On decompositions for noncontrollable nonlinear systems[*]

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RESUMO - O problema de linearização parcial (PLP) é tratado neste artigo. Neste problema, desejamos decompor um sistema não controlável em um subsistema linear controlável e um subsistema autônomo, que faz o papel de subsistema não controlável. Para a classe de sistemas parcialmente linearizáveis, mostramos que a influência do subsistema não controlável pode ser sempre eliminada via realimentação de estado. No caso mais geral (em que o PLP não é necessariamente solúvel), mostramos uma decomposição em parte controlável/não controlável diretamente relacionada com as integrais primeiras.

ABSTRACT - We define and solve the *Partial Linearization Problem* by static state feedback *(PLP)*. Our notion of linearization is weaker than the one found in the literature. In fact, we want to transform a given system, via static-state feedback and coordinate change, into a controllable linear system that is affected by a autonomous system, which plays the role of its noncontrollable part.

For the class of nonlinear systems which is PLP solvable, we show that the influence of the noncontrollable part in the linear part can be always removed by a convenient static state feedback. We construct another decomposition that holds for a general (time-varying) nonlinear noncontrollable systems consisting in a controllable subsystem and a noncontrollable one that is generated by a set of conservation laws.

Key words: nonlinear systems, time-varying systems, static-state feedback, feedback linearization, controllability, canonical forms, linear systems.

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1 Introduction and Motivation

It is well known that, from the "input-to-state" point of view, a linear control system can be decomposed, after a convenient coordinates change, into two interconnected subsystems, namely the noncontrollable and the controllable subsystems (Kailath, 1980), (Wonham, 1985), (Fliess, 1990) (see also (Isidori, 1989), (Nijmeijer and van der Schaft, 1990) for nonlinear systems). The noncontrollable subsystem is completely autonomous, *i. e.*, it is not affected either by the input or by the controllable subsystem. The situation can be illustrated by the structure of figure 1.

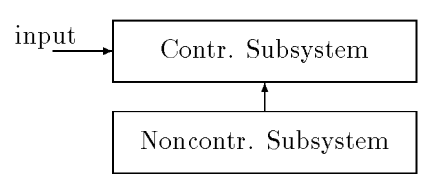

Figura 1 - Structure of a noncontrollable system.

Consider now the class of linear time-invariant systems with the state evolving on the linear space $\mathcal{X}$. Then the controllable and the noncontrollable subsystems that appear in this decomposition have, in some sense, an intrinsic meaning[1].In fact, the controllable subsystem is the "restric-

[1]It is important to stress that the noncontrollable subsystem in the state space approach corresponds to a choice of a subspace $\hat{\mathcal{X}}$ such that the direct sum of $\hat{\mathcal{X}}$ with the controllable space is the entire space. This choice is not unique, but is unique up to an isomorphism. In the approach of (Fliess, 1990), the noncontrollable space is represented by the torsion submodule, being intrinsic. The controllable subsystem is in this case nonunique, but unique up to an isomorphism.

tion" of the system to the controllable subspace $\mathcal{R}$ and the noncontrollable one is the "quotient" subsystem induced in the quotient space $\frac{\mathcal{X}}{\mathcal{R}}$, which is not affected by the input (Wonham, 1985). However, the influence of the noncontrollable subsystem on the controllable one can be changed a lot by a similarity transformation that preserves the controllable subspace (*i.e.* , it transforms a basis for the controllable subspace into another basis for the controllable subspace). In fact, consider the following example.

Example 1 *(Pereira da Silva, 1996b) Consider the linear system with state* $x(t) = (x_1(t),\ x_2(t),\ x_3(t))^T \in I\!R^3$ *and input* $u(t) \in I\!R$ *given by*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + x_3(t) \\ \dot{x}_2(t) &= x_3(t) + u(t) \\ \dot{x}_3(t) &= -2x_3(t) \end{aligned}$$

Let $z^1 = (x_1, x_2)^T$ *and* $z^2 = x_3$. *Then the last equation can be rewritten as*

$$\dot{z}^1 = \tilde{A}_{11} z^1 + \tilde{A}_{12} z^2 + B_1 u \qquad (1.1a)$$

$$\dot{z}^2 = \tilde{A}_{22} z^2 \qquad (1.1b)$$

where $\tilde{A}_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tilde{A}_{22} = \begin{pmatrix} -2 \end{pmatrix}$, $\tilde{A}_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *and* $B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

It is easy to see that the subsystem (1.1a) *is the controllable subsystem of the given system and* (1.1b) *is the noncontrollable one. The matrix* $\tilde{A}_{12}$ *represents the influence of the noncontrollable subsystem on the controllable subsystem.*

Now let $\xi_1 = x_1$, $\xi_2 = \dot{x}_1 = x_2 + x_3$ *and* $\xi_3 = x_3$. *Set* $\xi = (\xi_1, \xi_2, \xi_3)$. *Then,* $\xi = Tx$, *where* T *is a nonsingular matrix and we can write*

$$\left\{ \begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\xi_3 + u \\ \dot{\xi}_3 &= -2\xi_3 \end{aligned} \right. \qquad (1.2)$$

In particular we see that the influence of the noncontrollable subsystem on the controllable one can be canceled by the regular state feedback $u = \xi_3 + v$.

In (Pereira da Silva, 1996b) it is shown that the procedure of the example 1 can be done for an arbitrary noncontrollable linear system, *i.e.* , they have all the structure depicted in figure 2. This linear result is not explicitly stated in the literature, but it is not difficult to verify that this is a consequence of theorem 4.1 of (Morse, 1973) in the case of $C = 0$.

In this paper we will generalize these results, showing that "essentially linear" systems have a similar structure. Our results are closely related to the ones of (Marino et al., 1985) and (Marino, 1986) for nonlinear affine systems (see proposition 10 of (Marino et al., 1985) and consider that $\lambda = \mu$, *i. e.*, the subvector $\tilde{x}_2$ is absent). However, our approach is completely different. Furthermore, we are interested in the

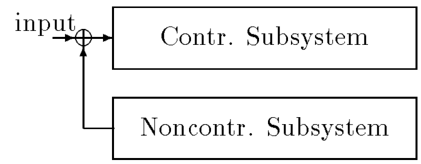

Figura 2 - Structure of noncontrollable linear systems.

connections of these results with the ones of (Morse, 1973), (Pereira da Silva, 1996b) about canonical forms for linear systems, which are not made clear in (Marino et al., 1985).

To state precisely what we mean by "essentially linear", let us consider the Problem of *exact linearization* of nonlinear systems, which has attracted a lot of attention over the last two decades. This problem was completely solved in its static-state feedback version (see (Jakubczyk and Respondek, 1980), (Hunt et al., 1983)) but it remains open in the dynamic feedback case (Charlet et al., 1989), (Shadwick, 1990), (Charlet et al., 1991). Closed related to this problem is the notion of flatness (see (Fliess et al., 1995b), (Fliess et al., 1995a)). It is important to point out the usefulness of the techniques of exterior calculus (Briant et al., 1991) for the problem of exact linearization (see (Shadwick, 1990), (Gardner and Shadwick, 1992), (Sluis, 1992), (Martin and Rouchon, 1993), (Murray, 1993), (Tilbury et al., 1995), (Shadwick and Sluis, 1993), (Aranda-Bricaire et al., 1995), (van Nieuwstadt et al., 1994)).

In the usual terminology, "exact linearization" means that we want to transform the system, via state-feedback and coordinate change, into a completely controllable linear system. In this paper we shall consider a weaker form of this problem, which allows the transformation of the system into a controllable linear system that is affected by an autonomous[2] nonlinear system. We will show that the necessary and sufficient conditions to solve this problem are the same as the ones given in (Sluis, 1992), (Shadwick and Sluis, 1993) apart from the assumption of considering the system to be maximally nonholonomic (*i.e.* , strongly controllable). Under these conditions, we will show that the influence of the noncontrollable subsystem on the "exact linearized" subsystem can be always removed by a convenient static state feedback. The techniques of the proof of the last result are different from the ones in (Shadwick and Sluis, 1993), (Sluis, 1992) since we do not use *Goursat* canonical forms. Furthermore, similar techniques may be useful to study the structure of implicit nonlinear systems (Pereira da Silva, 1996c).

It is known (see (Fliess et al., 1993)) that a system is controllable if and only if it does not admit conservation laws. We construct a decomposition for (time varying) nonlinear noncontrollable systems consisting in a controllable subsystem (in the sense that it does not contain any conservation

[2] It is not reasonable to expect that the noncontrollable subsystem of a nonlinear system can be exactly linearized since it is not affected by the input and thus it is feedback-invariant.

law) and a subsystem with state variables $w = (w_1, \ldots, w_l)$ formed by conservation laws. Furthermore, any conservation law of the system is a function of $w_1, \ldots, w_l$.

The paper is organized as follows. In section 2 we present the notations and some introductory remarks about differential geometry. In section 3 we consider some decompositions of linear systems. In section 4, we give some geometric definitions of feedbacks and we state the *Partial Linearization Problem (PLP)*. In section 5, the PLP is solved for non-linear time-varying systems. In section 6 we present some remarks about the time-invariant case. Finally, in section 7 we consider a decomposition that holds for general noncontrollable nonlinear systems.

2 Mathematical background and notations

The field of real numbers will be denoted by $I\!R$. The set of real matrices of n rows and m columns is denoted by $I\!R^{n\times m}$. The matrix M^T stands for the transpose of M. The set of natural numbers $\{1, \ldots, k\}$ will be denoted by $\lfloor k \rceil$.

We will use the standard notations of differential geometry and exterior algebra (Warner, 1971), (Briant et al., 1991). Given a smooth manifold $\mathcal{P}$ of dimension δ, $C^\infty(\mathcal{P})$ denotes the set of smooth maps from $\mathcal{P}$ to $I\!R$. Let $\mathcal{Q}$ be a smooth manifold of dimension γ and let $\phi : \mathcal{P} \mapsto \mathcal{Q}$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_p\mathcal{P} \mapsto T_{\phi(p)}\mathcal{Q}$ and $\phi^* : T^*_{\phi(p)}\mathcal{Q} \mapsto T^*_p\mathcal{P}$. Given a field f and a 1-form ω on $\mathcal{P}$, we denote $\omega(f)$ by $\langle f, \omega\rangle$. The set of smooth k-forms on $\mathcal{P}$ will be denoted by $\Lambda_k(\mathcal{P})$ and $\Lambda(\mathcal{P}) = \cup_{k\in I\!N}\Lambda_k(\mathcal{P})$.

Given two forms η and ξ in $\Lambda(\mathcal{P})$, then $\eta \wedge \xi$ denotes their *wedge* multiplication. The *exterior derivative* of $\eta \in \Lambda(\mathcal{P})$ will be denoted by $d\eta$. Note that the graded algebra $\Lambda(\mathcal{P})$, as well as their homogeneous elements $\Lambda_k(\mathcal{P})$ of degree k, have a structure of $C^\infty(\mathcal{P})$-module.

A smooth codistribution J is a $C^\infty(\mathcal{P})$-submodule $J \subset \Lambda_1(\mathcal{P})$. Let $p \in \mathcal{P}$ be a regular point of J and take a local basis $\{\eta_1, \ldots, \eta_k\}$ where $\{\eta_1(q), \ldots, \eta_k(q)\}$ are $I\!R$-linearly independent covectors for all q in some open neighborhood U of p. Denoting two different local basis around p by column vectors η and $\tilde\eta$, it is easy to verify that, for some open neighborhood V of p, $\tilde\eta = \mathcal{M}\eta$ where $\mathcal{M}$ is a matrix with elements in $C^\infty(\mathcal{P})$ such that $\mathcal{M}(q)$ is nonsingular for all $q \in V$.

It is worth recalling that one may regard the exterior differentiation operator d as a map $d : \Lambda_1(\mathcal{P}) \mapsto \Lambda_2(\mathcal{P})$. Associated to a codistribution is the submodule $S(J) \subset \Lambda_2(\mathcal{P})$ defined by $S(J) = \mathrm{span}_{C^\infty(\mathcal{P})}\{\eta \wedge \xi \mid \eta \in \Lambda_1(\mathcal{P}) \text{ and } \xi \in J\}$. The quotient module $\frac{\Lambda_2(\mathcal{P})}{S(J)}$ plays an important role for us and this will be clear very soon. By simplicity, we denote the elements of $\frac{\Lambda_2(\mathcal{P})}{S(J)}$ by (ζ mod J), instead of (ζ mod $S(J)$). Note that (ζ mod J) $\equiv$ (θ mod J) if and only if $\zeta = \theta + \sum_{\text{finite}} \eta_i \wedge \xi_i$ for convenient 1-forms $\eta_i \in J$ and $\xi_i \in \Lambda_2(\mathcal{P})$. Given a smooth codistribution J, define the map $\tilde d : J \mapsto \frac{\Lambda_2(\mathcal{P})}{S(J)}$ by $\tilde d\eta = d\eta \bmod J$. It is easy to show that $\tilde d(\alpha_1\eta_1 + \alpha_2\eta_2) = \alpha_1\tilde d(\eta_1) + \alpha_2\tilde d(\eta_2)$. In other words, $\tilde d$ is a morphism of $C^\infty(\mathcal{P})$-modules. The *kernel* of the morphism $\tilde d : J \mapsto \frac{\Lambda_2(\mathcal{P})}{S(J)}$ is a $C^\infty(\mathcal{P})$-submodule of J and so is a smooth codistribution, which is denoted by $\tilde J$. It follows that

$$\tilde J = \mathrm{span}_{C^\infty(\mathcal{P})}\{\omega \in J | d\omega \bmod J \equiv 0\}. \qquad (2.1)$$

Note that, by the Frobënius theorem, a nonsingular codistribution J is integrable if and only if it is involutive, *i.e.* , $\tilde J = J$.

Let p be a nonsingular point of a codistribution J and let the pointwise linearly independent one forms $\{\eta_1, \ldots, \eta_k\}$ be a basis for J in an open neighborhood U of p. Let $\{\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_\delta\}$ be a basis for T^*U for some open neighborhood U of p. Note that

$$\mathcal{B} = \{\eta_i \wedge \eta_j \mid i \in \lfloor\delta\rceil,\ i < j \le \delta\}$$

is a basis for $\Lambda_2(U)$ (as a $C^\infty(U)$-module). So we can locally write :

$$d\eta_\rho = \sum_{i=1}^{k}\sum_{j=i+1}^{\delta}\alpha^\rho_{ij}\eta_i\wedge\eta_j + \sum_{i=k+1}^{\delta}\sum_{j=i+1}^{\delta}\beta^\rho_{ij}\eta_i\wedge\eta_j \quad (\rho \in \lfloor k\rceil) \qquad (2.2)$$

where α^ρ_{ij} and β^ρ_{ij} are convenient smooth functions and equation (2.2) corresponds to the expression of $d\eta_\rho$ in this basis. Note also that

$$\tilde{\mathcal{B}} = \{\eta_i \wedge \eta_j \bmod J \mid i \in \{k+1, \ldots, \delta\},\ i < j \le \delta\} \qquad (2.3)$$

is a basis for $\frac{\Lambda_2(U)}{S(J)}$. Hence, for each $\rho \in \lfloor k\rceil$ we can represent $\tilde d\eta_\rho$ in this basis by column vectors β^ρ for $\rho = 1, \ldots, k$ with elements obtained from the functions β^ρ_{ij} of equation (2.2). We can also construct a matrix $\beta = [\beta^1 \ldots \beta^k]$ formed by the column vectors β^ρ, for $\rho = 1, \ldots, k$. Note that β is a matrix with $C^2_{\delta-k}$ rows and k columns that represents the morphism $\tilde d : J \mapsto \frac{\Lambda_2(U)}{S(J)}$, *i.e.* , if $\omega = \sum_{i=1}^k a_i\eta_i \in J$, then $\tilde d\omega$ when represented in the basis (2.3) is given by the column vector βa, where $a = (a_1, \ldots, a_k)^T$. In particular, $\omega \in \tilde J = \ker \tilde d$ if and only if

$$\beta a = 0. \qquad (2.4)$$

Giving a submodule S of $\Lambda(\mathcal{P})$ and $p \in \mathcal{P}$, $S(p)$ denotes the $I\!R$-linear subspace of $\Lambda_p(\mathcal{P})$ giving by $\mathrm{span}_{I\!R}\{\zeta(p)|\zeta \in S\}$. In particular, if J is a codistribution, $J(p)$ denotes the subspace of $T^*_p\mathcal{P}$ giving by $\mathrm{span}_{I\!R}\{\omega(p)|\omega \in J\}$.

For every $p \in \mathcal{P}$, let $J^{(1)}(p)$ be the subspace of $T^*_p\mathcal{P}$ given by

$$J^{(1)}(p) = \mathrm{span}_{I\!R}\{\omega(p) \mid \omega \in J, \text{ such that } d\omega(p) \bmod J(p) \equiv 0\}. \qquad (2.5)$$

where $d\omega(p) \bmod J(p)$ denotes the canonical projection of $d\omega(p)$ in $\frac{\Lambda_2(p)}{S(J)(p)}$. In other words, $d\omega(p) \bmod J(p) = 0$ if and only if $d\omega(p) = \sum_{finite} \eta_i \wedge \xi_i\big|_p$ for convenient 1-forms $\eta_i \in J$ and $\xi \in \Lambda_2(\mathcal{P})$.

It is now clear that, if the rank of β is locally constant around some $p \in \mathcal{P}$, then p is a nonsingular point of $J^{(1)}$

and $\dim J^{(1)}(q)$ is equal to $\dim\ker\beta(q)$ for q in some open neighborhood of p. Note also that J is involutive if and only if $\beta = 0$.

The rank of the matrix $\beta(q)$ in a point $q \in U$ does not depend on the particular basis $\eta = \{\eta_1, \ldots, \eta_\delta\}$ for T^*U chosen, since a different choice of basis, say $\tilde{\eta}$, will produce a matrix $\tilde{\beta} = A\beta B$ where $A(p), B(p)$ are nonsingular for p in an open neighborhood of q. So the matrix β is "an intrinsic" object.

It is easy to show from equation (2.4) that, if q is a regular point of $J^{(1)}$ then, for any p in an open neighborhood V of q, we have $J^{(1)} = \operatorname{span}\left\{\omega(p) | \omega \in \tilde{J}\right\}$. In other words, apart from singular points of $J^{(1)}$, the equations (2.1) and (2.5) define the same codistribution.

The following useful result is known as "Cartan Lemma" ((Warner, 1971), p.80 ex. 16). Let $\omega_1, \ldots, \omega_r \in \Lambda_1(\mathcal{P})$ be independent pointwise. Assume that there exist one forms $\eta_1, \ldots, \eta_r$ such that $\sum_{i=1}^r \eta_i \wedge \omega_i = 0$. Then there exist functions $a_{ij} \in C^\infty(\mathcal{P})$, with $a_{ij} = a_{ji}$, such that $\eta_i = \sum_{j=1}^r a_{ij}\omega_j \quad (i = 1, \ldots, r)$. The same result is also valid pointwise, *i.e.* , $\sum_{i=1}^r \eta_i \wedge \omega_i\big|_p = 0$ implies that $\eta_i(p) = \sum_{j=1}^r a_{ij}\omega_j(p) \ \ (i = 1, \ldots, r)$ for convenient $a_{ij} = a_{ji} \in I\!\!R$.

3 Linear Systems

We open this section by showing that we can always construct a decomposition of linear control system in the controllable and noncontrollable subsystems in such a way that the influence of the noncontrollable subsystem in the controllable subsystem can be canceled by a convenient static-state feedback. This result can be deduced from theorem 4.1 of (Morse, 1973) when $C = 0$. However, it seems interesting to give an elementary proof of this fact.

Theorem 1 *(Pereira da Silva, 1996b) Given a linear system (A, B) which is not completely controllable, there exists a similarity transformation T and a regular static-state feedback $u = Fx + Gv$, where G is a nonsingular square matrix, such that*

$$T^{-1}(A + BF)T = \begin{pmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{pmatrix} \qquad (3.1a)$$

$$T^{-1}BG = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \qquad (3.1b)$$

where the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable and is in the Brunovsky canonical form (Brunovsky, 1970).

Proof. Let $\mathcal{B}_\mathcal{R} = \{r_1, \ldots, r_k\}$ be a basis[3] for $\mathcal{R} = \operatorname{Im}\left[B\ AB \ldots A^{n-1}B\right]$ and complete this basis to a basis $\mathcal{B}_\mathcal{X} = \{r_1, \ldots, r_k, \ldots, \hat{x}_1, \ldots, \hat{x}_{n-k}\}$ for the entire state space $\mathcal{X}$. It is well known that, when written in the basis $B_\mathcal{X}$, the pair (A, B) is of the form (1.1a)–(1.1b) and the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable. Up to an application of a convenient regular static-state feedback, we can assume that this pair is in the Brunovsky canonical form *i.e.* , the equations of the system in a new basis are of the form (see (Brunovsky, 1970)) :

$$\begin{aligned} \dot{z}^i_j &= z^i_{j+1} + \alpha^i_j z^{m+1} \quad j \in \lfloor \kappa_i - 1 \rceil && (3.2a) \\ \dot{z}^i_{\kappa_i} &= \alpha^i_{\kappa_i} z^{m+1} + v_i && (3.2b) \\ \dot{z}^{m+1} &= \tilde{A}_{22} z^{m+1} && (3.2c) \end{aligned}$$

where $i \in \lfloor m \rceil$, $z = \left((z^1)^T, \ldots, (z^m)^T, (z^{m+1})^T\right)^T$ is the state vector, $z^i = \left(z^i_1, \ldots,\ z^i_{\kappa_i}\right)^T \in I\!\!R^{\kappa_i}, i \in \lfloor m \rceil$, $z^{m+1} \in I\!\!R^{(n-k)}$, $v = (v_1, \ldots, v_m)^T$ is the input, and $\alpha^i_j \in I\!\!R^{1\times(n-k)}$, $j \in \lfloor \kappa_i \rceil$ are convenient row vectors.

Let $\xi^i_1 = z^i_1$ and set $\xi^i_{j+1} = \dot{\xi}^i_j$ for $i \in \lfloor m \rceil$ and $j \in \lfloor \kappa_i - 1 \rceil$. Denote the vectors $(\xi^i_1, \ldots, \xi^i_{\kappa_i})^T$ by ξ^i for $i \in \lfloor m \rceil$ and set $\xi^{m+1} = z^{m+1}$. Then using (3.2a) and (3.2c) it is easy to verify that

$$\xi^i_j = z^i_j + \beta^i_j z^{m+1} \quad : j \in \lfloor \kappa_i \rceil \qquad (3.3)$$

for convenient row vectors $\beta^i_j \in I\!\!R^{1\times(n-k)}$ and $i \in \lfloor m \rceil$. By the definition of the ξ^i_j's, from (3.3) with $j = \kappa_i$ and from (3.2b) and (3.2c), we can write for $i \in \lfloor m \rceil$:

$$\begin{aligned} \dot{\xi}^i_j &= \xi^i_{j+1} \quad : j \in \lfloor \kappa_i - 1 \rceil && (3.4a) \\ \dot{\xi}^i_{\kappa_i} &= \gamma^i \xi^{m+1} + v_i && (3.4b) \\ \dot{\xi}^{m+1} &= \tilde{A}_{22}\xi^{m+1} && (3.4c) \end{aligned}$$

for convenient row vectors $\gamma^i \in I\!\!R^{1\times(n-k)}$ and $i \in \lfloor m \rceil$.

Let $T \in I\!\!R^{n\times n}$ be the square matrix such that $\xi = Tz$ where $\xi = \left((\xi^1)^T, \ldots,\ (\xi^m)^T,\ (\xi^{m+1})^T\right)^T$ and $z = \left((z^1)^T, \ldots, (z^m)^T, (z^{m+1})^T\right)^T$. From (3.3) and from the fact that $\xi^{m+1} = z^{m+1}$ it is clear that the matrix T is block triangular and the blocks of the diagonal are identity matrices of adequate dimension. In particular, T is nonsingular. The proof may be completed by noting from equation (3.4b) that the static state feedback $v_i = -\gamma^i\xi^{m+1} + \nu_i$, $i \in \lfloor m \rceil$, where $\nu = (\nu_1, \ldots, \nu_m)^T$ is the new input, furnishes a closed loop system with state ξ and ouput ν of the form (3.1a)-(3.1b). □

The theorem 1 may be restated using the terminology of the geometric approach of (Wonham, 1985), in the following form :

Corollary 1 *(Pereira da Silva, 1996b) Let (A, B) be a noncontrollable linear system and denote its state space by $\mathcal{X}$. Let $\mathcal{R} = \operatorname{Im}B + \operatorname{Im}AB + \ldots + \operatorname{Im}A^{n-1}B$. Then there exists an (A, B)-invariant subspace $\hat{\mathcal{R}}$ such that $\mathcal{X} = \mathcal{R} \oplus \hat{\mathcal{R}}$.*

Remark 1 *In the module-theoretic approach of Fliess (Fliess, 1990), (Fliess, 1994), a time-invariant linear system is by definition a $I\!\!R[\frac{d}{dt}]$-module Λ. Since Λ is a principal ideal domain, it can be decomposed as a direct sum $\Lambda = L \oplus T$ where L is a free module and T is a torsion*

[3] Recall that $\mathcal{R}$ is feedback invariant (Wonham, 1985).

module. The submodule L plays the role of its controllable subsystem and T the noncontrollable one. The direct sum of this decomposition means that L and T are completely independent from each other. So this can be considered as a very elegant version of the theorem 1 under endogenous feedback transformations *(Fliess et al., 1995b), (Fliess et al., 1992). However, some extra work, already done in the proof of theorem 1, is necessary to show that the same result is valid when one considers only the class of regular static-state feedback transformations.*

Remark 2 *A system is controllable if and only if it does not have conservation laws*[4] *(see (Fliess et al., 1993)). Note that every noncontrollable nonlinear system of the form* (1.1a)-(1.1b) *have conservation laws of the form* $w^2 = e^{-\tilde{A}_{22}t}z^2$. *In fact, since* $e^{-\tilde{A}_{22}t}\tilde{A}_{22} = \tilde{A}_{22}e^{-\tilde{A}_{22}t}$ *note that* $\frac{d}{dt}w^2 = 0$. *Note also that the transformation* $w = T(t)z$ *such that :*

$$\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & e^{-\tilde{A}_{22}t} \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$$

can be considered as a time-varying similarity transformation. We can write

$$\begin{aligned} \dot{w}^1 &= \tilde{A}_{11}w^1 + \tilde{A}_{12}e^{\tilde{A}_{22}t}w^2 + B_1 u && (3.5a) \\ \dot{w}^2 &= 0 && (3.5b) \end{aligned}$$

We will show that a general nonlinear system can be always decomposed in similar way.

4 The Partial Linearization Problem

A (time varying) nonlinear system is a set of differential equations of the form[5]

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x}_i(t) &= f_i(t, x_1(t), \ldots, x_n(t), u(t)) \ , \quad i \in \lfloor n \rceil \end{aligned} \tag{4.1}$$

where $f_i : I\!R \times \mathcal{X} \times \mathcal{U} \mapsto I\!R$ are smooth functions of the state $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathcal{X}$ and the input $u(t) = (u_1(t), \ldots, u_m(t)) \in \mathcal{U} \cong I\!R^m$ for $i \in \lfloor n \rceil$. For simplicity, we shall consider that the state $x(t) = (x_1(t), \ldots, x_n(t))$ evolves in[6] an open subset $\mathcal{X}$ of $I\!R^n$.

Let $M = I\!R \times \mathcal{X}$ and $\mathcal{N} = I\!R \times \mathcal{X} \times \mathcal{U}$ be smooth manifolds with canonical coordinates given respectively by (t, x) and (t, x, u). Let f be the *Cartan* field on $\mathcal{N}$ given by

$$\frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(t, x, u)\frac{\partial}{x_i} \tag{4.2}$$

By definition, $\langle dt, f \rangle \equiv 1$. Now, let $\pi : \mathcal{N} \mapsto M$ be the canonical projection. Using the coordinates (t, x, u) for $\mathcal{N}$ and (t, x) for M we have $\pi(t, x, u) = (t, x)$. Let $p : (-\epsilon, \epsilon) \mapsto \mathcal{N}$ be a smooth curve on $\mathcal{N}$. Denote, as usual[7], $(\pi \circ p(t))_*(d/dt)$ by $\dot{\pi}(p(t))$. Notice that the equations (4.1) corresponds to the differential equation given by $\dot{\pi}(p(t)) = \pi_* f(p)$, where $p \in \mathcal{N}$, when written with the same choice of coordinates for $\mathcal{N}$ and M above.

Definition 1 *A (time varying)* state transformation *is a new local chart of* $M = I\!R \times \mathcal{X}$ *of the form* (t, z). *A (local)* static state feedback *transformation, or (local)* feedback transformation *for short, is a new local chart of* $\mathcal{N} = I\!R \times \mathcal{X} \times \mathcal{U}$ *of the form* (t, z, v) *where* (t, z) *is a (local) state transformation.*

Note that the last definition is a notion of regular feedback, since (locally) there exists smooth maps $u = \alpha(t, z, v)$ and $v = \beta(t, x, u)$. The differential equation $\dot{\pi}(p(t)) = \pi_* f(p(t))$, written in the coordinates (t, z, v) for $\mathcal{N}$ and (t, z) for M is called the closed loop system. For instance, when $z = x$, we have that the the closed loop system is given by the equations

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x}_i(t) &= f_i(t, x_1(t), \ldots, x_n(t), \alpha(t, x(t), v(t))) \ , \quad i \in \lfloor n \rceil \end{aligned} \tag{4.3}$$

We stress that our notion of state feedback corresponds to a change of coordinates of $\mathcal{N}$. Hence the Cartan field f and all the notions related to f are feedback-invariant by definition. It is important to point out that our notions of state transformation and static-state feedback are time-varying and local around some operation point (t_0, x_0, u_0). In particular, these definitions are also local in time.

We are now able to state our version of the the problem of exact linearization :

Definition 2 (Partial Linearization Problem – PLP) *Giving a nonlinear system, the* PLP *is the problem of finding a local feedback transformation* $(t, (w, z), v)$ *in a such a way that the closed loop system, locally around an operation point* (t_0, x_0, u_0) *be of the form :*

$$\begin{aligned} \dot{t} &= 1 \\ \dot{w}(t) &= A_{11}w(t) + B_1 v(t) + B_2 y(t) \\ \dot{z}(t) &= \mathcal{A}_{22}(t, z(t)) \\ y(t) &= \mathcal{A}_{12}(t, z(t)) \end{aligned} \tag{4.4}$$

where $(w(t), z(t)) \in I\!R^n$, $v(t) \in I\!R^m$, $y(t) \in I\!R^s$, A_{11} B_1 *and* B_2 *are real matrices of convenient dimensions*[8]*, the pair* (A_{11}, B_1) *is controllable and* $\mathcal{A}_{12}$ *and* $\mathcal{A}_{22}$ *are smooth mappings depending on* t *and* z.

Note that the equation $\dot{z}(t) = \mathcal{A}_{22}(t, z(t))$ plays the role of a (perhaps nonlinear) noncontrollable subsystem. The vector $y(t)$ can be considered as an output of the noncontrollable subsystem and as an input for the linear subsystem.

[4] A conservation law here means a *first integral*.

[5] Since our aim here is to develop a canonical form under a class of transformations, we do not care about the functional space of the input $u(\cdot)$.

[6] Since we will develop local results, one may consider that $x(t)$ evolves on a smooth manifold without any problem.

[7] See def. 1.41, p. 35 of (Warner, 1971).

[8] There is no loss of generality, by choosing $s = n$, if one considers B_2 equal to the indentity matrix.

5 Solution of the PLP — time varying systems

It is well known from the literature that the derived flag is a geometric invariant of a nonlinear system that is closed related to the exact linearization problem. The derived flag is a sequence of smooth codistributions on $\mathcal{N}$ that can be defined in the following way[9].

$$I^{(-1)} = \operatorname{span}\{f\}^{\perp} \tag{5.1a}$$
$$I^{(0)} = I^{(-1)} \cap \operatorname{span}\{dt, dx\} \tag{5.1b}$$
$$I^{(i)}(p) = \operatorname{span}\Big\{\omega(p) \in I^{(i-1)} | \omega \in I^{(i-1)} \text{ and } d\omega(p) \bmod I^{(i-1)}(p) \equiv 0\Big\} \quad i \geq 1. \tag{5.1c}$$

Note that equation (5.1c) corresponds to the successive applications of (2.5) with $J = I^{(i-1)}$ and $J^{(1)} = I^{(i)}$ Note also that $I^{(-1)}$ is of dimension $n+m$, since is has codimension 1 and the manifold $\mathcal{N}$ is of dimension $n+m+1$. Furthermore it is clear that for any function θ in $C^\infty(\mathcal{N})$, then the 1-form $\omega = (d\theta - L_f\theta dt)$ is in $I^{(-1)}$. Hence it is easy to verify that

$$\begin{aligned} I^{(-1)} &= \operatorname{span}\{(dx_i - f_i(t,x,u)dt), du_j : i \in \lfloor n \rceil, j \in \lfloor m \rceil\} \\ I^{(0)} &= \operatorname{span}\{(dx_i - f_i(t,x,u)dt) : i \in \lfloor n \rceil\} \end{aligned}$$

Notice that $I^{(0)}$ coincides with the definition of the Pfaffian system derived from a given nonlinear system (Gardner and Shadwick, 1992), (Sluis, 1992), (Tilbury et al., 1995). We stress that, by definition, $I^{(k+1)} \subset I^{(k)}$. If the codistributions $I^{(k)}$ are all nonsingular for $k \in I\!N$, then there exists some N, called the holonomy index, such that $I^{(N)} = I^{(N+1)}$. The codistribution $I^{(N)}$ is called the bottom derived system and it is easy to show that $I^{(N)}$ is the maximal involutive codistribution contained in $I^{(0)}$. If $I^{(N)} = 0$, the system is said to be *maximally nonholonomic*. In this paper we will consider the case where $I^{(N)}$ is non trivial, *i.e.* , the system is not controllable[10] (Aranda-Bricaire et al., 1995), (Tilbury et al., 1995).

The system is said to be *well formed* if all the inputs affect the system independently, *i.e.* , that $\frac{\partial f}{\partial u}$ has rank m where f is the column vector of functions f_i of (4.1), that represents the Cartan field written in coordinates. It is easy to show that this is equivalent to say that

$$\operatorname{span}\{dt, dx_i, df_i : i \in \lfloor n \rceil\} = \operatorname{span}\{dt\} + I^{(-1)} \tag{5.2}$$

or equivalently[11]

$$I^{(0)} + L_f I^{(0)} = I^{(-1)} \tag{5.3}$$

[9] Here we give a geometric definition that is slightly different from the one given in the literature. Note that $\operatorname{span}\{f\}^{\perp}$ denotes the kernel of the morphism of $C^\infty(\mathcal{N})$-modules $\alpha : T^*\mathcal{N} \mapsto C^\infty(\mathcal{N})$ such that $\alpha(\omega) = \langle f, \omega\rangle$, and hence is a codistribution.

[10] Since $I^{(N)}$ is involutive, locally around its regular points it is spanned by exact covector fields $\{d\theta_i : i = 1,\dots,k\}$. Since, $I^{(N)} \subset I^{(-1)} = \operatorname{span}\{f\}^{\perp}$ we have $\langle d\theta_i, f\rangle = 0$ for $i = 1,\dots,k$. Hence $\dot\theta_i = \langle d\theta_i, f\rangle = 0$ and so the system cannot be strongly controllable because it possesses a nontrivial conservation law (Fliess et al., 1993).

[11] The equation obtained from (5.3) by summing, in both sides, the codistribution $\operatorname{span}\{dt\}$ is easily seen to be equivalent to the equation (5.2). After that, one obtains (5.3) by observing that $L_f I^{(-1)} \subset I^{(-1)}$ and $I^{(-1)} \cap \operatorname{span}\{dt\} = \{0\}$ (see lemma 1, part (ii) in the Appendix).

If we assume that the matrix B_1 in the equation (4.4) has full column rank m, it follows easily that (4.4) is well formed. So we shall assume, without loss of generality that the system is well formed, since it is a necessary condition to solve the PLP.

We can assume that, after some feedback transformation, the pair (A_{11}, B_1) in the equation (4.4) is given in a Brunovsky canonical form, *i.e.* , this system can be rewritten in the form[12]

$$\begin{cases} \dot t = 1 \\ \dot w^1_{k,i_k} = w^2_{k,i_k} + \phi^1_{k,i_k}(t,z) \\ \dot w^2_{k,i_k} = w^3_{k,i_k} + \phi^2_{k,i_k}(t,z) \\ \quad\vdots \\ \dot w^{\tilde N-k+1}_{k,i_k} = v_{k,i_k} + \phi^{\tilde N-k+1}_{k,i_k}(t,z) \\ \dot z = \mathcal{A}_{22}(t,z) \end{cases} \tag{5.4}$$

where $k \in \lfloor \tilde N \rceil$ and $i_k \in \lfloor s_{\tilde N-k} \rceil$ for some integer $\tilde N$ for which $\sum_{k=1}^{\tilde N} s_{\tilde N-k} = m$ and the integers $s_{\tilde N-k} \geq 0$. Computing the derived flag for the system (5.4) is a tedious but straightforward work and it is easy to verify that

$$\begin{aligned} \operatorname{span}\{dt\} + I^{(\tilde N+r)} &= \operatorname{span}\{dt, dz\}\ , \quad r \geq 0 \\ \operatorname{span}\{dt\} + I^{(\tilde N-r)} &= \Big\{dt, dz, dw^j_{k,i_k} \mid k \in \lfloor r \rceil, \\ &\qquad i_k \in \lfloor s_{\tilde N-k} \rceil, j \in \lfloor r-k+1 \rceil\Big\}, \\ &\qquad 0 < r \leq \tilde N \end{aligned}$$

In particular the codistributions $\operatorname{span}\{I^{(k)}, dt\}$ are nonsingular and involutive for all $k \in I\!N$. These are in fact the necessary and sufficient conditions for the solution of the PLP.

Theorem 2 *The PLP is locally solvable around an operation point (t_0, x_0, u_0) if and only if the codistributions* $\operatorname{span}\{I^{(k)}, dt\}$ *are nonsingular and involutive for all $k \in I\!N$ in some open neighborhood of (t_0, x_0, u_0).*

To prove the theorem 2 we need some auxiliary results that are presented in the following proposition. For convenience, their proofs are deferred to the Appendix.

Proposition 1 *Under the conditions of the theorem 2, we have :*

(i) *For all $k \in I\!N$ there exists a set of pointwise independent covector fields $\{\omega_1,\dots,\omega_{r_k}\}$ where $\omega_i = (d\theta_i - L_f\theta_i dt)$, with $\theta_i \in C^\infty(\mathcal{N})$, such that we locally have $I^{(k)} =$* $\operatorname{span}\{\omega_1,\dots,\omega_{r_k}\}$.

(ii) *If ω is of the form $(d\theta - L_f\theta dt)$ for a function $\theta \in C^\infty(\mathcal{N})$, then $\omega \in I^{(k+1)}$ if and only if $L_f\omega \in I^{(k)}$. In particular $I^{(k)} \supset I^{(k+1)} + L_f I^{(k+1)}$.*

(iii) *Let $\{\omega_1,\dots,\omega_r\} \subset I^{(k-1)}$ be a set of 1-forms such that $\omega_i = (d\theta_i - L_f\theta_i dt)$, where $\theta_i \in C^\infty(\mathcal{N})$. Assume*

[12] The choice of the notation used to represent the Brunovsky canonical form here is compatible with the one of the proof of theorem 2.

that the set $\{\omega_1(p), \ldots, \omega_r(p)\}$ is linearly independent[13] mod $I^{(k)}(p)$. *Then* $\{L_f\omega_1(p), \ldots, L_f\omega_r(p)\} \subset I^{(k-2)}(p)$ *is linearly independent* mod $I^{(k-1)}(p)$.

Proof. *(Of theorem 2)* We need only to prove the sufficiency.

Let $N \in \mathbb{N}$ be the smallest integer such that $I^{(k)} = I^{(k+1)}$ for all $k \geq N$. Since span $\{I^{(N)}, dt\}$ is involutive, by the Frobënius theorem there exists a set of functions $z = (z_1, \ldots, z_l) \subset C^\infty(\mathcal{N})$ such that $\{dz_1, \ldots, dz_l, dt\}$ is a basis for span $\{I^{(N)}, dt\}$. By proposition 1 part (i) we have that the set

$$\mathcal{B}_N = \{dz_i - L_f z_i dt \mid i \in \lfloor l \rceil\}$$

is a basis for $I^{(N)}$. By proposition 1 part (ii), note that $L_f(dz_i - L_f z_i dt) \in I^{(N)} = I^{(N+1)}$. Then, $dL_f z_i \in$ span $\{I^{(N)}, dt\}$ and we can (locally) write

$$\dot{z}_i = \mathcal{A}^i_{22}(t, z_1, \ldots, z_l)$$

for convenient smooth functions $\mathcal{A}^i_{22}$ defined on an open neighborhood of (t_0, x_0, u_0). By proposition 1 part (i), we can complete the set $\mathcal{B}_N$ to a basis $\mathcal{B}_{N-1}$ for $I^{(N-1)}$ of the form

$$\begin{aligned}\mathcal{B}_{N-1} \;=\; & \{(dz_i - L_f z_i dt), (d\theta_{1,i_1} - L_f\theta_{1,i_1}dt) \mid i \in \lfloor l \rceil, \\ & i_1 \in \lfloor s_{N-1} \rceil\}\end{aligned}$$

By proposition 1 parts (ii) and (iii) it is clear that the set

$$\begin{aligned}&\{(dz_i - L_f z_i dt), (d\theta_{1,i_1} - L_f\theta_{1,i_1}dt), (dL_f\theta_{1,i_1} - L_f^2\theta_{1,i_1}dt) \mid \\ & i \in \lfloor l \rceil, i_1 \in \lfloor s_{N-1} \rceil\}\end{aligned}$$

is in $I^{(N-2)}$ and is linearly independent. So, by proposition 1 part (i), the last set may be completed to a basis

$$\begin{aligned}\mathcal{B}_{N-2} \;=\; & \{(dz_i - L_f z_i dt), (d\theta_{1,i_1} - L_f\theta_{1,i_1}dt), \\ & (dL_f\theta_{1,i_1} - L_f^2\theta_{1,i_1}dt), (d\theta_{2,i_2} - L_f\theta_{2,i_2}dt) \mid \\ & i \in \lfloor l \rceil, i_1 \in \lfloor s_{N-1} \rceil, i_2 \in \lfloor s_{N-2} \rceil\}\end{aligned}$$

for $I^{(N-2)}$. Continuing in this way, we will construct in the r-th step a basis for $I^{(N-r)}$ of the form

$$\begin{aligned}\mathcal{B}_{N-r} \;=\; & \{(dz_i - L_f z_i dt), (dL_f^{j-1}\theta_{k,i_k} - L_f^j\theta_{k,i_k}dt) \mid \\ & i \in \lfloor l \rceil, k \in \lfloor r \rceil, i_k \in \lfloor s_{N-k} \rceil, j \in \lfloor r-k+1 \rceil\}\end{aligned} \tag{5.5}$$

In particular, it follows for $r = N$ that $\dim I^{(0)} = l + \sum_{k=1}^N (N-k+1)s_{N-k}$. Note that $L_f(d\theta - L_f\theta dt) \in I^{(-1)}$ for any smooth function θ. Hence, from the equation (5.3), we conclude that

$$\begin{aligned}I^{(-1)} \;=\; & \mathrm{span}\{\mathcal{B}_0, (dL_f^{N-k+1}\theta_{k,i_k} - L_f^{N-k+2}\theta_{k,i_k}dt), \\ & k \in \lfloor N \rceil, i_k \in \lfloor s_{N-k} \rceil\}\end{aligned} \tag{5.6}$$

where $\mathcal{B}_0$ is a basis for $I^{(0)}$ of the form (5.5) for $r = N$. On the other hand the set

$$\left\{(dL_f^{N-k+1}\theta_{k,i_k} - L_f^{N-k+2}\theta_{k,i_k}dt), k \in \lfloor N \rceil, i_k \in \lfloor s_{N-k} \rceil\right\}$$

must be independent mod $I^{(0)}$, otherwise, by proposition 1 part (iii) the set

$$\left\{(dL_f^{N-k}\theta_{k,i_k} - L_f^{N-k+1}\theta_{k,i_k}dt), k \in \lfloor N \rceil, i_k \in \lfloor s_{N-k} \rceil\right\}$$

which is a subset of $\mathcal{B}_0$, would be dependent mod $I^{(1)}$. We conclude that the set on the left hand side of equation (5.6) is a basis for $I^{(-1)}$. In particular we have $\dim I^{(-1)} = \dim I^{(0)} + \sum_{k=1}^N s_{N-k}$ Since $\dim I^{(0)} = n$ and $\dim I^{(-1)} = n + m$, we conclude that $\sum_{k=1}^N s_{N-k} = m$. Now define the set of m (input) functions

$$v = \left\{v_{k,i_k} \mid v_{k,i_k} = L_f^{N-k+1}\theta_{k,i_k} : k \in \lfloor N \rceil, i_k \in \lfloor s_{N-k} \rceil\right\}$$

and the set of (state) functions (w, z), where

$$\begin{aligned} w \;=\; & \{w^j_{k,i_k} = L_f^{j-1}\theta_{k,i_k} : k \in \lfloor N \rceil, \\ & i_k \in \lfloor s_{N-k} \rceil, j \in \lfloor N-k+1 \rceil\}\end{aligned}$$

and $z = (z_1, \ldots, z_l)$. By construction of $\mathcal{B}_0$ (see eq. (5.5) for $r = N$) it is clear that the functions $(t, (w, z))$ form a new local chart for M and the functions $(t, (w, z), v)$ form a new local chart for $\mathcal{N}$. So, these local charts define a regular static-state feedback transformation and in these coordinates the system (4.1) reads :

$$\begin{aligned}&\dot{t} = 1 \\ &\left\{\begin{array}{rcl}\dot{w}^1_{k,i_k} &=& w^2_{k,i_k} \\ \dot{w}^2_{k,i_k} &=& w^3_{k,i_k} \\ &\vdots& \\ \dot{w}^{N-k+1}_{k,i_k} &=& v_{k,i_k}\end{array}\right. \quad , k \in \lfloor N \rceil, \; i_k \in \lfloor s_{N-k} \rceil \\ &\dot{z} = \mathcal{A}_{22}(t, z)\end{aligned} \tag{5.7}$$

□

Note that, if the PLP is solvable, the equation (5.7) means that the influence of the noncontrollable subsystem (represented by the functions $\phi^j_{k,i_k}(t, z)$ in equation (5.4) or the map $\mathcal{A}_{12}(t, z)$ in the equation (4.4)) can be removed by a convenient choice of a static-state feedback transformation.

Example 2 *Consider the equations of* spacecraft attitude control *(Nijmeijer and van der Schaft, 1990), (Crouch, 1984) :*

$$\dot{A} = AS(\varpi) \tag{5.8}$$

$$J\dot{\varpi} = -S(\varpi)J\varpi + u \tag{5.9}$$

where $A(t)$ is a 3×3 matrix with row vectors given by $a_i = (a_{i_1}, a_{i_2}, a_{i_3})^T, i \in \{1, 2, 3\}$, J is the inertia matrix, which is symmetric and positive definite, $\varpi = (\varpi_1, \varpi_2, \varpi_3)^T$ is the angular velocity, $u \in \mathbb{R}^3$ is the input (control torques), and $S(\varpi)$ is the skew-symmetric matrix

$$S(\varpi) = \begin{bmatrix} 0 & -\varpi_3 & \varpi_2 \\ \varpi_3 & 0 & -\varpi_1 \\ -\varpi_2 & \varpi_1 & 0 \end{bmatrix}.$$

Note that $\frac{d}{dt}(AA^T) = \dot{A}A^T + A\dot{A}^T = AS(\varpi)A^T + AS(\varpi)^T A^T = 0$. Hence, if the initial condition $A(t_0)$ is an orthogonal matrix (this is true for the real problem

[13] The linear independence of the set $\{\omega_1(p), \ldots, \omega_r(p)\}$ mod $I^{(k)}(p)$ for some $p \in \mathcal{N}$ means that $\left(\sum_{i=1}^r \alpha_i\omega_i(p) + \omega(p)\right)\Big|_p = 0$ for $\omega \in I^{(k)}$ and $\alpha_i \in \mathbb{R}$ implies that $\omega(p) = 0$ and $\alpha_i = 0$.

of attitude control), then $A(t)$ will be orthogonal for all $t \geq t_0$. Let us consider the equations (5.8) – (5.9) from a formal point of view, i.e. , we will not assume that the initial condition $A(t_0)$ is orthogonal. So the state vector is $(a_{1_1}, \ldots, a_{3_3}, \varpi_1, \varpi_2, \varpi_3)$, being of dimension 12. Let $\eta_{ij} = a_i a_j{}^T$, $i \in \{1,2,3\}, j \in \{1,\ldots,i\}$. Note that $\dot\eta_i = 0$.

Calculating the derived flag for this system one obtains

$$
\begin{aligned}
I^{(0)} &= \text{span}\left\{(da_{i_j} - \dot a_{i_j} dt), (d\varpi_i - \dot\varpi_i dt) | i,j \in \{1,2,3\}\right\} \\
I^{(1)} &= \text{span}\left\{(da_{i_j} - \dot a_{i_j} dt) |\ i,j \in \{1,2,3\}\right\} \\
I^{(2)} &= \text{span}\left\{d\eta_{ij} \mid i \in \{1,2,3\}, j \in \{1,\ldots,i\}\right\} \\
&= I^{(3)}
\end{aligned}
$$

Generically (for every point for which A is nonsingular), we have $\dim I^{(0)} = 12$, $\dim I^{(1)} = 9$, $\dim I^{(2)} = 6$. Furthermore we see that the distributions $\text{span}\left\{I^{(k)}, dt\right\}$ are spanned by exact covector fields and hence are involutive. From the proof of theorem 1, equation (5.7) and from the fact that $\dim I^{(r)} = \dim I^{(N)} + \sum_{k=1}^{N-r}(N-k+1)s_{N-k}$ we see that $s_1 = 3$ and $s_2 = 0$ and so, around any point such that A is a nonsingular matrix, this system has a canonical form given by

$$
\begin{array}{ll}
\dot t = 1 & \\
\left\{\begin{array}{rcl} \dot w_i^1 &=& w_i^2 \\ \dot w_i^2 &=& v_i \end{array}\right. & ,\ i \in \{1,2,3\} \\
\dot\eta_{ij} = 0 & ,\ i \in \{1,2,3\}, j \in \{1,\ldots,i\}
\end{array}
\tag{5.10}
$$

Remark 3 [14] *Example 2 may be considered using the differential algebraic approach of Fliess (Fliess, 1989). In fact let $\mathbf{k} = I\!R$ be the ground field and consider the system (5.8)-(5.9) denoted by $\mathcal{K}/\mathbf{k}$. Denote by $\mathcal{L}$ the subfield of $\mathcal{K}$ formed by the elements that are differentially algebraic over $\mathbf{k}$. The elements $\eta = \{\eta_{ij},\ (i = 1,2,3,\ j = 1,\ldots,i)\}$ are in $\mathcal{L}$. Furthermore, η is a (nondifferential) transcendence basis for $\mathcal{L}/\mathbf{k}$. Since $\mathcal{L}$ is not algebraic over $\mathbf{k}$, it is clear that the system $\mathcal{K}/\mathbf{k}$ is not flat (Fliess et al., 1992), (Fliess et al., 1995b). However if we take $\bar{\mathbf{k}} = \mathbf{k}\langle\eta\rangle$ as a new ground field, then $\mathcal{K}/\bar{\mathbf{k}}$ is a flat system. In fact, from the equation $AA^T = \eta$ it is easy to show that we can choose three elements $\alpha_1, \alpha_2, \alpha_3$ of the matrix A in a such way that all the other elements can be determined from η and α_i (this is a simple problem of Euclidian Geometry since the matrix η contains the information of the lenghts and angles between the row vectors of A). After that, it is easy to show that $\alpha_1, \alpha_2, \alpha_3$ is a flat output of the system $\mathcal{K}/\bar{\mathbf{k}}$.*

6 Solution of the PLP — time-invariant systems

For time invariant systems, *i.e.* , systems of the form (4.1) for which the functions f_i do not depend[15] on t, then it is easy to show that, if the PLP is solvable, the state transformation and the state feedback that solve the PLP are also time invariant (and in particular are *not local in time*). This means that *there is no gain in generality of seeking time-varying solutions for time-invariant systems.* Although this is not surprising, let us sketch the proof of this fact as a corollary of our previous results.

Examining the proof of theorem 2, it is easy to see that it suffices to show that the functions θ_i of proposition 1 part (i) are such that $d\theta_i \in \text{span}\{dx\}$. In fact we only have to prove the following result :

Proposition 2 *Assume that the system (4.1) is time-invariant. Then, under the conditions of the theorem 2, for all $k \in I\!N$ there exists a set of pointwise independent covector fields $\{\omega_1, \ldots, \omega_{r_k}\}$ where $\omega_i = (d\theta_i - L_f\theta_i dt)$, $\theta_i \in C^\infty(\mathcal{N})$, $\text{span}\{d\theta_i\} \subset \text{span}\{dx\}$ and we locally have $I^{(k)} = \text{span}\{\omega_1, \ldots, \omega_{r_k}\}$*

Proof. Assume that this is true for k, *i.e.* , $I^{(k)} = \text{span}\{d\theta_i - L_f\theta_i dt \mid i \in \lfloor r_k \rceil\}$, and $d\theta_i \in \text{span}\{dx\}$. As $dL_f\theta_i \in \text{span}\{dx, du\}$, in order to compute a basis $\mathcal{B} = \{\omega_i \mid i = 1, \ldots, r_{k+1}\}$ for $I^{(k+1)}$, we must solve the equation (2.4) in the case where the components of β are time invariant. Hence we can construct a basis $\mathcal{B}$ in such a way that

$$
\omega_i = \sum_{j=1}^{r_k} a_{ij}(d\theta_j - L_f\theta_j dt) \tag{6.1}
$$

where a_{ij} are time invariant smooth functions, *i.e.* , $da_{ij} \in \text{span}\{dx, du\}$.

Now define the canonical insertion $\iota_t : \mathcal{X}\times\mathcal{U} \mapsto I\!R\times\mathcal{X}\times\mathcal{U} = \mathcal{N}$ such that $\iota_t(x,u) = (t,x,u)$. As the "pull back" $\iota_t^* : \Lambda(\mathcal{N}) \mapsto \Lambda(\mathcal{X}\times\mathcal{U})$ is a map that commutes with the exterior derivation d (see proposition 2.23 of (Warner, 1971)), it follows that it maps involutive codistributions into involutive codistributions. By construction, for all $\xi \in \mathcal{X}\times\mathcal{U}$ we have that $\iota_t^*|_{\iota_t(\xi)} : T^*_{\iota_t(\xi)}\mathcal{N} \mapsto T^*_\xi(\mathcal{X}\times\mathcal{U})$ is a linear mapping with $\ker \iota_t^*|_{\iota_t(\xi)} = \text{span}\{dt\}|_p$. So, by lemma 1 part (ii) it is easy to show that ι_t^* maps $(I^{(k+1)} + \text{span}\{dt\})$ into a nonsingular codistribution. Let $\tilde I^{(k+1)} = \iota_t^*(I^{(k+1)} + \text{span}\{dt\})$. It follows that $\tilde I^{(k+1)}$ is integrable. Since we have a basis for $I^{(k+1)}$ of the form (6.1) with $da_{ij} \in \text{span}\{dx, du\}$ and $d\theta_i \in \text{span}\{dx\}$, it follows that $\tilde I^{(k+1)} \subset \text{span}\{dx\}$ and $\tilde I^{(k+1)}$ is time invariant, *i.e.* , $\tilde I^{(k+1)} = \iota_{t_1}^*(I^{(k+1)} + \text{span}\{dt\}) = \iota_{t_2}^*(I^{(k+1)} + \text{span}\{dt\})$ for all $t_1, t_2 \in I\!R$. By the Frobënius theorem, there exists smooth functions $\tilde\gamma_i, (i = 1, \ldots r_{k+1})$ such that we locally have $\tilde I^{(k+1)} = \text{span}\left\{d\tilde\gamma_1, \ldots, d\tilde\gamma_{r_{k+1}}\right\}$. Let $\tau : I\!R\times\mathcal{X}\times\mathcal{U} \mapsto \mathcal{X}\times\mathcal{U}$ be the canonical projection. Let $\gamma_i = \tau^*\tilde\gamma_i = \tilde\gamma_i \circ \tau$. Then it is easy to show that $\left\{d\gamma_i, \ldots d\gamma_{r_{k+1}}, dt\right\}$ is a local basis for $(I^{(k+1)} + \text{span}\{dt\})$. The application of the idea of the proof of proposition 1 part (i) shows the desired result for $k+1$ and this finishes the proof. □

7 Controllability and conservation laws

As claimed in the remark 2 of section 3, a reasonable definition of controllability is to say that a system does not possess any conservation law (Fliess et al., 1993).

[14] The subject of this remark is based in a conversation between Michel Fliess, Michel Petitot and me in the Laboratoire des Signaux et Systémes CNRS – Gif-sur-Yvette, France in 1995.

[15] Let $\tau : I\!R \times \mathcal{X}\times\mathcal{U} \mapsto \mathcal{X}\times\mathcal{U}$ be the canonical projection. Then f is time invariant if and only if f is τ-related with some field $\bar f$ on $\mathcal{X}\times\mathcal{U}$.

Assume that a system is given in the form of its state representation (4.1). A conservation law is a function $\phi(t, x, u, \dot{u}, \ldots, u^{(k)})$ such that $\frac{d}{dt}\phi = \frac{\partial\phi}{\partial t} + \sum_{i=1}^{n} \frac{\partial\phi}{\partial x_i} f_i(t, x, u) + \sum_{i=1}^{m}\sum_{j=0}^{k} \frac{\partial\phi}{\partial u_i^{(j)}} u^{(j+1)} = 0$ for any admissible trajectory of the system. Assuming that the inputs are differentially independent, which is a quite natural asumption (Fliess et al., 1993), (Fliess, 1989) it is not difficult to show that there is no conservation law depending on the input u or on its derivatives of any order. Hence, any conservation law of a system in the form (4.1) is a function $\phi(t, x)$, depending only on time and state. The next proposition shows that the codistribution $I^{(N)}$ is spanned by the differentials of the conservation laws.

Proposition 3 *Consider a system* (4.1) *for which* $I^{(N)}$ *of section 5 is well defined and is nonsingular. Then, the function* ϕ *is a conservation law for the system 4.1 if and only if* $d\phi$ *is in* $I^{(N)}$.

Proof. Since $I^{(N)} \subset I^{(-1)} = \mathrm{span}\,\{f\}^{\perp}$, then if $d\phi \in I^{(N)}$ we see that $\dot{\phi} = \langle d\phi, f\rangle = 0$. Now assume $\dot{\phi} = L_f\phi = 0$ Then by proposition 2 part (iii) it follows easily that $d\phi = d\phi - L_f\phi dt$ is such that $d\phi$ is in $I^{(j)}$ for $j = 0, 1, 2, \ldots$. In particular $d\phi \in I^{(N)}$. As $I^{(N)}$ is integrable, it follows that it is spanned by the differentials of the conservation laws. $\square$

Proposition 4 *Consider a system* (4.1) *for which* $I^{(N)}$ *of section 5 is well defined and is nonsingular. Then there exists a local state transformation* $(t, (z^1, z^2))$ *in a way that the system written in these coordinates reads*

$$\begin{array}{rcl} \dot{t} & = & 1 \\ \dot{z}^1(t) & = & g(t, z^1(t), z^2(t), u(t)) \\ \dot{z}^2(t) & = & 0 \end{array} \tag{7.1}$$

Furthermore there is no conservation law depending on z^1. *In other words, for each initial condition* $z_0^2 = z^2(t_0)$, *the system*

$$\begin{array}{rcl} \dot{t} & = & 1 \\ \dot{z}^1(t) & = & g(t, z^1(t), z_0^2, u(t)) \end{array} \tag{7.2}$$

is controllable.

Proof. By the last proposition, there exists a set $z^2 = \{z_1^2, \ldots, z_l^2\}$ of conservation laws in a way that $I^{(N)} = \mathrm{span}\,\{dz_1^2, \ldots, dz_l^2\}$. We can complete, by choosing a convenient subset $z^1 = \{x_{i_1}, \ldots, x_{i_{n-l}}\}$ of the state functions, the set $\{dt, dz^2\}$ to a basis $\{dt, dz^2, dz^1\}$ of span $\{dt, dx\}$. It is clear that (t, z^1, z^2) is a local state transformation having the claimed properties. $\square$

Remark 4 *Applying the proposition 4 to any autonomous system* $\dot{\xi}(t) = \mathcal{A}_{22}\ (t, \xi(t))$, *one can always find a state-transformation* (t, z) *such that the corresponding equations are of the form* $\dot{z} = 0$ *In fact, it is easy to see that, in this case* $I^{(0)} = I^{(k)}$ *are involutive and nonsingular for all* $k \in \mathbb{N}$. *Applying this ideas to the noncontrollable subsystem of* (5.7), *if the PLP is solvable, we obtain a canonical form given by*

$$\begin{array}{l} \dot{t} = 1 \\ \left\{ \begin{array}{rcl} \dot{w}_{k,i_k}^1 & = & w_{k,i_k}^2 \\ \dot{w}_{k,i_k}^2 & = & w_{k,i_k}^3 \\ & \vdots & \\ \dot{w}_{k,i_k}^{N-k+1} & = & v_{k,i_k} \end{array} \right. \quad , k \in \lfloor N \rceil,\ i_k \in \lfloor s_{N-k} \rceil \\ \dot{z} = 0 \end{array} \tag{7.3}$$

REFERENCES

Aranda-Bricaire, E., Moog, C. H., and Pomet, J. B. (1995). A linear algebraic framework for dynamic feedback linearization. *IEEE Trans. Automat. Control*, 40:127–132.

Briant, R., Chern, S., Gardner, R., Goldschmidt, H., and Griffths, P. (1991). *Exterior Differential Systems*. Springer Verlag.

Brunovsky, P. (1970). A classification of linear controllable systems. *Kybernetika (Prague)*, 6:173–188.

Charlet, B., Lévine, J., and Marino, R. (1989). On dynamic feedback linearization. *Systems Control Lett.*, 13:143–151.

Charlet, B., Lévine, J., and Marino, R. (1991). Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control Optim.*, 29:38–57.

Crouch, P. E. (1984). Spacecraft attitude control and stabilization : applications of geometric control theory to rigid body models. *IEEE Trans. Automat. Control*, 29:321–331.

Fliess, M. (1989). Automatique et corps différentiels. *Forum Math.*, 1:227–238.

Fliess, M. (1990). Some basic structural properties of generalized linear systems. *Systems Control Lett.*, 15:391–396.

Fliess, M. (1994). Une intreprétation algébrique de la transformation de Laplace et des matrices de transfert. *Linear Algebra Appl.*, 203–204:429–442.

Fliess, M., Lévine, J., Martin, P., Ollivier, F., and Rouchon, P. (1995a). Flatness and dynamic feedback linearizability: two approaches. In *Proc. 3rd European Control Conference*.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1992). Sur les systèmes non linéaires différentiellement plats. *C. R. Acad. Sci. Paris Sér. I Math.*, 315:619–624.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1993). Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund. *C. R. Acad. Sci. Paris Sér. I Math.*, 317:981–986.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1995b). Flatness and defect of non-linear systems: introductory theory and examples. *Internat. J. Control*, 61:1327–1361.

Gardner, R. B. and Shadwick, W. F. (1992). The GS algorithm for exact linearization to Brunovsky normal form. *IEEE Trans. Automat. Control*, 37:224–230.

Hunt, L. R., Su, R., and Meyer, G. (1983). Design for multi-input nonlinear systems. In Brocket, R., Millmann, R., and Sussmann, H. J., editors, *Differential Geometric Methods in Nonlinear Control Theory*, pages 268–298.

Isidori, A. (1989). *Nonlinear Control Systems*. Springer–Verlag, 2nd edition.

Jakubczyk, B. and Respondek, W. (1980). On linearization of control systems. *Bull. Acad. Pol. Sc., Ser. Sci. Math.*, 28:517–522.

Kailath, T. (1980). *Linear Systems*. Prentice Hall Inc., Englewood Cliffs, NJ.

Marino, R. (1986). On the largest feedback linearizable subsystem. *Systems Control Lett.*, 7:345–351.

Marino, R., Boothby, W. M., and Elliot, D. L. (1985). Geometric properties of linearizable control systems. *Mathematical Systems Theory*, 18:97–123.

Martin, P. and Rouchon, P. (1993). Feedback stabilization and tracking of constrained robots. *CAS*, Tech. Report 446.

Morse, A. S. (1973). Structural invariants of linear multivariable systems. *SIAM J. Control*, 11:446–465.

Murray, R. M. (1993). Applications and extensions of Goursat normal form to the control of nonlinear systems. In *Proc. IEEE Conf. Decision and Contr.*, pages 3425–3430.

Nijmeijer, H. and van der Schaft, A. J. (1990). *Nonlinear Dynamical Control Systems*. Springer-Verlag, New York.

Pereira da Silva, P. S. (1996a). Canonical forms for a class of noncontrollable nonlinear systems. In *Proc. IEEE Conf. Dec. Control*, Kobe. to appear.

Pereira da Silva, P. S. (1996b). On canonical forms for noncontrollable linear systems. In *Proc. CBA - Congresso Brasileiro de Automática*, São Paulo, Brazil.

Pereira da Silva, P. S. (1996c). On implicit systems and differential flatness. Manuscript in Preparation.

Shadwick, W. F. (1990). Absolute equivalence and dynamic feedback linearization. *Systems Control Lett.*, 15:35–39.

Shadwick, W. F. and Sluis, W. M. (1993). Dynamic feedback for classical geometries. *Tech. Report FI93-CT23, The Fields Inst.*

Sluis, W. M. (1992). *Absolute equivalence and its application to control theory*. Phd Thesis, Univ. of Waterloo.

Tilbury, D., Murray, R. M., and Sastry, S. R. (1995). Trajectory generation for the n-trailer problem using Goursat normal form. *IEEE Trans. Automat. Control*, 40:802–819.

van Nieuwstadt, M., Rathinam, M., and Murray, R. M. (1994). Differential flatness and absolute equivalence. *Tech. Report CIT-CDS 94-006.*

Warner, F. W. (1971). *Foundations of differentiable manifolds and Lie Groups*. Scott, Foresman and Company, Glenview, Illinois.

Wonham, W. M. (1985). *Linear Multivariable Control: a Geometric Approach*. Springer-Verlag, New York, 3rd edition.

8 Appendix - auxiliary results and proofs

Lemma 1 *For any integer $k \geq -1$ and $p \in \mathcal{N}$, we have :*

(i) $\left(I^{(k)}+\text{span}\,\{dt\}\right)\big|_p \cap I^{(-1)}(p) \subset I^{(k)}(p)$.

(ii) $I^{(k)}(p) \cap \text{span}\,\{dt\}|_p = \{0\}$

Proof. (i) Let $\omega(p) \in (I^{(k)}+\text{span}\,\{dt\})\big|_p \cap I^{(-1)}(p)$. Then $\omega = \varpi(p) + \beta dt|_p$ where $\varpi \in I^{(k)}$ and $\beta \in C^\infty(\mathcal{N})$. Then $\langle \omega, f\rangle|_p = \langle \varpi, f\rangle|_p + \beta\langle dt, f\rangle|_p$. Since $\langle dt, f\rangle = 1$ and $I^{(k)} \subset I^{(-1)} = \text{span}\,\{f\}^\perp$, it follows that $\beta(p) = 0$ and hence $\omega(p) = \varpi(p) \in I^{(k)}(p)$.
(ii) Let $\omega(p) \in I^{(k)} \cap \text{span}\,\{dt\}$. Hence $\omega(p) = \beta(p)dt$. So $\langle \omega, f\rangle|_p = \beta(p) = 0$. □

Proof of proposition 1.
Part (i). By the Frobënius theorem we see that $\text{span}\left\{I^{(k)}, dt\right\}$ is spanned by some linearly independent 1-forms $\{dt, d\theta_1, \ldots, d\theta_{r_k}\}$, where $\theta_i \in C^\infty(\mathcal{N})$ are convenient functions. Note that $\omega_i = (d\theta_i - L_f\theta_i dt) \in I^{(-1)}$. From lemma 1, it follows that $\omega_i \in I^{(k)}$. Since $I^{(k)} \subset I^{(-1)}$ and $I^{(-1)} \cap \text{span}\,\{dt\} = \{0\}$ we see that $\dim I^{(k)} = r_k$. To complete the proof of (i) it suffices to show that the set $\{\omega_1, \ldots \omega_k\}$ is linearly independent pointwise. In fact, assume that $\sum_{i=1}^{r_k} a_i(d\theta_i - L_f\theta_i dt) = 0$ in some $p \in \mathcal{N}$. This implies that the set $\{dt, d\theta_1, \ldots, d\theta_{r_k}\}$ is linearly dependent at this point.

Part (ii). We show first that we have

$$d\omega(p) \bmod I^{(k)}(p) \equiv -\, L_f\omega \wedge dt\big|_p \bmod I^{(k)}(p) \qquad (8.1)$$

for all $p \in \mathcal{N}$. For, by (i), note that $\omega = \sum_{i=1}^{r_k} \alpha_i(d\theta_i - L_f\theta_i dt)$ for convenient functions $\alpha_i, \theta_i \in C^\infty(\mathcal{N})$. So,

$$\begin{aligned} d\omega(p) \bmod I^{(k)}(p) \quad &= \quad [\textstyle\sum_{i=1}^{r_k} d\alpha_i \wedge (d\theta_i - L_f\theta_i dt) - \\ &\qquad \alpha_i dL_f\theta_i \wedge dt]\big|_p \bmod I^{(k)}(p) \end{aligned}$$

Note that $d\alpha_i \wedge (d\theta_i - L_f\theta_i dt)\big|_p \bmod I^{(k)}(p) \equiv 0$. Hence, $d\omega(p) \bmod I^{(k)}(p) = \sum_{i=1}^{r_k} -\alpha_i dL_f\theta_i \wedge dt\big|_p \bmod I^{(k)}(p)$. Now observe that

$$\begin{gathered} L_f\omega \wedge dt = L_f[\textstyle\sum_{i=1}^{r_k} \alpha_i(d\theta_i - L_f\theta_i dt)] \wedge dt \\ = [\textstyle\sum_{i=1}^{r_k} L_f\alpha_i(d\theta_i - L_f\theta_i dt)] \wedge dt + \\ \{\textstyle\sum_{i=1}^{r_k} \alpha_i[dL_f\theta_i - (L_f\theta_i)dL_f t - (L_f^2\theta_i)dt]\} \wedge dt. \end{gathered}$$

Since $[\sum_{i=1}^{r_k} L_f\alpha_i(d\theta_i - L_f\theta_i dt)]\big|_p \in I^{(k)}(p)$, we see that

$$[\sum_{i=1}^{r_k} L_f\alpha_i(d\theta_i - L_f\theta_i dt)] \wedge dt\Bigg|_p \bmod I^{(k)}(p) = 0.$$

As $dL_f t = d(1) = 0$ and $L_f^2\theta_i dt \wedge dt = 0$, we conclude that (8.1) is true.

Now we will show that

$$\begin{gathered} \omega(p) \in I^{(k+1)}(p) \text{ if and only if} \\ L_f\omega(p) \in \text{span}\left\{I^{(k)}, dt\right\}(p) \text{ for all } p \in \mathcal{N}. \end{gathered} \qquad (8.2)$$

For this, notice that, $L_f\omega \wedge dt\big|_p \bmod I^{(k)}(p) \equiv 0$ means that $L_f\omega \wedge dt\big|_p + \sum_{i=1}^{r_k} \eta_i \wedge \omega_i\big|_p = 0$ for convenient 1-forms η_i and $\omega_i = (d\theta_i - L_f\theta_i dt)$, as in (i). Since $\{dt, \omega_i : i \in \{1, \ldots, r_k\}\}$ is a basis for $\text{span}\left\{I^{(k)}, dt\right\}$, from the Cartan Lemma (see section 2), we conclude that $L_f\omega(p) \in \text{span}\left\{I^{(k)}, dt\right\}(p)$. Then, (8.2) follows from (8.1) and equation (2.5) for $J = I^{(k)}$ and $J^{(1)} = I^{(k+1)}$.

If $\omega = d\theta - L_f\theta dt$ then $\mathrm{L}_f\omega \in I^{(-1)}$. By (8.2) and from lemma 1 part (i), it follows that $L_f\omega \in I^{(k)}$. To complete the proof of (ii) it suffices to note that, by (i), $I^{(k+1)}$ has a basis of this particular form.

(iii) We show first that

$$\begin{gathered} \text{The set } \{L_f\omega_1(p), \ldots, L_f\omega_r(p)\} \subset \text{span}\left\{I^{(k-2)}, dt\right\}(p) \\ \text{is independent } \bmod \text{span}\left\{I^{(k-1)}, dt\right\}(p). \end{gathered} \qquad (8.3)$$

For this, assume that there exists ω in $I^{(k-1)}$ and functions $\alpha_i \in C^\infty(\mathcal{N})$ such that, for $p \in \mathcal{N}$:

$$(\omega + \alpha_0 dt + \sum_{i=1}^{r} \alpha_i L_f\omega_i)\Bigg|_p = 0$$

Hence

$$\{[\omega - \sum_{i=1}^{r}(L_f\alpha_i)\omega_i] + \alpha_0 dt + L_f(\sum_{i=1}^{r} \alpha_i\omega_i)\}\Bigg|_p = 0$$

Since $[\omega - \sum_{i=1}^{r}(L_f\alpha_i)\omega_i](p) \in I^{(k-1)}(p)$, it follows that $L_f(\sum_{i=1}^{r} \alpha_i\omega_i)\big|_p \in \text{span}\left\{I^{(k-1)}, dt\right\}(p)$. It follows from (8.2) that $(\sum_{i=1}^{r} \alpha_i\omega_i)(p) \in I^{(k)}(p)$ and hence the set $\{\omega_1, \ldots \omega_r\}$ is not linearly independent $\bmod I^{(k)}$ in $p \in \mathcal{N}$. To conclude the proof, note that (iii) is a straightforward consequence of lemma 1,the condition (8.3) and from the fact that $L_f(d\theta_i - L_f\theta_i dt) \in I^{(-1)}$. □