (iii) follows easily from Def. 4.1 and conditions (B.1).

To show that two output subsystems are Lie-Bäcklund isomorphic, let $\pi_i : V_\xi^i \rightarrow Y_i$ be local output subsystems for $i = 1, 2$. Assume that $V_\xi^1 \cap V_\xi^2 \neq \emptyset$. We will show that there exist a local Lie-Bäcklund isomorphism $\delta : W_1 \rightarrow W_2$ where $H$ is some open neighbourhood of $\xi$ for which $H \subseteq V_\xi^1 \cap V_\xi^2$ and $W_i = \pi_i(H), i = 1, 2$.

Since the $\pi_i$ are Lie-Bäcklund submersions for $i = 1, 2$, there exists local charts of $\phi_i = (t, X_i, Z_i), i = 1, 2$, defined in some $H \subseteq S$ and local charts $\psi_i = (t, X_i)$, of $Y_i, i = 1, 2$, defined on $W_i = \pi_i(H)$ such that, in these coordinates $\phi_i \circ \pi_i^{-1} \circ
$\psi_i(t, X_i, Z_i) = (t, X_i)$, $i = 1, 2$. Since $Y_1$ and $Y_2$ are both local subsystems we have \( \text{span} \{dt, dX_i\} = \text{span} \{dt, dy^{(j)} : k \in [N] \} \), for $i = 1, 2$. In particular, it follows that the local coordinate change $(t, X_1, Z_1) = \varphi_1 \circ \varphi_2^{-1}(t, X_2, Z_2)$ is such that $X_1 = \theta(t, X_2)$ and $X_2 = \theta(t, X_1)$. So the map $\mu$ defined by $(t, X_2) \mapsto (t, \theta(t, X_2))$ is a local diffeomorphism. Let $\delta : W_2 \subset Y_2 \to W_1 \subset Y_1$ be the local diffeomorphism defined by $\delta = \psi^{-1}_1 \circ \mu \circ \psi_2$. To complete the proof it suffices to show that $\delta$ is Lie-Bäcklund. For this, we show first that $\delta \circ \pi_2|_H = \pi_1|_H$. In fact, note that

$$
\psi_1 \circ (\delta \circ \pi_2) \circ \phi_1^{-1}(t, X_1, Z_1) = \psi_1 \circ (\delta \circ \pi_2 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})(t, X_1, Z_1) = \\
(\psi_1 \circ \delta) \circ \pi_2 \circ \phi_2^{-1}(t, X_2, Z_2) = (\mu \circ \psi_2) \circ \pi_2 \circ \phi_2^{-1}(t, X_2, Z_2) = \\
\mu(t, X_2) = (t, X_1) = \psi_1 \circ \pi_1 \circ \phi_1^{-1}(t, X_1, Z_1)
$$

From the first and the last terms above, we have that $\delta \circ \pi_2|_H = \pi_1|_H$. Denote by $\partial_i$ the Cartan fields respectively of $Y_i$, for $i = 1, 2$. By definition $\pi_i^* \frac{d}{dt} = \partial_i \circ \pi_i$. In particular $\partial_1 \circ \delta \circ \pi_2 = \partial_1 \circ \pi_1 = (\pi_1)^* \frac{d}{dt} = (\delta \circ \pi_2)^* \frac{d}{dt} = \delta_*(\pi_2)^* \frac{d}{dt} = \delta_* \partial_2 \circ \pi_2$. As $\pi_2$ is surjective it follows that $\partial_1 \circ \delta = \delta_* \partial_2$, showing that $\delta$ is Lie-Bäcklund.

B.3. Proof of Prop. 4.5. Let $\rho = p$ be the output rank and let $\{\sigma_1, \ldots, \sigma_n\}$ be the structure at infinity of the system. By Lemma C.1, the sequence $\sigma_i$ is nonincreasing. Since $\sigma_i$ converges to $\rho$, we must have $\sigma_i = \rho, i \in [n]$. Thus, the differentials $\{dt, dy^{(j)} : j \in \mathbb{N}\}$ must be independent in $U$. In particular, in the proof of Theorem 4.3, a possible choice of $z_a$ and $u_a$ can be $z_a = (y, \ldots, y^{(m)})$ and $u_a = y^{(m+1)}$. It is then clear that the subsystem $Y$ (locally) flat with flat output $y$.

B.4. Proof of equation (8.2). From (B.5) it follows that $\omega \in \mathcal{I}(0)$ if and only if $\omega \in \mathcal{I}(-1)$ and $\hat{\omega} \in \mathcal{I}(-1) + J + \text{span} \{dt\} = \text{span} \{dx, du, dx, (du)^{(j)} : j \in \mathbb{N}, dt\}$. Let $\omega \in \mathcal{I}(-1)$. Then $\omega = \sum_i \alpha_i(dx_{h_i} - \hat{x}_{h_i} dt) + \sum_j \beta_j(du_{b_j} - \hat{u}_{b_j} dt)$ for convenient functions $\alpha_i, \beta_j$ defined on $S$. Hence $\omega = \sum_i \alpha_i(dx_{h_i} - \hat{x}_{h_i} dt) + \sum_j \beta_j(du_{b_j} - \hat{u}_{b_j} dt)$ and $\omega = \sum_i \gamma_i(dx_{a_i} + \sum_j \delta_j du_{a_j} + \sum_k \epsilon_k dx_{b_k} + \theta dt) + \sum_l \beta_l du_{b_l}$ where $\gamma_i, \delta_j, \epsilon_k, \beta_l$ and $\theta$ are convenient functions. Notice that $\{t, x, (u)^{(k)} : k \in \mathbb{N}\}$ is a local chart, where $x = (x, u, v)$ and $u = (u, u, u)$. Hence $\omega \in \mathcal{I}(0)$ if and only if $\beta_j = 0$ for all $j \in [m]$.

B.5. Proof of Proposition 8.3. In order to prove Prop. 8.3 we need the following lemmas:

**Lemma B.1.** For all integers $k \geq 0$, $r \geq 0$ and for every point $p \in S$, we have:

(i) $\left( \mathcal{I}^{(k)} + J_r + \text{span} \{dt\} \right)|_p \cap \text{span} \left\{ \frac{d}{dt} \right\} \subset \mathcal{I}^{(k)}(p) + J_r$. The same result also holds when replacing $J_r$ by $J_r$.

(ii) $\text{span} \left\{ \mathcal{I}^{(k)}(p) \cap J(p) \right\} \cap \text{span} \{dt\}(p)$

**Proof.** (i) Assume that $\omega(p) \in \left( \mathcal{I}^{(k)} + J_r + \text{span} \{dt\} \right)|_p \cap \text{span} \left\{ \frac{d}{dt} \right\}$. Then $\omega(p) = \varpi(p) + \beta dt$ where $\varpi \in \mathcal{I}^{(k)} + J_r$ and $\beta \in C^\infty(S)$. Then $\langle \omega, \frac{d}{dt} \rangle|_p = \langle \varpi, \frac{d}{dt} \rangle|_p + \beta \langle dt, \frac{d}{dt} \rangle|_p$. Since $\langle dt, \frac{d}{dt} \rangle = 1$ and $\mathcal{I}^{(k)} + J \subset \text{span} \left\{ \frac{d}{dt} \right\}$, it follows that $\beta(p) = 0$ and hence $\omega(p) = \varpi(p) \in \mathcal{I}^{(k)}(p) + J_r(p)$.

(ii) Let $\omega(p) \in \left\{ \left( \mathcal{I}^{(k)} + J \right) \cap \text{span} \{dt\} \right\}|_p$. We have $\omega(p) = \beta(p) dt$. So $\langle \omega, \frac{d}{dt} \rangle|_p = \beta(p) = 0$ and so $\beta(p) = 0$.

---

22 We stress that we are not using the Inverse Function Theorem, but only the existence of the inverse of the coordinate change map.
Let $\omega(p) \in \{(I^k + \text{span} \{dt\}) \cap J\}_p$. We have $\omega(p) = \omega(p) + \beta(p) dt$, where $\omega(p) \in I^k \subset \text{span} \{\frac{d\theta}{\theta}\}$. Since $J \subset \text{span} \{\frac{d\theta}{\theta}\}$, then $\langle \omega(p), \frac{d\theta}{\theta}\rangle = \langle \omega(p), \frac{d\theta}{\theta}\rangle + \beta(p) dt$, $\frac{d\theta}{\theta} = 0$. Hence $\omega(p) \in I^k$. Since $I^k \cap J = 0$, it follows that $\omega(p) = 0$.

Let $\omega(p) \in \{(J + \text{span} \{dt\}) \cap I^k\}_p$. Using the same reasoning above, one verify easily that $\omega(p) = 0$. □

**Lemma B.2.** Assume the conditions of the Theorem 8.2 are satisfied on an open neighbourhood $V_\xi$ of $\xi$ in $S$. Then $I^k, k \in \mathbb{N}$ is a smooth codistribution and for every $p \in V_\xi$ and $k \in \mathbb{N}$ we have:

(i) For all $k \in \mathbb{N}$ there exists a set of covector fields $\Omega = \{\omega_1, \ldots, \omega_r\} \subset I^k + J$, where $r_k = \dim I^k$, $\omega_i = (d\theta_i - \theta_i dt)$, with $\theta_i \in C^\infty(S)$, and an open neighbourhood $V$ of $\xi$ such that the canonical projections of the elements of $\Omega(\nu)$ form a basis for $(I^k(\nu) + J(\nu)) \mod J(\nu)$ for all $\nu \in V$.

(ii) If $\omega$ is of the form $\left(\frac{d\theta - \theta dt}{\theta}\right)$ for a function $\theta \in C^\infty(S)$, then $\omega \in I^k + J$ if and only if $\omega \in I^k + J$. In particular $I^k + J \supset I^k + J + I^k(k+1) + J$.

(iii) Let $\{\omega_1, \ldots, \omega_r\} \subset I^k + J$ be a set of 1-forms such that $\omega_i = (d\theta_i - \theta_i dt)$, with $\theta_i \in C^\infty(S)$. Assume that the set $\{\omega_1(p), \ldots, \omega_r(p)\}$ is linearly independent\(^{23}\) mod $I^k(p) + J(p)$. Then $\{\omega_1(p), \ldots, \omega_r(p)\} \subset (I^k(p) + J(p))$ is linearly independent mod $\{I^k(p) + J + \text{span} \{dt\}\}_p$.

**Proof.** Assume by induction that $I^{(j)}, j = -1, 0, \ldots, k$ is smooth. We will show first that (i) and (ii) holds. (i) We show now that, for an integer $I^k$ big enough then span $\{I^k, J_1, dt\}$ is involutive (see eq. (8.3)). In fact, since $I^k$ is nonsingular and finite dimensional, there exist a local basis $\left\{\omega_i : i \in [r_k]\right\}$ of $I^k$. By part (ii) of Lemma B.1, it follows that the set $\left\{(\omega_i : i \in [r_k]), dx_a, du_a, \ldots, du_a^{(l_k)}(k), dt\right\}$ is a local basis of $I^k + J_1 + \text{span} \{dt\}$. Since the codistribution span $\{I^k, J_1, dt\}$ is involutive, then $d\omega_i = \sum_j \eta_{ij} \wedge \nu_{ij}$ for convenient 1-forms $\eta_{ij}, \nu_{ij}$ with $\nu_{ij} \in \text{span} \{I^k, J_1, dt\}$. Hence $\nu_{ij} \in \text{span} \left\{(\omega_i : i \in [r_k]), dx_a, du_a, \ldots, du_a^{(s_j)}, dt\right\}$. Let $l^*_k = \max_{i,j} \{s_j\}$.

Then span $\{I^k, J_1, dt\}$ is involutive for every $l^*_k \geq l_k$. By the Frobenius theorem and part (ii) of Lemma B.1, we see that span $\{I^k, J_1, dt\}$ is spanned by linearly independent 1-forms $\{dt_1, \ldots, dt_r, dx_a, du_a, \ldots, du_a^{(l_k)}, dt\}$, where $\dim I^k = r_k$. Now note that $\omega_i = (d\theta_i - \theta_i dt) \in \text{span} \{\frac{d\theta}{\theta}\}$. Since $\omega_i \in \text{span} \{I^k, J_1, dt\}$, by Lemma B.1 part (i) it follows that $\omega_i \in I^k + J_1$. Let $K = \text{span} \{J_1, dt\}_p$ and $L = \text{span} \{J, dt\}_p$.

By the construction of the canonical projection of the set $\{\omega_i : i \in [r_k]\}$ on $(I^k(p) + K)/K$ form a basis of $(I^k(p) + K)/K$. By part (ii) of Lemma B.1, it is easy to see that the map $\Psi : (I^k(p) + K)/K \to (I^k(p) + L)/L$ such that $\omega(p) \mod K \mapsto \omega(p) \mod L$ is an isomorphism. In particular the canonical projections of the $\omega_i$ on $(I^k(p) + L)/L$ also form a basis.

(ii) We show first that we have

$$\omega(p) \mod (I^k(p) + J(p)) = \omega(p) \mod (I^k(p) + J(p))$$

for all $p \in S$. For, by (i), note that $\omega = \sum_{i=1}^{r^*} \alpha_i \omega_i + \beta \eta + \sum_{j=0}^{l_k} \gamma_j \mu_j$ for $\omega_i = d\theta_i - \theta_i dt$, $\eta = (dx_a - \dot{x}_a dt)$, and $\mu_j = (du_a^{(j)}(p) - u_a^{(j+1)}(p) dt)$ for convenient smooth functions $\alpha_i, \beta$ and $\gamma_j$. Hence, $\omega \mod (I^k + J) = \left\{\sum_{i=1}^{r^*} \alpha_i d\omega_i + d\alpha_i \wedge \omega_i + (\beta dt + d\beta \wedge \eta) + \right.$

\(^{23}\)The linear independence of the set $\{\omega_i(p), \ldots, \omega_r(p)\}$ mod $(I^k(p) + J)$ for some $p \in S$ means that $\{\sum_{i=1}^{r^*} \alpha_i \omega_i(p) + \omega_r(p)\}_p = 0$ for $\omega \in I^k + J$ and $\alpha_i \in \mathbb{R}$ implies that $\omega(p) = 0$ and $\alpha_i = 0$.}
\[
\sum_{j=1}^{l_k} (\gamma_j d\mu_j + d\gamma_j \wedge \mu_j) \] \mod \(I(k) + J\). Since \(d(\theta dt - \dot{\theta} dt) = -d\theta \wedge dt\) for all \(\theta \in C^\infty(S)\), and \(\sum_{a} a \delta \omega_i + d\beta \wedge \eta + \sum_{j} d\gamma_j \eta_j \) \mod \(I(k) + J\) \equiv 0, we see that

\[
(B.3) \quad d\omega \mod (I(k) + J) = -\sum_{i} \alpha_i d\theta_i \wedge dt - \beta d\bar{x}_a \wedge dt - \sum_{j} \gamma_j d\bar{x}_a \wedge dt
\]

Now note that \(\omega \wedge dt = \sum_{i} \alpha_i \omega_i d\theta_i + (\beta \eta + \dot{\beta} \eta) + \sum_{j} \gamma_j \mu_j \wedge dt\). Since \(\{\sum_{i} \alpha_i \omega_i d\theta_i + (\beta \eta + \dot{\beta} \eta) + \sum_{j} \gamma_j \mu_j \wedge dt\} \mod (I(k) + J) \equiv 0\), it follows by direct computation that \(\omega \wedge dt\) is also given by the right hand side of equation \((B.3)\) and so \((B.2)\) holds.

Now we shall show that, for all \(p \in S\) and \(\omega \in I(k)\), we have

\[
(B.4) \quad \omega(p) \in I^{k+1}(p) \Leftrightarrow \omega(p) \in \text{span} \{I(k), J, dt\} (p)
\]

Let \(\{dt, (\omega_i : i \in [r_k]), \eta, (\mu_j : j \in [l_k])\}\) be a basis for span \(\{I(k), J, dt\}\). Notice that \(\omega \wedge dt\) \mod \(I(k) + J) \equiv 0\) means that \(\omega \wedge dt\) \equiv 0 \mod \(I(k) + J) \equiv 0\) for convenient 1-forms \(\zeta_i, \xi, \eta_j\). From the Cartan Lemma (see section 2), we conclude that \(\omega(p) \in \text{span} \{I(k), J, dt\} (p)\). Then, \((B.4)\) follows from \((B.2)\) and the Definition 8.1. It is easy to show that the same arguments and the fact that \(J\) is involutive imply that

\[
(B.5) \quad \omega(p) \in I^{k+1}(p) + J(p) \Leftrightarrow \omega(p) \in \text{span} \{I(k), J, dt\} (p)
\]

If \(\omega = d\theta - \dot{\theta} dt\) then \(\omega \in \text{span} \{\frac{d}{dt}\}\). By \((B.5)\) and from Lemma B.1 part (i), it follows that \(\omega \in I(k) + J\). Now note that, by \((i)\), \((I(k) + J)\) has a basis for this particular form. This completes the proof of \((ii)\). We show now that our induction hypothesis \((i.e., \text{that } I(j)\text{ is smooth for } j = -1, 0, \ldots, k)\) implies that \(I(k+1)\) is smooth. In fact, by the proof of \((i)\), given a local basis \(\{\omega_i : i \in [r_k]\}\) of \(I(k)\), there exists a local basis \(\{d\bar{x}_k, (\bar{x}_a : k \in \mathbb{N}), dt\}\) of \(W_k = \text{span} \{I(k), J, dt\}\). Notice that \(W_k \subset W_0 = \text{span} \{dx_k, J, dt\}\). In particular we have \(\omega_i = \hat{\omega}_i + \gamma_i\), where \(\omega_i \in \text{span} \{dx_k\}\) and we may replace \(\hat{\omega}_i\) by \(\omega_i\) in the basis of \(W_k\), obtaining another basis of \(W_k\). Note also that there exist a subset \(x_k\) of \(x_k\) such that \(\{dx_k, (\hat{\omega}_i : i \in [r_k]), \eta, (\mu_j : k \in \mathbb{N}, dt)\}\) is a basis of \(W_{-1}\). Let \(z = (\bar{x}_k, x_k)\). Let \(\omega_i = \hat{\omega}_i + \gamma_i = \sum_{j} \gamma_j \bar{x}_k + \mu_j\), where \(\mu_j \in \text{span} \{J, dt\}\). Denote the matrix formed by the functions \(\alpha_j\) by \(A\). By \((B.4)\), \(\alpha(p) = \sum_{j} \alpha_j \gamma_j \in I^{k+1}(p)\) if and only if \(\sum_{j} (\gamma_j \hat{\omega}_i + \gamma_i \hat{\omega}_j) \in \text{span} \{J, dt\}\). Denoting the column vector with components \(\alpha_i\), then \(\alpha(p) \in I^{k+1}(p)\) if and only if \(A(p) \alpha(p) = 0\). Then if \(I^{k+1}\) is nonsingular if and only if \(A(p)\) is (locally) constant rank and in this case it is clear that \(I(k)\) is smooth\(^{24}\).

\textbf{(iii)} To prove \((iii)\), assume that there exists \(\omega \in I^{k+1}(p) + J\) and functions \(\alpha_i \in C^\infty(S)\) such that, for every \(p \in S\) then \(\omega + \alpha_i \omega_i + \sum_{i=1}^{r} \alpha_i \omega_i \in I^{k+1}(p) + J(p)\). Hence, \(|\omega - \sum_{i=1}^{r} \alpha_i \omega_i + \sum_{i=1}^{r} \alpha_i \omega_i|^p = 0\). Since \(|\sum_{i=1}^{r} \alpha_i \omega_i|^p \in \text{span} \{I^{k+1}(p) + J(p)\}\), it follows that \(\sum_{i=1}^{r} \alpha_i \omega_i\) \in \(I^{k+1}(p) + J(p)\). It follows from \((B.5)\) that \(\sum_{i=1}^{r} \alpha_i \omega_i\) \in \(I^{k+1}(p)\) and hence the set \(\{\omega_1, \ldots, \omega_r\}\) is not linearly independent \(\text{mod } I^{k+1}(p) + J(p)\) in \(p \in S\). \(\square\)

\(^{24}\)This proof shows also that one may compute the relative derived flag by solving linear equations.
Proof (of Prop. 8.3). Since \( \omega_t = d\theta_t - \hat{\theta} dt, i \in [s] \), by part (ii) of Lemma B.1, the set \( B = \{ (d\theta^j i, j \in \{0, \ldots, r_i\}, i \in [s]) \}, dx_a, du_a, \ldots, du_{(l_{i-1})}, dt \} \) is a local basis of \( I^{(k-1)} + J_{k-2} + \text{span} \{ dt \} \), for any \( l_{k-1} > l_{k-2} \) for some \( l_{k-2} \).

By part (iii) of Lemma B.2, \( \hat{B} \) is linearly independent mod \( \{ I^{(k-1)} + J + \text{span} \{ dt \} \} \). Hence \( B \cup \hat{B} = \{ (d\theta^j i, j \in \{0, \ldots, r_i\}, i \in [s]) \}, dx_a, du_a, \ldots, du_{(l_{i-1})}, dt \} \) is linearly independent for every \( l_{k-1} \).

From the proof of part (i) of Lemma B.2, we also have that there exists a local basis \( \{ (d\theta^j i, i \in [r]) \}, dx_a, du_a, \ldots, du_{(l_{i-2})}, dt \} \) of \( I^{(k-2)} + J_{k-2} + \text{span} \{ dt \} \) for every \( l_{k-2} \). Let \( l_{k-2} = l_{k-2} = \max \{ l_{k-1}, l_{k-2} \} \).

As \( I^{(k-1)} \subseteq I^{(k-2)} \), we may complete \( B \cup \hat{B} \) with a subset \( \hat{B} = \{ \theta_i, i = s+1, \ldots, \sigma \} \) of \( \{ \theta_i, i \in [r] \} \) in order to form a basis of \( I^{(k-2)} + J_{k-2} + \text{span} \{ dt \} \). By the same reasoning of the end of the proof of part (i) of Lemma B.2, it follows that \( B \cup \hat{B} \) is a basis of \( I^{(k-2)} + J \). The fact that \( B \cup \hat{B} \) is linearly independent mod \( (I^{(k-1)} + J + \text{span} \{ dt \}) \) implies that \( B \cup \hat{B} \) is also linearly independent mod \( (I^{(k-1)} + J) \).

Appendix C. Geometric Interpretation of the Dynamic Extension Algorithm.

In [14] it was shown, using an algebraic approach, that the output rank (the number of differentially independent outputs) [15] can be computed by the structure algorithm [51] and the dynamic extension algorithm [13, 35]. This interpretation was developed further in [11] in order to study control synthesis problems by quasi-static state feedback. In [39], the algebraic results of [14, 11] are translated to the differential geometric approach of [20], giving the following Lemma:

**Lemma C.1.** [39, Lemma 8.2] Consider the analytic (explicit) system \( S \) defined by (1.1a) with analytic output \( y = h(t, x, u) \). Let \( S_k \) be the open and dense set of regular points of the codistributions \( Y_i = \text{span} \{ dt, dy_1, \ldots, dy_i \} \) and \( \hat{Y}_i = \text{span} \{ dt, dx_d, dy_1, \ldots, dy_i \} \). In the \( k \)th step of the dynamic extension algorithm, one may construct a partition of \( y = (\hat{y}_k, \hat{y}_k) \) and a new local classical state representation \( (x_k, u_k) \) of the system \( S \) with state \( x_k = (x, \hat{y}_k) \) and input \( u_k = (\hat{y}_k, \hat{u}_k) \), defined in an open neighbourhood \( V_k \) of \( \xi \in S_k \), such that

1. \( \text{span} \{ dt, dx_k \} = \text{span} \{ dt, dx, dy_1, \ldots, dy_{k+1} \} \).
2. \( \text{span} \{ dt, dx_k, du_k \} = \text{span} \{ dt, dx_d, dy_1, \ldots, dy_{k+1}, du \} \).
3. It is always possible to choose \( \hat{y}_k^{(k+1)} \) in a way that \( \hat{y}_k^{(k+1)} \subseteq \hat{y}_k^{(k+1)} \).
4. It is always possible to choose \( \hat{u}_k^{(k+1)} \subseteq \hat{u}_k^{(k+1)} \).
5. Let \( D(C) \) denote the generic dimension of a codistribution \( C \) generated by the differentials of a finite set of analytic functions. The sequence \( \sigma_k = D(Y_k - D(Y_{k-1})) \) is nondecreasing, the sequence \( \rho_k = D(Y_k) - D(Y_{k-1}) \) is nonincreasing, and both sequences converge to the same integer \( \rho \), called the output rank, for some \( k^* < n = \text{dim} x \).

6. \( S_k = S_{k^*} \) for \( k \geq k^* \).
7. \( Y_{k^*} \cap \text{span} \{ dx \} = Y_{k^*} \cap \text{span} \{ dx \} \) for every \( \nu \in S_{k^*} \) and \( k \geq k^* \).
8. Around \( \xi \in U_k \), one may choose \( \hat{y}_k = \hat{y}_k^{(k+1)} \) for \( k \geq k^* \). Furthermore, \( Y_{k+1} = Y_k + \text{span} \{ \hat{y}_k^{(k+1)} \} \) for \( k \geq k^* \).

**Proof.** A complete proof of this result can be found in [39]. (see [14, Thm. 2.5] and [11, Lemma 4.1.6] for similar results in algebraic contexts.)
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