Flatness-based position/force control for under-actuated constrained robots

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RESUMO: Neste trabalho mostramos que um robô sub-atuado com n graus de liberdade, r restrições e l juntas ativas é genericamente "*flat*". Um sistema de controle que assegura rastreamento simultâneo de $r-n+l$ forças de contato e da posição do robô ao longo da restrição é construído baseado nesta propriedade
Palavras chave: Robótica, sistemas não lineares, sistemas implícitos, linearização por realimentação.

ABSTRACT: In this paper we show that a constrained under-actuated robot with n degrees of freedom, r constraints and l active joints, where $l \geq n-r$ is flat (under generical *non-orthogonality* conditions). We construct a flatness-based control that assure simultaneous tracking of $r-n+l$ contact forces and the position of the robot along the constraint surface.
Keywords. Robotics, nonlinear systems, implicit systems, feedback linearization.

1 Introduction

Constrained robots are robots whose movement is restricted by some physical contact surfaces. Such restrictions can be represented by adding r holonomic constraints $\phi_i(q) = 0 \ \ (i = 1, \ldots, r)$ to its original equations.

The following model can be obtained by taking into account the contact forces (McClamroch & Wang 1988, Krishnan & McClamroch 1994) :

$$\begin{aligned} M(q)\ddot{q} + H(q,\dot{q}) &= (J\phi)^T(q)\lambda + \tau & \text{(1.1a)} \\ \phi_i(q) &= 0 \ \ (i = 1, \ldots, r) & \text{(1.1b)} \end{aligned}$$

where $q \in I\!R^n$, $J\phi(q) = \partial\phi/\partial q$, $\lambda = (\lambda_1, \ldots, \lambda_r)^T$ is a vector corresponding to the contact forces, $M(q)$ is the symmetric positive definite mass matrix, and $H(q, q^{(1)})$ corresponds to Coriolis and gravity forces. We will assume that $\partial\phi/\partial q$ has rank r for all q in the operation region of the robot.

Simultaneous force/position control for constrained robots was studied by (McClamroch & Wang 1988, Lozano & Brogliato 1992, Reithmeier & Leitmann 1992, Jean & Fu 1993, You & Chen 1993, Krishnan & McClamroch 1994, Leviner & Dawson 1995, Pereira da Silva 1996, Siciliano & Villani 1997, Liu & Chen 1998, Lian & Lin 1998, Vukobratovic, Stojic & Ekalo 1998) for the case of active joints. In this paper we develop a flatness-based control for constrained under-actuated robots with simultaneous tracking of some components of the contact forces based on the infinite dimensional geometric approach of (Fliess, Lévine, Martin & Rouchon 1999). We suppose that position and velocity measurements are available and the perfect knowledge of the model and the constraint surface is also assumed. In the case of model and/or constraint surface uncertainties, these results have at least some philosophical importance. Our main result (Theorem 2) is a generalization for the under-actuated case of previous results (Pereira da Silva 1996) that are valid in the full-actuated case.

Flatness is a notion of control systems theory that corresponds to a complete and finite parametrization of all solutions of a control system by a differentially independent family of functions (Fliess, Lévine, Martin & Rouchon 1992, Fliess, Lévine, Martin & Rouchon 1995). Many models are found to be flat in control applications, showing the relevance of this concept in pratice. Since flatness implies feedback linearization by dynamic (or static) feedback, a linearizing feedback can be used to assure stabilization or asymptotic tracking (see for instance (Isidori 1989)). In many applications, the flat output is the set of functions that one is interested to control. In this case, asymptotic tracking of the flat output is always possible by a flatness-based control.

Singular (or implicit) systems are an important class of control systems. Solvability of nonlinear implicit differential equations is considered in (Brenan, Campbell & Petzold 1995, Rheinboldt 1991). Some results on feedback linearization and flatness of implicit systems can be found on (Liu 1993, Liu & Zhang 1993, Kawaji & Taha 1994, Pereira da Silva & Corrêa Filho 1998*a*, Schlacher, Kugi & Haas 1998, Pereira da Silva & Corrêa Filho 1998*b*).

In (Pereira da Silva & Corrêa Filho 1998*a*, Pereira da Silva & Corrêa Filho 1998*b*) its shown that the infinite dimensional geometric setting recently introduced in control theory (see (Fliess, Lévine, Martin & Rouchon 1993, Pomet 1995, Fliess, Lévine, Martin & Rouchon 1997, Fliess et al. 1999, Fliess, Lévine, Martin & Rouchon 1998)) is useful for studying flatness of implicit systems. Consider the system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) && (1.2\text{a}) \\ y(t) &= h(x(t)) = 0 && (1.2\text{b}) \end{aligned}$$

where $x(t) \in I\!\!R^n$, $y(t) \in I\!\!R^r$, $u(t) \in I\!\!R^m$ and all the components of $f(x)$ and $g(x)$ are analytical functions of x. The following result is shown in in (Pereira da Silva & Corrêa Filho 1998*a*, Pereira da Silva & Corrêa Filho 1998*b*)

Theorem 1 *(Pereira da Silva & Corrêa Filho 1998*a*, Pereira da Silva & Corrêa Filho 1998*b*) Consider the unconstrained system S given by* (1.2a) *and assume that this system is locally flat with flat output (y, ψ),* i. e. *the restriction functions $y_j = h_j(x), j = 1, \ldots, r$ are part of a local flat output of system* (1.2a)*. Then the implicit system* (1.2a)–(1.2b) *is locally flat with flat output ψ.*

Remark. *Theorem 1 is a simplified version of the result presented in (Pereira da Silva & Corrêa Filho 1998*a*, Pereira da Silva & Corrêa Filho 1998*b*). In fact, it is shown in these papers that there exists a system Γ (i. e. a diffiety with Cartan field and a notion of time (Fliess et al. 1999, Fliess et al. 1997)) and a Lie-Bäcklund immersion $\iota : \Gamma \to S$ such that every solution $\sigma(t)$ of S such that $y(\sigma(t))$ is identically zero is of the form $\sigma = \iota \circ \gamma(t)$, where $\gamma(t)$ is a solution of Γ. Furthermore Γ is flat with flat output $\{\psi_1 \circ \iota, \ldots, \psi_{n-r} \circ \iota\}$.*

Based on Theorem 1, a sufficient condition for flatness of constrained under-actuated robots is easily obtained in this paper. It is shown shown that a robot with n degrees of freedom, r holonomic constraints and only l active joints, where $l \geq n - r$, is generically flat. A possible choice of flat output is the position along the constraint surface and $r - n + l$ contact forces, showing that the problem of simultaneous tracking of these outputs is solvable. Some computer simulations are also presented for an academic example.

The paper is organized as follows. An application of the results of (Pereira da Silva & Corrêa Filho 1998*a*, Pereira da Silva & Corrêa Filho 1998*b*) to robotics is presented in section 2. A flatness based control of this (under-actuated) robot is studied in detail in the academic example of section 3.

2 Control of constrained under-actuated robots

A representation of the system (1.1a)–(1.1b) in the form (1.2a)–(1.2b) is given by

$$\begin{pmatrix} \dot{q} \\ \dot{q}^{(1)} \end{pmatrix} = \begin{pmatrix} q^{(1)} \\ -M^{-1}(q)H(q, q^{(1)}) \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ M^{-1}(q)(J\phi)^T \end{pmatrix} \lambda + \begin{pmatrix} 0 \\ M^{-1}(q) \end{pmatrix} \tau \quad (2.3)$$

$$0 = \phi_i(q), \; i = 1, \ldots, r \quad (2.4)$$

where the unconstrained system (2.3) has input $u = (\lambda, \tau)$, state $x = (q, q^{(1)})$ and output $y = \phi = (\phi_1, \ldots, \phi_r)$. Now suppose that the robot is under-actuated, *i. e.*, we have only l actuators, where $l \geq n - r$. In other words we have that $\tau_{i_1} = \ldots \tau_{i_{n-l}} = 0$ for convenient integers $i_1, \ldots, i_l$. Let P be formed by convenient columns of the identity matrix in a way that

$$\tau = P\bar{\tau} \quad (2.5)$$

where $\bar{\tau}$ is the l-vector of the torques applied by the actuators.

Non-orthogonality assumption. *Let $q_0 \in I\!\!R^n$ and assume that*

$$\dim \operatorname{Im} \left[J(\phi)^T(q_0) \;\; P \right] = n.$$

In other words, we can complete the matrix P with columns of the matrix $(J\phi)^T$ in order to construct a nonsingular matrix in an open set containing q_0.

Roughly speaking, this means that no vector of the tangent space of the constrained surface is is orthogonal to the directions of the torque actuation (represented by the columns of P). Note that this assumption can be always satisfied if one can choose what are the l active joints). Let $\bar{R}(q)$ be a matrix formed by convenient columns of the identity matrix such that

$$[(J\phi)^T\bar{R}\;\; P] \tag{2.6}$$

is locally nonsingular around q_0. Let $\widehat{R}$ be a matrix formed by the columns of the identity not present in $\bar{R}$. Then we can write

$$\lambda = \bar{R}\bar{\lambda} + \widehat{R}\widehat{\lambda} \tag{2.7}$$

where $\bar{\lambda}$ and $\widehat{\lambda}$ are formed respectively by $n-l$ and $r-n+l$ components of λ, conveniently reordered. Now we can rewrite the unconstrained robot equation (2.3) by extension of the state[1], obtaining a state representation with input $(\widehat{\lambda}^{(1)}, \bar{\lambda}, \bar{\tau})$ and state $(\widehat{\lambda}, q, q^{(1)})$ given by

$$\begin{pmatrix} \dot{q} \\ \dot{q}^{(1)} \\ \dot{\widehat{\lambda}} \end{pmatrix} = \begin{pmatrix} q^{(1)} \\ M^{-1}(q)[-H(q,q^{(1)}) + (J\phi)^T\widehat{R}\widehat{\lambda}] \\ 0 \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & M^{-1}(J\phi)^T\bar{R} & M^{-1}P \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{\lambda}^{(1)} \\ \bar{\lambda} \\ \bar{\tau} \end{pmatrix} \tag{2.8}$$

Now choose a set of functions $\psi = (\psi_1, \ldots, \psi_{n-r})$ in a way that the Jacobian matrix $\left(J\phi^T J\psi^T\right)^T$ is nonsingular[2].

Proposition 1 *The unconstrained robot equations* (2.8) *are (locally) state-feedback linearizable and $(\widehat{\lambda}, \phi, \psi)$ is a (local) flat output this system.*

Proof. Considering the output $(\phi, \psi, \widehat{\lambda})$ for the unconstrained system (2.8), we will show that its decoupling matrix is always nonsingular and the sum of their characteristic numbers gives the dimension of the state. Then the result will follow from (Isidori 1995, Lemma 5.2.1, p. 230). For this note that, for $\xi = (\phi, \psi)$, $\dot{\xi} = (J\xi)\dot{q}$ and so $\ddot{\xi} = (J\xi)\ddot{q} + F(q, \dot{q})$, where $F(q, \dot{q})$ is a vector with n components given by $F_j = \dot{q}^T \mathcal{H}_j \dot{q}$ and $\mathcal{H}_j = \frac{\partial^2 \xi_j(q)}{\partial q, \partial q}$ is the Hessian matrix of ξ_j, $j \in \{1, \ldots n\}$. A simple computation gives

$$\begin{pmatrix} \dot{\widehat{\lambda}} \\ \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} = \alpha + A \begin{pmatrix} \widehat{\lambda}^{(1)} \\ \bar{\lambda} \\ \bar{\tau} \end{pmatrix} \tag{2.9}$$

where

$$\alpha = \begin{pmatrix} 0 \\ F_\phi(q, q^{(1)}) + J\phi M^{-1}(q)[-H(q, q^{(1)}) + (j\phi)^T\widehat{R}\widehat{\lambda}] \\ F_\psi(q, q^{(1)}) + J\psi M^{-1}(q)[-H(q, q^{(1)}) + (j\phi)^T\widehat{R}\widehat{\lambda}] \end{pmatrix} \tag{2.10}$$

$$A = \begin{pmatrix} A_{\widehat{\lambda}} \\ A_\phi \\ A_\psi \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & J\phi M^{-1}(J\phi)^T\bar{R} & J\phi M^{-1}P \\ 0 & J\psi M^{-1}(J\phi)^T\bar{R} & J\psi M^{-1}P \end{pmatrix} \tag{2.11}$$

The non-orthogonality assumption implies that

$$(J\xi)M^{-1}[(J\phi)^T\bar{R}\;\; P] = \begin{pmatrix} J\phi M^{-1}(J\phi)^T\bar{R} & J\phi M^{-1}P \\ J\psi M^{-1}(J\phi)^T\bar{R} & J\psi M^{-1}P \end{pmatrix}$$

is a nonsingular matrix. So the decoupling matrix A is also nonsingular. Since the the sum of characteristic numbers gives the dimension of the state, the result follows. □

The following theorem means that one can easily control simultaneously the position of the constrained under-actuacted robot (2.3)–(2.4)–(2.5) along the constrained surface and $r-n+l$ contact forces.

Theorem 2 *Under the* non-orthogonality *assumption, i. e., the local nonsingularity of matrix* (2.6)*), the constrained under-actuated robot* (2.3)–(2.4)–(2.5) *is locally flat with flat output $(\psi, \widehat{\lambda})$.*

Proof. By Proposition 1, the unconstrained robot is locally flat with flat output $(\phi, \psi, \widehat{\lambda})$. Restricting to a open set V around the operation point, the application of Theorem 1 leads to the desired result. □

Flatness-based control. Now we show how to develop a control system for tracking simultaneously the position of the robot (2.3)–(2.4)–(2.5) along the constrained surface and $r-n+l$ contact forces.

For, let $\zeta(t) = (\psi_{ref}(t), \widehat{\lambda}_{ref}(t))$ be a reference trajectory. Let $e(t) = \psi(t) - \psi_{ref}(t)$. Let $\bar{\tau}$ be defined by (see equations (2.9)–(2.10)–(2.11))

$$\begin{pmatrix} \dot{\widehat{\lambda}}_{ref} \\ 0 \\ -\Lambda_1\dot{e} - \Lambda_0 e + \ddot{\psi}_{ref}(t) \end{pmatrix} = \alpha(q, q^{(1)}, \widehat{\lambda})\big|_{\widehat{\lambda}=\widehat{\lambda}_{ref}} + \\ + A(q, q^{(1)}) \begin{pmatrix} \widehat{\lambda}^{(1)}_{ref} \\ \bar{\lambda} \\ \bar{\tau} \end{pmatrix} \tag{2.12}$$

where Λ_i are symmetric positive definite matrices for $i = 0, 1$. By the nonsingularity and the structure of A, it is easy to see that the last equation define a unique vector of functions $\bar{\tau}$ that depends only on $q, \dot{q}, \psi_{ref}(t), \dot{\psi}_{ref}(t), \ddot{\psi}_{ref}(t)$. Note

[1] This extension of the state is equivalent to the addition of integrators in series of the input components of $\widehat{\lambda}$.

[2] Note that, in this case, the restrictions of the functions Ψ_j, $j = 1, \ldots, r$ to the constraint surface form a local chart of this surface.

also that (2.9)–(2.10)–(2.11) for $\dot{\phi}(t) = 0$ implies that

$$\begin{aligned} 0 &= F_\phi(J\phi)M^{-1}H + (J\phi)M^{-1}(J\phi)^T P\bar{\tau} + \\ & \quad (J\phi)M^{-1}(J\phi)^T[\widehat{R}\ \bar{R}]\begin{pmatrix} \widehat{\lambda} \\ \bar{\lambda} \end{pmatrix} \end{aligned} \tag{2.13}$$

Since $J\phi$ has full row rank and M is positive definite then $(J\phi)^T M^{-1}(J\phi)$ is nonsingular, showing that the choice of $\bar{\tau}(t)$ determines unique vectors of functions $\bar{\lambda}$ and $\widehat{\lambda}$. Note that (2.12) and the last equation implies that $\widehat{\lambda} = \widehat{\lambda}_{ref}(t)$.

By (2.9), the application of $\bar{\tau}(t)$ previously defined on the constrained robot implies that

$$\ddot{e} + \Lambda_1 \dot{e} + \Lambda_0 e = 0 \tag{2.14}$$

and so $e(t)$ converges asymptotically to zero, showing that the tracking problem is solved in this way.

Remark. *Recall that $\bar{\tau}$ defined by* (2.12) *depends only on $q, q^{(1)}, \psi_{ref}, \dot{\psi}_{ref}, \ddot{\psi}_{ref}, \lambda_{ref}$. Hence, there is no need to measure the contact forces for implementing this control law. An adaptation of this control law for taking into account the contact force errors might be done in this context.*

When the model of the robot or the real shape of the constraint surface are not precisely known, such a method may produce bad results. Note that small errors in the constraint map ϕ may produce big deviations in the expected value of $J\phi$. The choice of the functions ψ should be made in a way to have, locally, a nonsingular Jacobian matrix $[(J\phi)^T (J\psi)^T]^T$. In many cases one may choose more than one local flat output "ψ" and switch the corresponding control law according the operation point.

3 Example

We now present an academic example. Consider the two link robot arm of figure 1 (Gras & Nijmeijer 1989) . The constraint surface is represented by the horizontal dashed line. The two dark disks represents unit masses and we can apply control torques to each degree of freedom corresponding to θ_1 and θ_2. The contact force is represented by the vector λ and the two arm lengths are equal to one meter. The corresponding model of the unconstrained robot is given by (2.3)–(2.4). Note that $q = (\theta_1, \theta_2)^T$ is the vector of angular displacements, $\tau = (\tau_1, \tau_2)^T$ is the vector of torques, and $H(q, q^{(1)}) = C(q, q^{(1)}) + K(q)$, where

$$M(q) = \begin{pmatrix} 3 + 2\cos\theta_2 & 1 + \cos\theta_2 \\ 1 + \cos\theta_2 & 1 \end{pmatrix}$$

$$C(q, q^{(1)}) = \begin{pmatrix} -\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 \\ \dot{\theta}_1^2 \sin\theta_2 \end{pmatrix}$$

$$K(q) = \begin{pmatrix} 2g\sin\theta_1 + g\sin(\theta_1 + \theta_2) \\ g\sin(\theta_1 + \theta_2) \end{pmatrix}$$

and g is the gravity constant. Consider constraint function

$$0 = \phi(q) = \cos\theta_1 + \cos(\theta_1 + \theta_2) - L \tag{3.15}$$

corresponding to restrict the trajectory of the endpoint of the second arm to the dashed line of figure 1. Note that L is the vertical distance of the dashed line from the top of figure 1. We can choose $\psi = \sin(\theta_1) + \sin(\theta_1 + \theta_2)$, corresponding to the position along the constraint (the dashed line in Figure 1). Note that the coordinates $(\psi, -\phi)$ are the cartesian coordinates (x, y) for the figure 1.

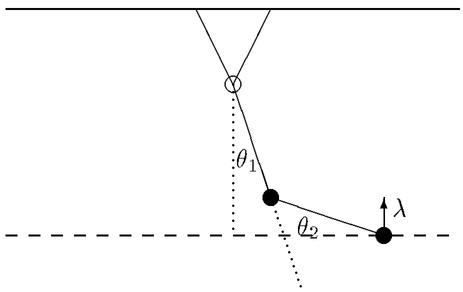

Figure 1: *Two link robot arm.*

A control system using the previous development was constructed and some computer simulations are presented for the full-actuated case (see figures 2 to 5) and for the under-actuated case (see figures 6 to 9). For all plotted curves, the horizontal axis represents time in seconds.

In the first case we have $l = n$, $\bar{\tau} = \tau$. Furthermore, $\bar{\lambda}$ and $\bar{R}$ are absent, $P = I$, $\widehat{R} = 1$. A flat output is given by (ψ, λ). So the position along the constrained surface and the contact force can be easily controlled. Note that there is a singular point of the Jacobian matrix $[(J\phi)^T \ (J\psi)^T]^T$ for $\theta_2 = 0$ that must be avoided.

In the second case we have that $l = 1$, $\bar{\tau} = \tau_1$, $\bar{\lambda} = \lambda$. Furthermore, $\widehat{\lambda}$ and $\widehat{R}$ are absent, $P = [1 \quad 0]^T$, $\bar{R} = 1$. A flat output is given by ψ. Hence, in this case only the position along the constrained surface may be controlled. In addition to the singularity of the first case, the matrix $[(J\phi)^T \bar{R} \quad P]$ is singular for $\theta_1 = -\theta_2$.

4 Conclusions

The example of the constrained robot illustrates how one can design a flatness-based control for simultaneous tracking of position and contact forces.

The measurement of the angular positions and velocities of the robot joints are needed to implement the flatness based control law. The feedback law here obtained is not of standard nature. The control of contact forces $\widehat{\lambda}$ can be regarded as a feedforward control, since the equation (2.13) establishes an explicit and instantaneous determination of $\widehat{\lambda}(t)$ by the choice of $\bar{\tau}(t)$. This explains why the measurement of the contact force is not needed, which may be an advantage or not, depending on the context.

The model of the robot could be improved by taking into account the friction that occurs along the constraint surface. In this case the model lacks smoothness, but similar results could be obtained.

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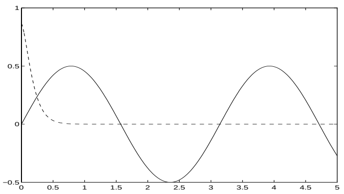

Figure 2: Reference position $\psi_{ref}(t)$ (continuous line) and tracking error e(t) (dashed line) in meters — full-actuated case.

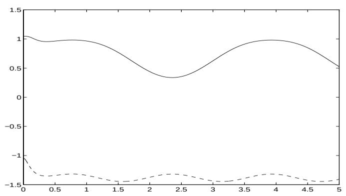

Figure 3: Angular positions θ_1 (continuous line) and θ_2 (dashed line) in radians — full-actuated case.

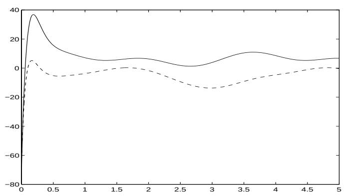

Figure 4: Control torques τ_1 and τ_2 in $N \times m$ — full-actuated case.

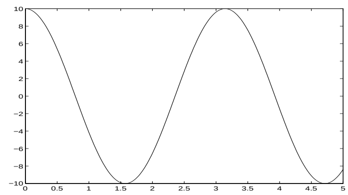

Figure 5: Contact force $\lambda(t)$ in N — full-actuated case.

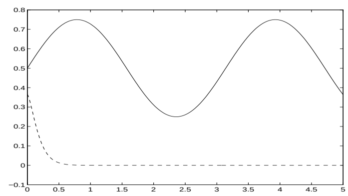

Figure 6: Reference position $\psi_{ref}(t)$ (continuous line) and tracking error e(t) (dashed line) in meters — under-actuated case.

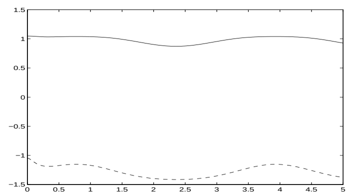

Figure 7: Angular positions θ_1 (continuous line) and θ_2 (dashed line) in radians — underactuated case.

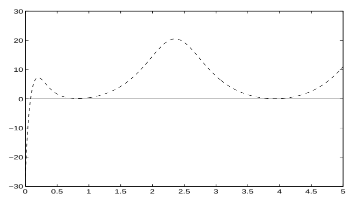

Figure 8: Control torques τ_1 and τ_2 in $N \times m$ — underactuated case.

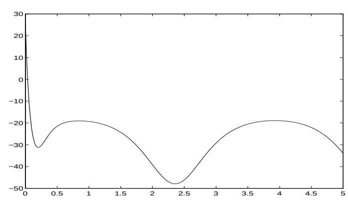

Figure 9: Contact force $\lambda(t)$ in N — underactuated case.