An infinite dimensional differential-geometric approach for nonlinear systems : Part II - System Theory

Paulo Sergio Pereira da Silva§
Hector Bessa Silveira§
Carlos Corrêa Filho†
Simone Batista‡

§ Escola Politécnica da USP
Av. Luciano Gualberto trav. 03, 158
05508-900 – São Paulo – SP BRAZIL
Email:paulo@lac.usp.br – Fax: +55.11.30915718

†IFSP Instituto Federal de São Paulo
Av. Salgado Filho, 3501
07115-000 – Guarulhos – SP – BRAZIL
‡ Centro Universitário da FEI
Av. Humberto de Alencar Castelo Branco, 3972
09850-901 - São Bernardo do Campo - SP - BRAZIL

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Abstract

This paper is second part of a survey on the basic notions and definitions of the infinite dimensional differential geometric approach of (Fliess, Lévine, Martin & Rouchon 1993, Pomet 1995). In this approach, a system is a infinite dimensional manifold. In the first part of this survey (Pereira da Silva, Silveira, Correa Filho & Batista 2008), one may find an elementary introduction about $I\!R^A$-manifolds and diffieties. In this second part, the notions of state representation and dynamic feedback are defined in an abstract manner, and then they reinterpreted in terms of coordinates and their corresponding equations. Instrumental in this setting are the different versions of the inverse function theorem, which are presented in this paper. The concept of subsystem is a key notion for establishing the definition of dynamic feedback and a new notion of regularity of implicit systems. This notion of regularity is shown to be useful for establishing a notion of equivalence between (implicit) systems.

Keywords. Nonlinear systems; implicit systems; input-output decoupling; disturbance decoupling; differential flatness; differential geometric approach; implicit systems.

1 Introduction

{sIntro}

The infinite dimensional geometric differential approach of nonlinear system was introduced in (Fliess et al. 1993, Pomet 1995). The main ingredient of this approach is the concept of Diffiety (Alekseevskij, Vinogradov & Lychagin 1991, Zharinov 1992), which is, roughly speaking, an infinite dimensional manifold induced by the infinite prolongation of a differential equation (that may be an ordinary or a partial differential equation).

In this approach, a control system

$$\dot{x}(t) = f(t, x(t), u(t))$$

is an infinite dimensional manifold with coordinates given by $\{t, x, u^{(0)}, u^{(1)}, u^{(2)}, \ldots\}$. This corresponds to the infinite prolongation of the original equation[1].

The main qualities of this point of view are:

1. One can define a system without choosing the input and the state a priori[2].
2. Geometric definitions of an input and of a state representation can be given.
3. The notion of dynamic feedback can be stated in a geometric manner.
4. Differential algebraic systems (DAEs) (or implicit systems) can be considered in this approach with a clear geometrical interpretation (for instance, systems with constraints are immersed submanifolds of the unconstrained system).
5. This approach leads to a geometric notion of equivalence between (implicit) systems.
6. In several contexts, a certain number of prolongations of the system are made, and this number is not fixed a priory, (for instance, the notion of flatness and the problem of decoupling). Hence, working with the infinite prolongation of the system may be very elegant in these cases.

Although the meaning of a state representation will be defined in a intrinsic manner, it is important to stress that the choice of a state representation can be regarded as a choice of particular set of (local) coordinates. This point of view is useful in the classification of feedback complexity and the study of implicit systems as well.

[1] The algebraic differential approach of (Fliess 1989, Fliess, Lévine, Martin & Rouchon 1995) is the algebraic version of this approach (see also (Conte, Moog & Perdon 1999))

[2] This point of view is shared by the behavioral approach (Willems 1992).

The notion of subsystem is instrumental in this setting, and allows a geometric definition of exogenous feedback. The system E is an exogenous extension of a system E, if S is a subsystem of E. The standard equations of dynamic feedback are recovered when one chooses particular state representations of E and S. In fact, those equations are the relations between these inputs and states. It is important to stress that the notions of dynamic feedback and state representation of this geometric approach mimics the notion of feedback of (Fliess 1989) and (Delaleau & Pereira da Silva 1998a).

It is well known that the inverse function theorem do not hold in this infinite dimensional approach (Zharinov 1992), at least considering its original statement. The main technical ingredient of the proofs of the results of this paper are some generalizations of the inverse function theorem. New results about this question and comparisons with existing ones are discussed in this work. It becomes clear that, in some sense, the *Dynamic Extension Algorithm (DEA)* is a version of the inverse function theorem and the results that are presented in this paper need less regularity assumptions than the DEA (see Appendix H).

In this work, an implicit system Δ is a pair (S, y), where S is a system and y is a set of constraints. The notion of subsystem is also a main ingredient of a new definition of regularity of an implicit system. It is shown that a regular implicit system Δ in the sense of our definition can be regarded as an immersed (embedded) submanifold $\tilde{\Delta}$ of S. One may regard $\tilde{\Delta}$ as a control system[3], and a state (local) representation of Δ can be defined as being a (local) state represetation of $\tilde{\Delta}$. This leads to a consistent definition of equivalence between two implicit systems Δ_1 and Δ_2, namely, they are equivalent if the corresponding immersed systems $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are equivalent by endogenous feedback.

This second part of this survey is organized as follows. In the present section. One will present the notations of the paper. In section 2, the notions of Diffities and Systems are recalled. In Section 3, the generalization of the inverse function theorem are introduced as well as the relationship with state space representations. In section 4 the notion of subsystem is discussed. In section 5, the notion of subsystem is used for establishing an intrinsic notion of dynamic feedback. In section 6 a new notion of regularity of implicit system is introduced. The appendices present some proofs and a geometric description and interpretation of the *dynamic extension algorithm (DEA)*. It includes also a comparison between the regularity assumptions of the DEA and the ones of our inverse function theorems, showing that the DEA (see lemma 5) needs stronger regularity assumptions when compared with theorems 3 and 5.

This work is mainly based on previous works (for instance, (Fliess, Lévine, Martin & Rouchon 1999, Pomet 1995). However, some results and definitions of this work are original. This is the case of theorems 3, 5, 8, 10, 11, and 12, and definitions 17 and 19. However, the authors believe that the the unifying spirit of the presentation of this work is also an important contribution. This work also establishes some important auxiliary results for (Pereira da Silva & Batista 2009, Pereira da Silva & Batista 2010).

[3]In the sense of a diffiety that admits a local state representation.

The reader may refer to the Part I of this paper for a survey on the main aspects of $I\!R^A$ manifolds and diffieties (Pereira da Silva et al. 2008) used in this Part II of this work. It introduces also some notations considered here. The following result, which is a consequence of Proposition 25 of Part I, is restated here in a less intrinsic manner.

{pDifferential}

Proposition 1 *Let S be a $I\!R^A$-manifold. Given a local chart (U, ϕ) with coordinate functions $\{x_i, i \in I\!N^*\}$, and a one-form ω defined on U, there exists some open neighborhood $V_\xi \subset U$ of ξ and a finite subset $F \subset I\!N^*$, such that, for all $x \in V_\xi$, one has*

$$\omega(x) = \sum_{i \in F} \phi_i(x) dx_i|_x$$

where $\phi_i(x)$ does not depend on x_i for $i \notin F$ for all $x \in V_\xi$.

{rRestrict}

Remark 1 *Many results of this paper relies on the last proposition. As a consequence, in many results that holds locally around some ξ, one must restrict the open neighborhood of ξ according to Proposition 1. Recall from Part I of this survey, that the differential $d\phi$ of a smooth function $\phi : U \to I\!R$ is a one-form. In particular, Prop. 1 generalizes a result that holds for differentials.*

One introduces now some further notations.

Context permitting, we will denote the Cartan field of an ordinary diffiety M simply by $\frac{d}{dt}$.

Given a smooth object ϕ (a smooth function, field or form) defined on a diffiety M with Cartan field $\frac{d}{dt}$, then $\dot{\phi}$ (or $\phi^{(1)}$ stands for the Lie-derivative $L_{\frac{d}{dt}}\phi$, and $\phi^{(n)}$ stands for $L^n_{\frac{d}{dt}}\phi = L_{\frac{d}{dt}} L^{n-1}_{\frac{d}{dt}}\phi, n \in I\!N$, where $L^0_{\frac{d}{dt}}\phi = \phi^{(0)} = \phi$.

Let x be a smooth function defined on M. Then dx denotes its differential. Let $x = (x_1, \ldots, x_n)$ be a vector of smooth functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$. If $u = (u_1, \ldots, u_m)$ is a set of functions then $u^{(k)} = (u_1^{(k)}, \ldots, u_m^{(k)})$ and $\bar{u}$ stands for the set of functions $\{u^{(k)} : k \in I\!N\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in Z\!\!Z^m$ be a multi-index. Then $\alpha - 1$ stands for $(\alpha_1 - 1, \ldots, \alpha_m - 1)$. Consider the compact notation

$$u^{\langle\langle\alpha\rangle\rangle} = \bigcup_{i=1}^{m} \bigcup_{0 \le j \le \alpha_i} \{u_i^{(j)}\}$$

For instance, if $u = (u_1, u_2, u_3)$ and $\alpha = (2, -1, 0)$ then $u^{\langle\langle\alpha\rangle\rangle} = \{u_1^{(0)}, u_1^{(1)}, u_1^{(2)}, u_3^{(0)}\}$, $du^{\langle\langle\alpha\rangle\rangle} = \{du_1^{(0)}, du_1^{(1)}, du_1^{(2)}, du_3^{(0)}\}$, $\alpha - 1 = (1, -2, -1)$, and $du^{\langle\langle\alpha-1\rangle\rangle} = \{du_1^{(0)}, du_1^{(1)}\}$.

2 State Representations, Flatness and Control Systems

{sDiffSyst}

In Part I of this survey, one have studied the notion of *diffieties and systems*. Now, one shall study *control systems*, that is, systems that admits (local) *state* *representations.*

The *trivial diffiety* $T^m(u)$, of differential dimension m is the space $I\!R^A$ of global coordinates $\{t, u_j^{(k)} | j \in \lfloor m \rceil, k \in I\!N\}$ equipped with the Cartan field

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}$$

the set $u = (u_1^{(0)}, \ldots, u_m^{(0)})$ (which is a subset of coordinate functions) is called input of the trivial diffiety. The diffiety $T^m(u)$ is also called[4] *trivial diffiety of flat output* u.

The *time-invariant trivial diffiety* $\mathcal{T}^m(u)$, of differential dimension m is the space $I\!R^A$ of global coordinates $\{u_j^{(k)} | j \in \lfloor m \rceil, k \in I\!N\}$ equipped with the Cartan field

$$\frac{d}{dt} = \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}$$

the set $u = (u_1^{(0)}, \ldots, u_m^{(0)})$. The diffiety $\mathcal{T}^m(u)$ is also called[5] *time-invariant trivial diffiety of flat output* u.

{dFiber}

Definition 1 *(Fiber) Given a Lie-Bäcklund submersion* $\pi : V \to T$ *between* $I\!R^A$ *manifolds, let* $\nu \in V$ *and let* $\tau = \pi(\nu) \in T$*. A* fiber *is a subset of* T *given by* $F_\tau = \pi^{-1}(\tau)$*. Recall that there exists local coordinates* $\tilde{\phi} = (\tilde{x}, \tilde{z})$ *defined on an open neighborhood* U *of* ν *and* $\tilde{\psi} = \tilde{z}$ *defined on an open neighborhood* W *of* τ *such that* π *locally reads* $(\tilde{x}, \tilde{z}) \mapsto \tilde{z}$*. Such coordinates are called* adapted charts*. Without loss of generality, assume that* $\tilde{z}(\tau) = 0$*. Then, one may write* $F_\tau \cap U = \{\xi \in U \mid \tilde{(}\xi = (\tilde{x}, 0)\}$*. In particular, it is easy to show that the fiber is an immersed submanifold of* V *with local coordinates* $\tilde{x}$ *called* fiber coordinates.

{pFiber}

Proposition 2 *The cardinal of the fiber coordinates* $\tilde{x}$ *is a local invariant, that is, it does not depend on the choice of the adapted coordinates.*

Proof. See appendix L. □

If $\operatorname{card} \tilde{x} = n$ is finite, then one says that the fiber is finite dimensional and the integer n is called *fiber dimension.*

An intrinsic definition of local state representation,which is similar to the ones of (Fliess, Lévine, Martin & Rouchon 1997b, Fliess, Lévine, Martin & Rouchon 1997a, Fliess, Lévine, Martin & Rouchon 1998, Fliess et al. 1999), is given below.

[4] See definition 5.

[5] The time-invariant trivial diffiety $\mathcal{T}^m(u)$ represents the space of free differential variables $u_1, \ldots, u_m$. The differential-algebraic version of $\mathcal{T}^m(u)$ is the differential field $k\langle u \rangle$, where $u_1, \ldots, u_m$ are differentially independent over the ground field k (Fliess 1989).

{d:stateinput:intrinsec}

Definition 2 *(input and state — geometric version) We say that u is a (local) input for a system S if there exists a (local) Lie-Bäcklund submersion $\pi : V \subset S \to T^m(u)$, where V is an open subset of S and the fiber is finite dimensional, with constant dimension n. Any set of fiber coordinates x is called a state for the input u.*

It is an easy exercise to show that the last definition is locally equivalent to the following definition:

{d:stateinput:nonintrinsec}

Definition 3 *(input and state — local coordinates version) A local state representation of a system $(S, I\!R, \tau)$ is a local coordinate system, (ψ, V), with $\psi = \{t, x, U\}$ where $x = \{x_i, i \in \lfloor n \rceil\}$, $U = \{u_j^{(k)} | j \in \lfloor m \rceil, k \in I\!N\}$ where $\tau \circ \psi^{-1}(t, x, U) = t$. The set of functions $x = (x_1, \ldots, x_n)$ is called* state *and $u = (u_1, \ldots, u_m)$ is called* input. *In these coordinates the Cartan field is locally written by*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{1}$$ {eCartanField}

It is important to point out that, in a first moment $u^{(k)}$ is only a notation describing the set of coordinates $\psi = \{t, x, U\}$. It follows from (1) that $L_{\frac{d}{dt}} u^{(k)} = \frac{d}{dt}(u^{(k)}) = u^{(k+1)}$. Another point of view is to include in the definition the fact that

$$u^{(k+1)} = \frac{d}{dt}(u^{(k)}), k \in I\!N \tag{2}$$ {eCartanU}

If one includes (2) in the definition, then (1) can be excluded, since it becomes a consequence of (2).

The first definition (Def. 2)is intrinsic, and it is in accordance with the definition of input and state given in the differential algebraic approach of (Fliess 1989). The second definition (Def. 3) is not intrinsic, but it is the coordinate counterpart of the first one. In many situations, it is easier to work with the second definition, and so this is the point of view adopted in this entire work.

Output. An *output* $y = (y_1, \ldots, y_p)$ of a system S is a set of functions defined on S.

{dProperInputs}

Definition 4 *A state representation of a system S is completely determined by the choice of the input u and the state x, and will be denoted by (x, u). If $span\{d\dot{x}\} \subset span\{dt, dx, du\}$ in V, the state representation is called classical (or proper), and f_i depends only on (t, x, u) for $i = 1, \ldots, n$. Given a local proper state representation (x, u) defined in V, an output y is called classical (or proper), if $span\{dy\} \subset span\{dt, dx, du\}$ in V. In this case, the output y depends only on (t, x, u).*

Flatness. In what follows, one may state the notion of flatness in three different directions[6]

[6] For the moment the precise statements will be ommited since one is interested only on the main ideas.

- Consider that a flat system is locally diffeomorphic to a trivial diffiety $T^s(y)$ with flat output (see Def. 5. This definition implies that the solutions of the system are parametrized in a unique way by the choice of the smooth map $y(t)$.

- Roughly speaking, a system is flat when one may compute the state and input as functions of the flat output and their derivatives. So, the open loop control for tracking a desired trajectory can be easily determined (for instance, computed torque method in robotics).

- A system is flat when it admits a state representation for which the state absent and the input is the *flat output*[7]. This means that the solutions of the system may be computed without any integration. All the variables of the system are completely determined by time differentiations (of the flat output).

The following definition is an intrinsic definition of Flatness.

{dFlatness1}

Definition 5 *A system is (locally) flat if it is (locally) Lie-Bäcklund isomorphic to the trivial diffiety, that is, there exists a Lie-Bäcklund difeomorphism $\phi : U \subset S \to V \subset T^m(y)$, where U and V are open subsets.*

This is equivalent to consider the dimension of the fiber to be zero in definition 2. In particular one may give the following equivalent (but not intrinsic) definition of flatness.

{dFlatness2}

Definition 6 *A system is (locally) flat if there exists a (local) state representation (x, u) with $x = \emptyset$. In this case, taking $y = u$, the Cartan field is locally given by*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} y_j^{(k+1)} \frac{\partial}{\partial y_j^{(k)}}, \tag{3}$$ {eCartanFlat2}

a particular case of (1) *in which y is called flat output.*

A third and fourth definitions of flatness consider that one may compute the input and the state a function of the flat output and its derivatives. These definitions depends on a particular choice of state representation, and so they cannot be intrinsic.

{dFlatness3}

Definition 7 *Let S be a system with state representation (x, u) and output U. Assume that (x, u) and y are defined on an open subset containing an open subset $U \subset S$. Let $s = \mathrm{card}\, y$. Recall that $T^s(y)$ is the trivial diffiety with flat output y (see the beginning of section 2). Consider the map $\Gamma : U \to T^s(y)$, defined by $\nu \mapsto (t(\nu), y^{(0)}(\nu), y^{(1)}(\nu), y^{(1)}(\nu), \ldots)$. The system S is flat (on U) with flat output y if*

[7] The name flat output is standard, but the flat output is in fact an input of the system.

1. *There exists a map $y : U \to I\!R^s$ and smooth maps $\chi : I\!R \times (I\!R^s)^{\alpha+1} \to I\!R^n$ and $\mu : I\!R \times (I\!R^s)^{\beta+1} \to I\!R^m$ such that, for every $\xi \in U$ one has $x(\xi) = \chi(t, y^{(0)}(\xi), \ldots, y^{(\alpha)}(\xi))$ and $u(\xi) = \mu(t, y^{(0)}(\xi), \ldots, y^{(\beta)}(\xi))$.*
2. *The image $\Gamma(U)$ contains an open neighborhood of $\Gamma(\xi)$.*

The next definition replaces the assumption that $\Gamma(U)$ contains an open neighborhood of ξ by the fact that the cardinal of the input and the flat output candidate y coincides.

{dFlatness4}

Definition 8 *Let S be a system with state representation (x, u) and output U. Assume that (x, u) and y are defined on an open subset containing an open subset $U \subset S$. Let $s = \mathrm{card}\, y$. The system S is flat (on U) with flat output y if*

1. *There exists a map $y : U \to I\!R^s$ and smooth maps $\chi : I\!R \times (I\!R^s)^{\alpha+1} \to I\!R^n$ and $\mu : I\!R \times (I\!R^s)^{\beta+1} \to I\!R^m$ such that, for every $\xi \in U$ one has $x(\xi) = \chi(t, y^{(0)}(\xi), \ldots, y^{(\alpha)}(\xi))$ and $u(\xi) = \mu(t, y^{(0)}(\xi), \ldots, y^{(\beta)}(\xi))$.*
2. *$s = \mathrm{card}\, y = \mathrm{card}\, u$.*

It is clear that the Definitions 5 and 6 are equivalent. The next result states a comparison with Definition 7. The proof is deferred to appendix K.

{pEquivalenceFlatness}

Proposition 3 *Consider a system S. Let $\xi \in S$. Let (x, u) be a local state representation defined on an neighborhood V of ξ. Consider the Definitions 6 and 7. Then*

- *If a system is locally flat around ξ according definition 6 then it is locally flat around ξ according Definition 7.*
- *If a system is locally flat around ξ according definition 7 then it is locally flat around ξ according Definition 6.*
- *If a system is locally flat around ξ according Definition refdFlatness2 it is locally flat according definition 8.*
- *If a system is flat on U according definition 8 then it is flat on an open and dense subset of U according definition 6.*

Remark 2 *To our best knowledge, there is no proof of equivalence of Definitions 6 and 8 without some regularity assumption (for instance, our proof holds on the set of regular points of some codistributions). In Corollary 2 (see section 4, this assumption is replaced by the independence of the differentials $\{dy^{(0)}, \ldots, dy^{(\gamma)}\}$ up to a convenient order γ.*

Differential Dimension. The number of components of the input of a (local) state representation (x, u) of a system S is called (local) *differential dimension*. Assume that system S a connected manifold and there exists local state representations around every point of S. Then, it can be shown that the differential dimension of S is a local invariant (Fliess et al. 1993, Pomet 1995, Fliess et al. 1999). For every connected component of S, the differential dimension is a global invariant (see (Pereira da Silva 2000)).

Control System. A control system S is a system that admits a local state representation around every point, and the differential dimension is a fixed, globally defined integer. In other words, the dimension of the input of every state representation is equal to a global invariant m.

Let S be a flat control system with flat output y. Because a flat output y is also an input u, then $\dim y$ is the differential dimension of the system.

System associated to differential equations. Now assume that a control system is given by a set of equations

$$\begin{array}{rcl} \dot{t} & = & 1 \\ \dot{x}_i & = & f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n \rceil \\ y_j & = & \eta_j(x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p \rceil \end{array} \qquad (4)$$ {eStatRep}

A diffiety S of global coordinates $\psi = \{t, x, U\}$, where $U = \{u^{(k)}, k \in I\!N\}$, and Cartan field given by (1) can be always associated to (4).

3 Inverse function theorems and state representations

{sInverse}

Recall that the inverse function theorem does not hold for $I\!R^A$-manifolds, at least when considering its standard statement (Zharinov 1992, Alekseevskij et al. 1991). For systems, the corresponding notion of coordinates is the concept of state representation. Hence, in this context our version of the inverse function theorem is devoted to establish conditions that assure that a set of functions (x, u) is a state representation of a system S. A known result about this question is in fact lemma 5 of the appendix C. Lemma 5 relies on the properties of the dynamic extension algorithm, and requires the nonsingularity of the codistributions (7). It will be shown that lemma 2 can replace the application of dynamic extension algorithm, with the need of less regularity assumptions (see appendix H).

3.1 Characterization of State representation

The next result is an auxiliary lemma that is useful in this paper. Its proof is deferred to appendix A.

{lLemmaMenos}

Lemma 1 *Let (x, u) and (z, v) be two proper state representations of the system S defined on an open neighborhood U of $\xi \in S$.*

1. *If $b \in I\!N$ is such that span $\{dv\} \subset$ span $\left\{dt, dx, du^{(0)}, \ldots, du^{(b)}\right\}$, then span $\{dz\} \subset$ span $\left\{dt, dx, du^{(0)}, \ldots, du^{(b-1)}\right\}$*[8]*.*

[8] When $b = 0$, then span $\left\{dt, dx, du^{(0)}, \ldots, du^{(b-1)}\right\}$ stands for span $\{dt, dx\}$.

2. *Let $\beta \in I\!N$ be the smallest non-negative integer such that there exists an open neighborhood V of ξ such that*[9] *span* $\{du\} \subset$ *span* $\left\{dt, dz, dv^{(0)}, \ldots, dv^{(\beta)}\right\}$ *on V. If span* $\{dz, dv\} \subset$ *span* $\left\{dt, dx, du^{(0)}, \ldots, du^{(\gamma)}\right\}$, *then* $\beta \leq n + m\gamma$, *where $n = \dim x$ and $m = \dim u$.*

The following result is a fundamental result of this work.

{lLemma3}

Lemma 2 *Let (x, u) be a local proper state representation of a system S around some $\xi \in S$, and let $z = (z_1, \ldots, z_s)$ and $v = (v_1, \ldots, v_r)$ be sets*[10] *of smooth functions defined on the diffiety S. Suppose that span* $\{dz, dv\} \subset$ *span* $\{dt, dx, du\}$. *Then (z, v) is a local state representation of S around ξ if and only if there exist $\alpha \in I\!N$ such that*

- *The set* $\mathbb{S} = \{dt, dz, dv, \ldots, dv^{(\alpha)}\}$ *is linearly independent at ξ.*
- *One has span* $\{dx\} \subset$ *span* $\left\{dt, dz, dv, \ldots, dv^{(\alpha-1)}\right\}$, *in an open neighborhood of ξ.*
- *One has span* $\{d\dot{z}, du\} \subset$ *span* $\left\{dt, dz, dv, \ldots, dv^{(\alpha)}\right\}$ *in an open neighborhood of ξ.*

The proof of the previous Lemma is deferred to the Appendix B.

The last two results can be generalized easily for the case where the orders of derivation of different components of the inputs are not the same. The proof of those results are similar to proof of Lemmas 1 and 2, and they are left to the reader.

{lLemmaMenosb}

Lemma 3 *Let (x, u) and (z, v) be two proper state representations of the system S defined on an open neighborhood U of ξ with* $\operatorname{card} u = m$. *If $\beta \in I\!N^m$ is such that*[11] *span* $\{dv\} \subset$ *span* $\left\{dt, dx, du^{\langle\langle\beta\rangle\rangle}\right\}$, *then span* $\{dz\} \subset$ *span* $\left\{dt, dx, du^{\langle\langle\beta-1\rangle\rangle}\right\}$.

{lLemma3b}

Lemma 4 *Let (x, u) be a local proper state representation of a system S around some $\xi \in S$, and let $z = (z_1, \ldots, z_s)$ and $v = (v_1, \ldots, v_r)$ be sets of functions defined on the diffiety S. Suppose that span* $\{dz, dv\} \subset$ *span* $\{dt, dx, du\}$. *Then (z, v) is a local state representation of S around ξ if and only if there exists some $\alpha \in I\!N^r$ such that*

- *The set* $\mathbb{S} = \{dt, dz, dv^{\langle\langle\alpha\rangle\rangle}\}$ *is linearly independent at ξ.*
- *One has span* $\{dx\} \subset$ *span* $\left\{dt, dz, dv^{\langle\langle\alpha-1\rangle\rangle}\right\}$, *in an open neighborhood of ξ.*
- *One has span* $\{d\dot{z}, du\} \subset$ *span* $\left\{dt, dz, dv^{\langle\langle\alpha\rangle\rangle}\right\}$ *in an open neighborhood of ξ.*

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[9] The integer β always exists (locally), since $\{t, z, (v^{(k)}, k \in I\!N)\}$ is a local coordinate system (see Proposition 1). If span $\{du\} \subset$ span $\{dt, dz\}$, one also chooses $\beta = 0$.

[10] A priori $r = \operatorname{card} v$ is not assumed to coincide with $m = \operatorname{card} u$. However, if the assumptions of Lemma 2 holds, then $\operatorname{card} v$ and $\operatorname{card} u$ must coincide (by Lemma 2 and the uniqueness of the differential dimension).

[11] See the notations in the end of section 1.

3.2 Comparison

In this work, the previous versions of the inverse function theorem will be compared with other results of the literature. We begin showing that the inverse function theorem of Pomet is a consequence of lemma 2. A comparison with the dynamic extension algorithm, which can be regarded as a version of the inverse function theorem, is deferred to Appendix H.

The following result is equivalent to a time-varying version of (Pomet 1995, Prop. 3).

Corollary 1 *Let S_1 and S_2 be two control systems and let $\phi : S_1 \to S_2$ be a Lie-Bäcklund map. Let (x_1, v_1) and (x_2, v_2) be two global state representations defined respectively around $\xi \in S_1$ and $\phi(\xi) \in S_2$. Assume that the set*

$$B_2 = \{dt, d(x_2 \circ \phi)\} \cup \{d(v_2^{(k)} \circ \phi), k \in I\!N\} \qquad (5)$$ {eAgora}

is a basis of the $C^\infty(S_1)$-module defined by $H = span\left\{dt, dx, dv^{(k)} : k \in I\!N\right\}$. Then there exists an open neighborhood U_1 of ξ such that the map $\phi|U_1 : U_1 \to \phi(U_1)$ is a diffeomorphism.

Proof. Let $z = x_2 \circ \phi$ and $v = v_2 \circ \phi$. We shall consider the abuse of notation of appendix F. Note now that one may assume, without loss of generality, that, locally, span $\{dz, d\dot{z}, dv\} \subset$ span $\{dt, dx_1, dv_1\}$, otherwise one may extend the state x_1 to $\tilde{x}_1 = (x_1, v_1, \ldots, v_1^{(\gamma)})$, with γ big enough, and take the new input $\tilde{v}_1$. Based on similar reasons, one may assume that $d\dot{z} =$ span $\{dt, dz, dv\}$. Since the set (5) is a basis of H, these 1-forms are independent pointwise at all $\nu \in S_1$ (see appendix D). Hence, the first assumption of Lemma 2 holds for every $\alpha \in I\!N$. Then As B_2 is a basis of H, there exists some α big enough such the third assumption of lemma 2 holds. By Lemma 1, the second assumption of 2 holds. □

4 Subsystems, decoupling and flatness

{sSubsystems}

The concept of subsystem is a key notion in the definition of dynamic feedback and in the study of implicit systems as well. We will show that lemma 2 is a tool for obtaining conditions for the existence of the output subsystem and the existence of adapted state equations for a given subsystem. The characterization of a flat output and a regularity notion for the decoupling problem (Martin 1993, Martin 1992) are easily obtained from this the last result.

{sSub}

Definition 9 *(Subsystem and adapted state equations) A (local)* subsystem S_a *of a given system S is a system S_a such that there exists a surjective*[12] *Lie–Bäcklund submersion $\pi : U \subset S \to S_a$, where U is an open subset of S. A (local) subsystem will be denoted by (S_a, π) or simply by S_a.*

[12] Since submersions are open maps, one can always consider that $S_a = \pi(U)$ by restricting S_a to the image of π.

Assume that there exists a local classical state representation (x, u) of a system S of the form

$$\dot{x}_a = f_a(t, x_a, u_a), \tag{6a}$$ {eSuba}

$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b), \tag{6b}$$ {eSubb}

where $x = (x_a, x_b)$ and $u = (u_a, u_b)$. Suppose that[13] (6a) *represents the state equations of a subsystem S_a and $\pi : S \to S_a$ is such that $\pi(t, x, U) = (t, x_a, U_a)$, where U denotes the set $(u^{(j)} | \ j \in I\!N)$ and U_a denotes the set $({u_a}^{(j)} | \ j \in I\!N)$. A state representation of S the form* (6a)–(6b) *is said to be adapted to the subsystem S_a.*

Later, under some regularity assumptions, it is show that state equations adapted to a subsystem can be generically constructed.

{sOS}

Definition 10 *(Output subsystem) Given a system S with output y, a (local)* output subsystem *is a (local) subsystem $\pi : U \subset S \to Y$ such that $\pi^*(T^*_{\pi(\xi)} Y)$ $=$ span $\left\{dt, dy^{(k)} : k \in I\!N\right\}|_\xi, \xi \in U$.*

It is shown in (Pereira da Silva & Corrêa Filho 2001) that the properties of dynamic extension algorithm can be used in order to study the existence of output subsystems of a system S. In this work it is shown that the results of section 3 may be used to obtain similar results, but with less regularity assumptions.

The next result shows that concept of (local) output subsystem is (locally) intrinsic.

Theorem 1 (Uniqueness of local output subsystems) *Let S be a system with output y. Two local output subsystems Y_1 and Y_2 defined around $\xi \in S$ are locally Lie-Bäcklund isomorphic.*

Proof. This result is shown in (Pereira da Silva & Corrêa Filho 2001). The proof is given here for the sake of completeness. Since the $\pi_i : U_i \subset S \to Y_i$ are Lie-Bäcklund submersions for $i = 1, 2$, there exists local charts of $\phi_i = (t, X_i, Z_i)$, $i = 1, 2$, defined in some common neighborhod $H \subset S$ of ξ and local charts $\psi_i = (t, X_i)$, of Y_i, $i = 1, 2$, defined on $W_i = \pi_i(H)$ such that, in these coordinates $\phi_i \circ \pi_i^{-1} \circ \psi_i(t, X_i, Z_i) = (t, X_i)$, $i = 1, 2$. Since Y_1 and Y_2 are both local subsystems we have span $\{dt, dX_i\} =$ span $\left\{dt, dy^{(k)} : k \in \lfloor I\!N \rceil\right\}$, for $i = 1, 2$. In particular, it follows that the local coordinate change $(t, X_1, Z_1) = \phi_1 \circ \phi_2^{-1}(t, X_2, Z_2)$ is such that $X_1 = \theta(t, X_2)$ and $X_2 = \tilde{\theta}(t, X_1)$. So the map μ defined by $(t, X_2) \mapsto (t, \theta(t, X_2))$ is a local diffeomorphism[14]. Let $\delta : W_2 \subset Y_2 \to W_1 \subset Y_1$ be the local diffeomorphism defined by $\delta = \psi_1^{-1} \circ \mu \circ \psi_2$. To

[13] Here one abuse notation, as explained in Appendix F.

[14] We stress that we are not using the Inverse Function Theorem, but only the existence of the inverse of the coordinate change map.

complete the proof it suffices to show that δ is Lie-Bäcklund. For this, we show first that $\delta \circ \pi_2|_H = \pi_1|_H$. In fact, note that

$$\begin{aligned} \psi_1 \circ (\delta \circ \pi_2) \circ \phi_1^{-1}(t, X_1, Z_1) &= \psi_1 \circ (\delta \circ \pi_2 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})(t, X_1, Z_1) = \\ (\psi_1 \circ \delta) \circ \pi_2 \circ \phi_2^{-1}(t, X_2, Z_2) &= (\mu \circ \psi_2) \circ \pi_2 \circ \phi_2^{-1}(t, X_2, Z_2) = \\ \mu \circ (\psi_2 \circ \pi_2 \circ \phi_2^{-1})(t, X_2, Z_2) &= \mu(t, X_2) = \\ (t, X_1) &= \psi_1 \circ \pi_1 \circ \phi_1^{-1}(t, X_1, Z_1) \end{aligned}$$

From the first and the last terms above, we have that $\delta \circ \pi_2|_H = \pi_1|_H$. Denote by ∂_i the Cartan fields respectively of Y_i, for $i = 1, 2$. By definition $\pi_i^* \frac{d}{dt} = \partial_i \circ \pi_i$. In particular $\partial_1 \circ \delta \circ \pi_2 = \partial_1 \circ \pi_1 = (\pi_1) * \frac{d}{dt} = (\delta \circ \pi_2)_* \frac{d}{dt} = \delta_* (\pi_2)_* \frac{d}{dt} = \delta_* \partial_2 \circ \pi_2$. As π_2 is surjective it follows that $\partial_1 \circ \delta = \delta_* \partial_2$, showing that δ is Lie-Bäcklund. $\square$

The following definition will be important in the study of implicit systems.

{dStronglyAdapted}

Definition 11 *(Strongly adapted state equations) Let S be a system with output y and let Y be a local output subsystem with corresponding Lie-Bäcklund submersion $\pi : U \subset S \to Y$. A state representation $((z_a, z_b), (v_a, v_b))$ is said to be strongly adapted if:*

- *It is adapted to Y according to Definition 9.*
- *One has[15] span $\left\{dy^{(k)} : k \in I\!N\right\}$ = span $\left\{dz_a, (dv_a^{(k)} : k \in I\!N)\right\}$.*
- *One has that z_a and v_a (and hence $v_a^{(k)}, k \in I\!N$) are subsets of $\{y^{(k)} : k \in I\!N\}$ [16].*

Consider the codistributions, called output filtrations: {eFiltrations}

$$\begin{aligned} \mathbb{Y}_{-1} &= \{0\} \\ \mathbb{Y}_k &= \text{span}\left\{dy^{(0)}, \ldots, dy^{(k)}\right\} && \text{(7a)} \end{aligned}$$ {eFiltrationsa}

$$\begin{aligned} Y_{-1} &= \text{span}\{dt\} \\ Y_k &= \text{span}\left\{dt, dy^{(0)}, \ldots, dy^{(k)}\right\} && \text{(7b)} \end{aligned}$$ {eFiltrationsb}

$$\begin{aligned} \mathcal{Y}_{-1} &= \text{span}\{dt, dx\} \\ \mathcal{Y}_k &= \text{span}\left\{dt, dx, dy^{(0)}, \ldots, dy^{(k)}\right\} && \text{(7c)} \end{aligned}$$ {eFiltrationsc}

The following result, adapted from (Pereira da Silva & Corrêa Filho 2001), is a consequence of the dynamic extension algorithm (see Lemma 5). It assures the existence of local output subsystems and their corresponding adapted state equations.

[15] The definition of output subsystem and the fact that (z_a, v_a) is a state representation of Y implies only that span $\left\{dt, (dy^{(k)} : k \in I\!N)\right\}$ = span $\left\{dt, dz_a, (dv_a^{(k)} : k \in I\!N)\right\}$

[16] This means that the components of z_a and v_a are functions belonging to these sets of functions.

{tY}

Theorem 2 (Existence of output subsystems) *Let S be a system and let (x,u) be a classical state representation defined on an open neighborhood $W \subset S$. Let y be a classical output defined on W. Let $n = \mathrm{card}\, x$. Let $U \subset W$ be the set of regular points of the codistributions $Y_k, \mathcal{Y}_k, k = 0, \dots, n$. Then, around any $\xi \in U$, there exists an open neighborhood V_ξ of ξ and a local* classical *state representation $(z, v) = ((z_a, z_b), (v_a, v_b))$ of the system S, defined on V_ξ, such that:*

{iY1}

1. *The (local) state equations are* {eYSub}

$$\begin{aligned} \dot{z}_a &= f_a(t, z_a, v_a), && \text{(8a)} \quad \{eYSuba\} \\ \dot{z}_b &= f_b(t, z_a, z_b, v_a, v_b). && \text{(8b)} \quad \{eYSubb\} \end{aligned}$$

{iY2}

2. *Let Y be the local subsystem associated to (8a) and let $\pi : V_\xi \to Y$ be the corresponding Lie–Bäcklund submersion. We have $\pi^*(T^*Y) = \mathrm{span}\, \{dt,\, dz_a,\, (dv_a^{(k)} : k \in I\!N)\} = span \left\{dt, dy^{(k)} : k \in I\!N\right\}$. In particular, Y is an output subsystem of S.*

3. *$Y_{n-1} = span\, \{dt, dz_a\}$ and $Y_n = span\, \{dt, dz_a, dv_a\}$. Furthermore, one may choose[17] $z_a \subset \{y^{(0)}, \dots, y^{(n-1)}\}$ and $v_a \subset \{y^{(0)}, \dots, y^{(n)}\}$.*

4. *Let[18] $\tilde{\Delta} = \{\xi \in S \mid y^{(k)}(\xi) = 0, k \in I\!N\}$. Assume that $\xi \in \tilde{\Delta}$. If $\mathbb{Y}_n$ and $\mathbb{Y}_{n-1}$ are nonsingular at ξ, then the local state representation (8) is* strongly adapted *to the output subsystem Y around ξ.*

Proof. See appendix E. □

The next result needs less regularity assumptions than the last theorem. It also assures the local existence of the output subsystem.

{toutputsubsystem}

Theorem 3 (Existence of output subsystems – invertible case) *Assume that S is a system with (a globally defined) output $y = (y_1, \dots, y_p)$. Assume that, around any $\xi \in S$ there exists a local proper state representation (x, u) and $\alpha = (\alpha_0, \dots \alpha_p) \in I\!N^p$ such that[19]*

{yintxu}

1. *$span\, \{dy\} \subset span\, \{dt, dx, du\}$.* {yxnaocresce}

2. *$span\, \{dx\} \cap span \left\{dt, dy^{\langle\langle \alpha - 1\rangle\rangle}\right\} = span\, \{dx\} \cap span \left\{dt, dy^{\langle\langle \alpha \rangle\rangle}\right\}$.* {yxnaosing1}

3. *$span \left\{dt, dx, dy^{\langle\langle \alpha - 1\rangle\rangle}\right\}$ is locally nonsingular around ξ.* {yxunaosing}

4. *$span \left\{dt, dx, du, dy^{\langle\langle \alpha \rangle\rangle}\right\}$ is locally nonsingular around ξ.*

[17]This means that the components of z_a and v_a are functions belonging to these sets of functions.

[18]This subset is important in the context of implicit systems (see (20)).

[19]Note that α may vary from each local choice of (x, u).

{tyindependent}

5. The set $\{dt, dy^{\langle\langle\alpha\rangle\rangle}\}$ *is independent at* ξ.

Then, around any $\xi \in S$ *there exists a local output subsystem* Y *and a strongly adapted state representation.*

Proof. By 3 and 5, one may construct a family $x_b \subset x$ of functions such that $\mathbb{B}_1 = \{dt, dx_b, dy^{\langle\langle\alpha-1\rangle\rangle}\}$ is a local basis of $\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha-1\rangle\rangle}\right\}$. Let $\beta \in I\!N^p$ be a multi-index. Then, at every $\nu \in S$ one may write

$$\begin{aligned}\dim\left(\operatorname{span}\left\{dt, dx, dy^{\langle\langle\beta\rangle\rangle}\right\}\right) &= \dim(\operatorname{span}\{dx\}) \\ &\quad + \dim\left(\operatorname{span}\left\{dt, dy^{\langle\langle\beta\rangle\rangle}\right\}\right) \\ &\quad - \dim\left(\operatorname{span}\{dx\} \cap \operatorname{span}\left\{dt, dy^{\langle\langle\beta\rangle\rangle}\right\}\right) \qquad (9)\end{aligned}$$ {eEstrela}

By 5, we have that $\operatorname{span}\left\{dt, dy^{\langle\langle\alpha-1\rangle\rangle}\right\}$ is locally nonsingular around ξ. Now by (9) for $\beta = \alpha - 1$, by 3, and by the nonsingularity of $\operatorname{span}\{dx\}$ it follows that $\operatorname{span}\{dx\} \cap \operatorname{span}\left\{dt, dy^{\langle\langle\alpha-1\rangle\rangle}\right\}$ is nonsingular around ξ.

By similar arguments, it is clear from 2 and (9) for $\beta = \alpha$ that

$$\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha\rangle\rangle}\right\} \text{ is nonsingular around } \xi. \qquad (10)$$ {eNonsingular}

It will be shown now that $\mathbb{B}_2 = \{dt, dx_b, dy^{\langle\langle\alpha\rangle\rangle}\}$ is a basis of $\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha\rangle\rangle}\right\}$. In fact, since $\{dt, dx_b, dy^{\langle\langle\alpha-1\rangle\rangle}\}$ is a basis of $\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha-1\rangle\rangle}\right\}$, it is clear that $\mathbb{B}_2$ generates $\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha\rangle\rangle}\right\}$. By 5 and (9) with $\beta = \alpha - 1$, and by 2 and (9) with $\beta = \alpha$, it follows easily that $\operatorname{card}\mathbb{B}_2 = \dim\left(\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha\rangle\rangle}\right\}\right)$. Hence $\mathbb{B}_2$ must be a basis of $\operatorname{span}\left\{dt, dx, dy^{\langle\langle\alpha\rangle\rangle}\right\}$. By 4 and (10), there exists a family $u_b \subset u$ of functions such that $\{dt, dx_b, du_b, dy^{\langle\langle\alpha\rangle\rangle}\}$ is a basis of $\operatorname{span}\left\{dt, dx, du, dy^{\langle\langle\alpha\rangle\rangle}\right\}$.

Now let $\beta = (\alpha, 0, \ldots, 0)$, where we have completed α with $m - p$ zeros in order to obtain a multiindex of dimension $p + \operatorname{card} u_b$. Let $v = (y, u_b)$. By construction, $v^{\langle\langle\beta-1\rangle\rangle} = y^{\langle\langle\alpha-1\rangle\rangle}$ and $v^{\langle\langle\beta\rangle\rangle} = (y^{\langle\langle\alpha\rangle\rangle}, u_b)$.

Then it follows[20] that $\operatorname{span}\{dx\} \in \operatorname{span}\left\{dt, dx_b, dv^{\langle\langle\beta-1\rangle\rangle}\right\}$, $\operatorname{span}\{du, d\dot{x}_b\} \in \operatorname{span}\left\{dt, dx_b, dv^{\langle\langle\beta\rangle\rangle}\right\}$ and $\operatorname{span}\{dx_b, dv\} \subset \operatorname{span}\{dt, dx, du\}$. Hence, lemma 4 implies that (x_b, v) is local state representation for S. In particular, by extending the state, one may take the local state representation $(\tilde{x}, \tilde{u})$ where $\tilde{x} = (x_a, x_b)$, $\tilde{u} = (u_a, u_b)$, $x_a = (y_1^{(0)}, \ldots, y_1^{(\alpha_1-1)}, \ldots, y_p^{(0)}, \ldots, y_p^{(\alpha_p-1)})$, and $u_a = (y_1^{(\alpha_1)}, \ldots, y_p^{(\alpha_p)})$. So, we may write the following state equations {eStateRepresentationY}

$$\begin{aligned}\dot{x}_b(t) &= g(t, x_a, x_b, u_a, u_b) \qquad (11a)\\ \dot{y}_1^{(0)} &= y_1^{(1)} \\ \dot{y}_1^{(1)} &= y_1^{(2)}\end{aligned}$$ {eZeroDynamics}

[20]Using the fact that the $\operatorname{span}\{d\dot{x}_b\} \subset \operatorname{span}\{d\dot{x}\} \subset \operatorname{span}\{dt, dx, du\}$.

$$
\begin{aligned}
& \vdots \\
\dot{y}_1^{(\alpha_1-1)} &= u_{a1} && \text{(11b)}\\
& \vdots \\
\dot{y}_p^{(0)} &= y_p^{(1)} \\
\dot{y}_p^{(0)} &= y_p^{(2)} \\
& \vdots \\
\dot{y}_p^{(\alpha_p-1)} &= u_{ap} && \text{(11c)}
\end{aligned}
$$

Note that (11) is of the form (8)(after replacing (z_a, z_b) by (x_a, x_b) and (v_a, v_b) by (u_a, u_b)).

Let V be the open neighborhood of ξ such that the state representation $(\tilde{x}, \tilde{u})$ is defined on V. Let $\pi : V \to Y$, where $Y = \pi(V)$, be the map defined by $\pi(t, x_a, x_b, (u_a^{(j)}, u_b^{(j)} : j \in I\!N)) = (t, x_a, (u_a^{(j)}, : j \in I\!N))$. As $(t, (y^{(k)} : k \in I\!N)) = (t, x_a, (u_a^{(j)}, : j \in I\!N))$ are global coordinates for Y, one may define the Cartan field ∂_Y by

$$\partial_Y = \frac{\partial}{\partial t} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor p \rceil}} y_j^{(k+1)} \frac{\partial}{\partial y_j^{(k)}}$$

With this definition it is easy to show that π is a Lie-Bäcklund submersion. Then it is clear that Y is a local output subsystem and the state representation $(\tilde{z}, \tilde{v})$ is strongly adapted to the local output subsystem (see Definition 11). □

{oOBS1}

Remark 3 *One sees that the state-feedback that is constructed in the proof of the last theorem is a solution of the input-output decoupling problem. The proof of theorem 3 shows that one can construct a state space representation* $((x_a, u_a), (x_b, u_b))$ *that is strongly adapted to the output subsystem* Y *such that:*

(A) $x_a = (y_1^{(0)}, \ldots, y_1^{(\alpha_1-1)}, \ldots, y_p^{(0)}, \ldots, y_p^{(\alpha_p-1)})$.

(B) $u_a = (y_1^{(\alpha_1)}, \ldots, y_p^{(\alpha_p)})$.

(C) x_b *completes* $\{dt, dy^{\langle\langle \alpha-1 \rangle\rangle}\}$ *to basis of* $\{dt, dx, dy^{\langle\langle \alpha-1 \rangle\rangle}\}$.

(D) One may choose u_b *in order to complete* $\{dt, dx_a, dx_b, du_a\}$ *to a basis* $\{dt, dx_a, dx_b, du_a, du_b\}$ *of span* $\left\{dt, dx, du, y^{\langle\langle \alpha \rangle\rangle}\right\}$.

The equation (11a) is in fact the *Zero Dynamics*. The dimension of the state x_b of the zero dynamics[21] is called by the *defect* of the output subsystem Y .

[21]The idea of zero dynamics was introduced in (Byrnes & Isidori 1988, Byrnes & Isidori 1991). The notion of defect is directly related to nonflatness, and can be found in (Fliess et al. 1995). Here the use of the word *defect* is associated to a particular output, whereas the real notion of defect is the minimal dimension of z for all the possible output choices.

Note that x_b is a set of fiber coordinates of the corresponding Lie-Bäcklund submersion $\pi : V \subset S \to Y$ (see definition 1 and Proposition 2). When the zero dynamics is absent, the system is flat with flat output $\tilde{y} = (y, u_b)$. The following result is a generalization of a result of (Martin 1992).

{cMartin2}

Corollary 2 *Assume that S is a system with a local proper state representation (x, u) with a proper output $y = (y_1, \dots, y_p)$. Let $\alpha = (\alpha_0, \dots \alpha_p)$ be a multiindex and consider the notations of the last theorem. Then y is a local flat output according Def. 6 if only if there exists some α such that, locally, one has*

{yxnaocresce1}

1. *span* $\{dx\} \subset$ *span* $\left\{dt, dy^{\langle\langle \alpha - 1 \rangle\rangle}\right\}$.

{yxunaosing1}

2. *span* $\{du\} \subset$ *span* $\left\{dt, dy^{\langle\langle \alpha \rangle\rangle}\right\}$.

{tyindependent1}

3. *The set of* 1*-forms* $\{dt, dy^{\langle\langle \alpha \rangle\rangle}\}$ *is independent at* ξ.

Proof. The necessity follows from the Definition 6 and part 1 of Lemma 1. The sufficiency is a straightforward application of Lemma 4. □

Now it is shown that the regularity[22] result of Martin (Martin 1993) for time-invariant systems *around an equilibrium point* may be obtained as a consequence of theorem 3.

{cMartin}

Theorem 4 *Let S be a system and assume that (x, u) is a local classical state representation defined on a neighborhood U of ξ. Let y be a classical output of S defined in U. Assume that state equations are of the form*

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) \end{aligned}$$

and suppose that[23]

$$\xi = (t_0, x_0, (u_0^{(j)}, j \in I\!N)), \text{ where } f(x_0, u_0) = 0 \text{ and } u_0^{(j)} = 0, j \in I\!N.$$

Assume that the components of f, g and h are analytical maps with respect to their arguments. Suppose that card $x = n$ *and* card $y =$ card $u = m$. *Let TS be the linearized system at ξ, given by*

$$\begin{aligned} \dot{x}(t) &= \left.\frac{\partial f}{\partial x}\right|_{x_0} x(t) + g(x_0)u(t) \\ y(t) &= \left.\frac{\partial h}{\partial x}\right|_{x_0} x(t) \end{aligned}$$

[22]The definition of regularity of Martin is more general than other regularity definitions, for instance the one of (Di Benedetto & Grizzle 1990), at least for square systems. However, (Di Benedetto & Grizzle 1990) also consider nonsquare systems.

[23]The time component t is irrelevant, as well as the presence of dt in $\mathcal{Y}_k$, since the system is time-invariant.

Consider the codistributions Y_k and $\mathcal{Y}_k, k = -1, 0, 1, 2, \ldots,$ obtained for system S by using (7)*. When written in local coordinates, $y^{(k)}$ is also an analytical map, and then one may define the generical dimensions*[24] *$\mathcal{D}(\mathcal{Y}_k)$. Similarly, let $T\mathcal{Y}_k, k = -1, 0, 1, 2, \ldots,$ be the codistributions obtained for system TS by using* (7)*. Assume that in U one has:*

- $\mathcal{D}(\mathcal{Y}_n) - \mathcal{D}(\mathcal{Y}_{n-1}) = m$
- *There exists $n_0 \geq n$ such that $\dim T\mathcal{Y}_{n_0} = \mathcal{D}(\mathcal{Y}_{n_0})$.*

Then there exists $\widehat{x} \subset x$ such that $(\widehat{x}, y)$ is a local state representation of S with state $\widehat{x}$ and input y.

Remark 4 *It will be clear from the proof of the last result that $\widehat{x}$ is the state of the zero dynamics.*

Proof. As ξ is an equilibrium point of the Cartan field, it will be shown that $T\mathcal{Y}_k$ and TY_k may be identified respectively with $\mathcal{Y}_k|_\xi$ and $Y_k|_\xi$, for $k \in I\!N$. Then it will follows that $\dim \mathcal{Y}_{n_0}|_\xi = \mathcal{D}(\mathcal{Y}_{n_0})$. In fact, let $A = \left.\frac{\partial f}{\partial x}\right|_{x_0}$, $B = g(x_0)$, $C = \left.\frac{\partial h}{\partial x}\right|_{x_0}$. Then, for the linearized system, it is easy to show that[25]

$$dy^{(k)} = CA^k dx + \sum_{j=0}^{k-1} CA^j B du^{(k-j-1)}$$

Now note that, for the nonlinear system

$$dy^{(0)} = Cdx + \phi dx$$

where $\phi = \frac{\partial h}{\partial x} - C$ is such that $\phi|_\xi = 0$.

Since ξ is an equilibrium point, for any function α that is locally defined around $\xi \in S$ one has $\dot{\alpha}|_\xi = 0$. Computing $dy^{(1)}$ one gets

$$\begin{aligned} dy^{(1)} &= [\frac{d}{dt}(C + \phi)]dx + (C + \phi)d\dot{x} \\ &= (CA + \dot{\phi} + \phi A)dx + (CB + \phi B)du \end{aligned}$$

Hence it is easy to show by induction that, for the nonlinear system:

$$dy^{(k)} = (CA^k + \phi_k)dx + \sum_{j=0}^{k-1}(CA^j B + \psi_{kj})du^{(k-j-1)}$$

where the matrices ϕ_k and ψ_{kj} are null at the point ξ. In particular this shows that the claimed identification between $T\mathcal{Y}_k$ (resp. TY_k) and $\mathcal{Y}_k|_\xi$ (resp. $Y_k|_\xi$), for $k \in I\!N$, makes sense.

[24]It is easy to show that these generical dimensions coincides with the dimensions taken over the meromorphic field of (Di Benedetto, Grizzle & Moog 1989).

[25]Abusing notation, and letting dx standing for the column vector $(dx_1, \ldots, dx_n)^T$, then Adx stands for $(\sum_{j=1}^n a_{1j}dx_j, \ldots, \sum_{i=1}^n a_{nj}dx_j)^T$ and so on.

Assuming without loss of generality that U is connected[26], recall that, given an analytical codistribution Ω, then $\mathcal{D}(\Omega) = \dim \Omega|_\nu$ if and only if $\nu \in U$ is a regular point of Ω. Now choose $\nu \in U$ such that ν is a regular point of the codistributions $\mathcal{Y}_k, Y_k, k = 0, \dots, n$ defined in (7b)-(7c). In particular, $\mathcal{D}(\mathcal{Y}_k) = \dim \mathcal{Y}_k|_\nu$, and the same property holds for Y_k.

From the fact that $\mathcal{D}(\mathcal{Y}_k) = \dim \mathcal{Y}_k|_\nu$, and from part 10 of Lemma 5 of Appendix C applied to system S at ν, and to system TS at ν, one concludes that there exist nonnegative integers δ, Δ, l, L, η and N, with $\eta \le n$ and $N \le n$, such that

$$\begin{aligned} \mathcal{D}(\mathcal{Y}_k) &= 1 + \delta + l(k+1), k \ge \eta, \\ \dim T\mathcal{Y}_k &= 1 + \Delta + L(k+1), k \ge N, \end{aligned}$$

Since $\dim T\mathcal{Y}_{n_0} = \mathcal{D}(\mathcal{Y}_{n_0})$, then $\delta - \Delta = (L-l)(n_0+1)$. Remember that $n_0 \ge n$, $n \ge \Delta \ge 0$, $n \ge \delta \ge 0$, then $L = l$ and $\delta = \Delta$. This implies that $|\delta - \Delta| \le n$, and so the equation $|\delta - \Delta| = |(L-l)|(n_0+1)$ has a unique solution $L = l$ and $\Delta = \delta$. This shows that $\dim T\mathcal{Y}_n = \mathcal{D}(\mathcal{Y}_n)$ and hence we have that $\dim T\mathcal{Y}_k = \dim \mathcal{Y}_k|_\xi = \mathcal{D}(\mathcal{Y}_k)$ for all $k \ge n$. In particular, ξ is a regular point of $\mathcal{Y}_k$ for $k \ge n$. Furthermore, since $\mathcal{D}(\mathcal{Y}_n) - \mathcal{D}(\mathcal{Y}_{n-1}) = m$, then $l = L = m$. In particular, from Lemma 5 part 5 applied to the linear system TS, we must have that $\dim TY_{k+1} - \dim TY_k = m$ for all $k \in I\!N$. Then $\dim Y_{k+1}|_\xi - \dim Y_k|_\xi = m$, for all $k \in I\!N$. In particular, this means that the set $\{dt, dy^{(0)}, \dots, dy^{(k)}\}$, (when computed for system S) is locally linearly independent around ξ for all $k \in I\!N$. In particular Y_k is also nonsingular at ξ for $k \ge n$. We show now that, locally around ξ, we have $\mathcal{Y}_n \cap \operatorname{span}\{dx\} = \mathcal{Y}_{n+1} \cap \operatorname{span}\{dx\}$. In fact, from Lemma 5 part 7 applied to system TS, one has $T\mathcal{Y}_n \cap \operatorname{span}\{dx\} = T\mathcal{Y}_{n+1} \cap \operatorname{span}\{dx\}$. Hence, $\mathcal{Y}_n \cap \operatorname{span}\{dx\}|_\xi = \mathcal{Y}_{n+1} \cap \operatorname{span}\{dx\}|_\xi$. As

$$\dim \mathcal{Y}_k|_\nu = \dim Y_k|_\nu + \dim \operatorname{span}\{dx\}|_\nu - \dim \mathcal{Y}_k \cap \operatorname{span}\{dx\}|_\nu$$

for all $\nu \in U$, the nonsingularity of Y_k and $\mathcal{Y}_k$ around ξ for $k \ge n$ implies the nonsingularity and the smoothness of $Y_k \cap \operatorname{span}\{dx\}$ around ξ for $k \ge n$. So, we locally have $\dim Y_n \cap \operatorname{span}\{dx\} = \dim Y_{n+1} \cap \operatorname{span}\{dx\}$. As $Y_n \cap \operatorname{span}\{dx\} \subset Y_{n+1} \cap \operatorname{span}\{dx\}$ then one must have $Y_n \cap \operatorname{span}\{dx\} = Y_{n+1} \cap \operatorname{span}\{dx\}$.

It is easy to show that[27] if $\mathcal{U}$ and $\mathcal{Y}$ are analytical codistributions such that $\mathcal{U} \subset \mathcal{Y}$ on an open and dense subset of S, then if ξ is a regular point of $\mathcal{Y}$, then, locally around ξ, one has $\mathcal{U} \subset \mathcal{Y}$.

Using the last remark, it will be shown now that, locally around ξ one has $\operatorname{span}\{du\} \subset \mathcal{Y}_n$. In particular $\operatorname{span}\{dx, du\} + \mathcal{Y}_n = \mathcal{Y}_n$ is locally nonsingular around ξ. In fact, from part 9 of Lemma 5, in an open and dense set, one has $\operatorname{span}\{du\} \subset \mathcal{Y}_n$. From the nonsingularity of $\mathcal{Y}_n$ at ξ, the claimed property follows. Summarizing:

1. $\operatorname{span}\{dy\} \subset \operatorname{span}\{dt, dx\}$

[26] Otherwise one may take the connected component of U containing ξ

[27] In fact, take a local basis $\{\omega_1, \dots, \omega_k\}$ of $\mathcal{Y}$ around ξ. Now take some $\omega \in \mathcal{U}$. If $\{\omega|_\nu, \omega_1|_\nu, \dots, \omega_k|_\nu\}$ were linearly independent, then $\mathcal{D}(\mathcal{Y} + \mathcal{U})$ would be greater than $\mathcal{D}(\mathcal{Y})$.

2. span $\{dx\} \cap Y_n = $ span $\{dx\} \cap Y_{n+1}$.
3. The codistribution $\mathcal{Y}_n = \text{span}\left\{dt, dx, dy^{(0)}, \dots, dy^{(n)}\right\}$ locally nonsingular around ξ.
4. The codistribution $\mathcal{Y}_n + \text{span}\{du\} = \text{span}\left\{dt, dx, dy^{(0)}, \dots, dy^{(n)}\right\}$ is locally nonsingular around ξ.
5. The set $\{dt, dy^{(0)}, \dots, dy^{(n)}\}$ is independent at ξ.

Then, the desired result follows from Theorem 3 in the particular case where α is the p-multi index $(n, n, \dots, n)$. $\square$

Remark 5 *An important example for which the regularity assumptions of theorems 4 and 3 hold, but not the ones of lemma 5 of Appendix C, is given in the end of this section (see* (12)*). Note that Theorem 3 assures that there exists a state representation of the form* (11)*, where, in that case, $\widehat{u}$ is absent.*

It is easy to see that the last proof shows that the following conditions are equivalent:

- *There exists $n_0 \geq n$ such that $\dim T\mathcal{Y}_{n_0} = \mathcal{D}(\mathcal{Y}_{n_0})$.*
- $\dim T\mathcal{Y}_n = \mathcal{D}(\mathcal{Y}_n)$.
- $\dim T\mathcal{Y}_k = \mathcal{D}(\mathcal{Y}_k)$ *for all $k \geq n$.*

The last condition is the notion of regularity adopted by Martin (Martin 1993).

The theorem 3 is generalized now for the case where the system is not right-invertible.

{tNoninvertible}

Theorem 5 (Existence of output subsystems – non-invertible case) *Let S be a system with proper state representation (x, u) and proper output y, both defined around some $\xi \in S$. Assume that there exists a partition $y = (\bar{y}, \widehat{y})$ such that, locally around ξ, one has*

{pyintxu}

1. *span $\{d\bar{y}\} \subset$ span $\{dt, dx, du\}$.*

{pyxnaocresce}

2. *span $\{dx\} \cap$ span $\left\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha-1)}\right\} =$ span $\{dx\} \cap$ span $\left\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha)}\right\}$.*

{pyxnaosing1}

3. *span $\left\{dt, dx, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha-1)}\right\}$ is locally nonsingular around ξ.*

{pyxunaosing}

4. *span $\left\{dt, dx, du, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha)}\right\}$ is locally nonsingular around ξ.*

{ptyindependent}

5. *The set $\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha)}\}$ is pointwise independent in an open neighborhood of ξ.*

6. *span* $\left\{dy^{(0)},\ldots,dy^{(\alpha-1)}\right\} \subset$ *span* $\left\{dt,dx,d\bar{y}^{(0)},\ldots,d\bar{y}^{(\alpha-1)}\right\}$

7. *span* $\left\{dt,dy^{(0)},\ldots,dy^{(k)}\right\}$ *is nonsingular for* $k=\alpha$ *and* $k=\alpha-1$.

8. *span* $\left\{dy^{(\alpha)}\right\} \subset$ *span* $\left\{dt,dy^{(0)},dy^{(1)},\ldots,dy^{(\alpha-1)},d\bar{y}^{(\alpha)}\right\}$

Then there exist a local output subsystem Y *that admits an adapted state representation* $(\tilde{x},\tilde{u})$, *where* $\tilde{x}=(x_a,x_b)$ *and* $\tilde{u}=(u_a,u_b)$, *with state equations*

$$\begin{aligned}\dot{x}_a(t) &= f_a(t,x_a(t),u_a(t))\\ \dot{x}_b(t) &= f_b(t,x_a(t),x_b,u_a(t),u_b(t))\end{aligned}$$

such that span $\{dt,dx_a\}=$ *span* $\left\{dt,dy^{(0)},\ldots,dy^{(\alpha-1)}\right\}$, *span* $\{dt,dx_a,du_a\}=$ *span* $\left\{dt,dy^{(0)},\ldots,dy^{(\alpha)}\right\}$, *span* $\{dt,d\tilde{x}\}=$ *span* $\left\{dt,dx,dy^{(0)},\ldots,dy^{(\alpha-1)}\right\}$, *and span* $\{dt,d\tilde{x},d\tilde{u}\}=$ *span* $\left\{dt,dx,du,dy^{(0)},\ldots,dy^{(\alpha)}\right\}$. *Furthermore, one may choose* $u_a=\bar{y}^{(\alpha)}$ *and* $x_a\subset\{y^{(0)},\ldots,y^{(\alpha-1)}\}$. *In particular, span* $\{dx\}+\mathcal{Y}=$ *span* $\{dx_b\}\oplus\mathcal{Y}$ *and span* $\{dx,du\}+\mathcal{Y}=$ *span* $\{dx_b,du_b\}\oplus\mathcal{Y}$. *Moreover, if the next condition holds*

9. *Let*[28] $\tilde{\Delta}=\{\xi\in S \mid y^{(k)}(\xi)=0,k\in I\!N\}$. *Assume that* $\xi\in\tilde{\Delta}$. *Suppose that span* $\left\{dy^{(0)},dy^{(1)},\ldots,dy^{(k)}\right\}$ *is nonsingular around* ξ *for* $k=\alpha-1$ *and* $k=\alpha$.

then the state representation $(\tilde{x},\tilde{u})$ *is strongly adapted to the (local) output subsystem* Y *around* ξ.

Remark 6 *The proof of Theorem 5 shows that*

(A) *One may choose* $u_a=\bar{y}^{(\alpha)}$.

(B) *One may choose* $x_a\subset\{y^{(0)},\ldots,y^{(\alpha-1)}\}$ *such that* $\{dt,dx_a\}$ *is a local basis of span* $\left\{dy,\ldots,dy^{(\alpha-1)}\right\}$.

(C) *One may chose* x_b *in a way that* dx_b *completes* $\{dt,dx_a\}$ *to a local basis of span* $\left\{dt,dx,dy,\ldots,dy^{(\alpha-1)}\right\}$.

(D) *One may chose* u_b *in order to complete* $\{dt,dx_a,dx_b,du_a\}$ *to a basis* $\{dt,dx_a,dx_b,du_a,du_b\}$ *of span* $\left\{dt,dx,du,dy,\ldots,dy^{(\alpha)}\right\}$.

Proof. The affirmations 1 to 5 says that the assumptions of Theorem 3 holds for the output $\bar{y}$. One may apply such result for the output $\bar{y}$, obtaining a state representation $(\tilde{x}^1,\tilde{u}^1)=((\tilde{x}^1_a,\tilde{x}^1_b),(\tilde{u}^1_a,\tilde{u}^1_b))$ as stated in the Remark 3, that is $x^1_a=(\bar{y}^{(0)},\ldots,\bar{y}^{(\alpha-1)})$, and $u^1_a=\bar{y}^{(\alpha)})$, for which span $\left\{dt,dx^1_a\right\}=$ span $\left\{dt,d\bar{y}^{(0)},\ldots,d\bar{y}^{(\alpha-1)}\right\}$, span $\left\{dt,dx^1_a,du_a\right\}=$ span $\left\{dt,d\bar{y}^{(0)},\ldots,d\bar{y}^{(\alpha)}\right\}$, span $\left\{dt,d\tilde{x}^1\right\}=$ span $\{dt,dx,\ d\bar{y}^{(0)},\ldots,d\bar{y}^{(\alpha-1)}\}$, and span $\left\{dt,d\tilde{x}^1,d\tilde{u}^1\right\}=$ span $\left\{dt,dx,du,d\bar{y}^{(0)},\ldots,d\bar{y}^{(\alpha)}\right\}$. From 7, one may locally complete $\{dt,dx^1_a\}$

[28]This subset is important in the context of implicit systems (see (20)).

to a basis $\{dt, dx_a\}$ of span $\{dt, dy, \ldots, dy^{(\alpha-1)}\}$. By 3 and 6 it follows that $\mathcal{Y}_{\alpha-1} = \text{span}\left\{dt, dx, dy^{(0)}, \ldots, dy^{(\alpha-1)}\right\} = \text{span}\left\{dt, dx, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha-1)}\right\}$ is nonsingular. So one may complete dx_a to a local basis $\{dt, dx_a, dx_b\}$ of $\mathcal{Y}_{\alpha-1}$. It follows that span $\left\{dt, dx_a^1, dx_b^1\right\} = \text{span}\,\{dt, dx_a, dx_b\}$. Let $\tilde{u} = (u_a, u_b) = \tilde{u}^1$. Hence, if $\tilde{x} = (x_a, x_b)$, then $(\tilde{x}, \tilde{u})$ is also a proper state representation (see Corollary 3 of section 5.1). By 8, it follows that span $\{d\dot{x}_a\} \subset \text{span}\,\{dt, dx_a, du_a\}$. From this it is easy to show that the state representation $(\tilde{x}, \tilde{u})$ is adapted to the output subsystem Y.

Now assume that 9 holds. Let $\mathbb{Y}_k = \text{span}\left\{dy^{(0)}, \ldots, dy^{(k)}\right\}$ and $Y_k = \text{span}\left\{dt, dy^{(0)}, \ldots, dy^{(k)}\right\}$. We show first that span $\{dt\} \cap \mathbb{Y}_k|_\xi = \{0\}$ for every point ξ of $\tilde{\Delta}$. In fact, let $\xi \in \tilde{\Delta}$ and let $\eta = \sum_{i=0}^{k} \sum_{j=1}^{r} \alpha_{ij} dy_j^{(i)}|_\xi = \beta dt|_\xi$. Then $\beta|_\xi = \langle \eta; \frac{d}{dt} \rangle|_\xi = \sum_{i,j} \langle \alpha_{ij} dy_j^{(i)}; \frac{d}{dt} \rangle|_\xi = \sum_{i,j} \alpha_{ij} \langle dy_j^{(i)}; \frac{d}{dt} \rangle|_\xi = \sum_{i,j} \alpha_{ij} y_j^{(i+1)}|_\xi = 0$.

Now note that

$$\dim Y_k = \dim\left(\text{span}\,\{dt\}\right) + \dim \mathbb{Y}_k - \dim\left(\text{span}\,\{dt\} \cap \mathbb{Y}_k\right)$$

The nonsingularity of span $\{dt\}$, Y_k and $\mathbb{Y}_k$ for $k = \alpha - 1$ and for $k = \alpha$ implies the nonsingularity of span $\{dt\} \cap \mathbb{Y}_{\alpha-1}$ and span $\{dt\} \cap \mathbb{Y}_\alpha$ around ξ. In particular, span $\{dt\} \cap \mathbb{Y}_{\alpha-1} = \text{span}\,\{dt\} \cap \mathbb{Y}_{\alpha-1} = \{0\}$ in an open neighborhood of ξ. We show now that one has span $\{dx_a\} = \mathbb{Y}_{\alpha-1}$ around ξ. Since $x_a \subset \{y, \ldots, y^{(\alpha-1)}\}$, it is clear that span $\{dz_a\} \subset \mathbb{Y}_{\alpha-1}$. To show the inverse inclusion, take some $\omega \in \mathbb{Y}_{\alpha-1}$. Then $\omega = \sum_{i=1}^{n_a} \alpha_i dx_{a_i}|_x i + \beta dt$ for convenient functions $\alpha_i, i \in \lfloor n_a \rceil$, and β. Let V_ξ be an open neighborhood of ξ for which span $\{dt\} \cap \mathbb{Y}_{\alpha-1} = 0$. If for some $\nu \in V_\xi$ one has $\beta|_\nu \neq 0$, then $\beta dt|_\nu$ will be in span $\{dt\} \cap \mathbb{Y}_{\alpha-1}|_\nu$. In particular, on V_ξ, ω belongs to span $\{dx_a\}$. By similar arguments, one shows that span $\{dx_a, du_a\} = \mathbb{Y}_\alpha$. By derivation, it follows easily that span $\left\{dx_a, (du_a^{(k)} : k \in I\!N)\right\} = \text{span}\left\{dy^{(k)} : k \in I\!N\right\}$. □

Now we present the tricky example of Respondek (Respondek 1992). Consider the control system {eRespondek}

$$\begin{aligned}
\dot{x}_1 &= x_5 + x_3 x_4 u_1 - x_3^2 u_2 && \text{(12a)}\\
\dot{x}_2 &= x_6 + x_4^2 u_1 - x_3 x_4 u_2 && \text{(12b)}\\
\dot{x}_3 &= a(x) && \text{(12c)}\\
\dot{x}_4 &= b(x) && \text{(12d)}\\
\dot{x}_5 &= u_1 && \text{(12e)}\\
\dot{x}_6 &= u_2 && \text{(12f)}\\
y_1 &= x_1 && \text{(12g)}\\
y_2 &= x_2 && \text{(12h)}
\end{aligned}$$

where $a(x)$ and $b(x)$ are both smooth functions such that $a(0) = b(0) = 0$. For this example the regularity assumptions of Lemma 5 do not hold (for instance the codistribution $\mathcal{Y}_1 = \text{span}\left\{dt, dx, dy, dy^{(1)}\right\}$ is singular around $x = 0$ and

$u^{(j)} = 0, j \in I\!N$). It is shown in (Martin 1993), that the assumptions of theorem 4 holds for this example. In particular, from the proof of Theorem 4 it follows that the regularity assumption of Theorem 3 holds.

5 The notion of dynamic feedback

{sDynamicFeedback}

The notion of (regular) dynamic feedback relies in the concept of feedback extension. A system E is a feedback extension of S if S is a subsystem of E (see for instance (Fliess et al. 1997b, Fliess et al. 1997a, Fliess et al. 1998, Fliess et al. 1999)). The following definitions establish a intrinsic notion of exogenous (or endogenous) dynamic feedback

Definition 12 *(Feedback Extension) A system E is a (local) feedbcak extension of a control system S around some $\xi \in S$, if*

- *There exists an open neighborhood $U \subset S$ of ξ and a Lie-Bäcklund submersion $\pi : E \to U$.*
- *The fiber dimension is finite and constant everywhere*[29].

The feedback extension is said to be endogenous *if π is a (local) Lie-Bäcklund isomorphism. If the feedback extension is not endogenous, then it is said to be* exogenous*.*

Remark 7 *Let $\xi \in \pi(E)$. There is no loss of generality of considering that $\pi(E) = U$, where U is an open neighborhood of ξ. In fact, as $\pi : E \to U$ is an open map, if it is not surjective, it suffices to take $\tilde{U} = \phi(E)$ and to consider $\pi : E \to \tilde{U}$.*

{pEndogenous}

Proposition 4 *Let (E, π) be a (local) feedback extension of a control system S. Then E is a control system of same differential dimension than S. Furthermore, the feedback extension is endogenous if and only if the dimension of the fiber is zero.*

Proof. Since S is a control system, one can choose a local state representation (x, u). Then $\{t, x, (u^{(k)} : k \in I\!N)\}$ is a local coordinate system of S. Denote the Cartan fields of E and S respectively by ∂_E and ∂_S. Abusing notation, one lets x and $u^{(k)}$ stand respectively for $x \circ \pi$ and $u^{(k)} \circ \pi$ for all $k \in I\!N$. As π is Lie-Bäcklund, such abuse of notation makes sense (see appendix F).

Since π is a submersion and the dimension of the fiber is finite, one may choose local coordinates (w, γ) for E and γ for S such that $\pi(w, \gamma) = \gamma$. Since the coordinate change map Γ such that $\gamma \mapsto (t, x, u^{(0)}, u^{(1)}, \ldots)$ is a local diffeomorphism, then the map $(w, \gamma) \mapsto (w, \Gamma(\gamma))$ is also a local diffeomorphism. In particular $\nu = \{t, w, x, (u^{(k)} : k \in I\!N)\}$ is a local coordinate system for E and so $((x, w), u)$ is a local state representation of E. In these coordinates one

[29] See definition 1 and Proposition 2.

may write $\pi(w, \nu) = \nu$. Since π is Lie-Bäcklund, from appendix F, it follows that $\partial_E(x \circ \pi) = \partial_S(x) \circ \pi = \dot{x} \circ \pi$. The following state representation of E is obtained: {eDynamicc}

$$\dot{x}(t) = f(t, x(t), u(t), \ldots, u^{(\alpha)}(t)) \quad (13a)$$ {eStateXX}

$$\dot{w}(t) = g(t, x(t), w(t), u(t), \ldots, u^{(\beta)}(t)) \quad (13b)$$ {eStateWW}

where (13a) is a local state representation of S. The first equation is equivalent to write $\partial_E(x \circ \pi) = \dot{x} \circ \pi$. In other words, (13a) is the state representation of S. The second equation only says that $\partial_E(w)$ may be written as a function of the the local coordinates.

In particular E is a control system of same differential dimension than S. It is also clear that, to say that π is a diffeomorphism is equivalent to say that the dimension of w is zero (which is the dimension of the fiber). □

The last definition is not clearly related to the classical feedback equations, since the state representations of the two control systems E and S are not chosen *a priori*. In fact, roughly speaking, a (dynamic) feedback in the classical sense is defined by the equations depending on the inputs and states of systems E and S. In the proof of the last proposition it was shown that the state of E is an extension of the state x of S. This is usually the case when one defines dynamic feedback from equations. Two things are not usual in equation (13). the first one is the fact that the input of system E is the same than one of system S. The second one is the fact that the state equations are not proper, that is, they depend on the derivatives of the inputs. We are ready to state our geometric definition of dynamic state feedback.

{dDynamicStateFeedback}

Definition 13 *(Dynamic State Feedback) Let (E, π) be a (local) dynamic extension of S. Choose local state representations (x, u) of S, defined on an open neighborhood V_ξ of $\xi = \pi(\zeta) \in S$, and (z, v) of E, defined on an open neighborhood W_ζ of $\zeta \in E$. Without loss of generality*[30]*, assume that $V_\xi = \pi(W_\zeta)$. Abusing notation, one may let x and u stand respectively for $x \circ \pi$ and $u \circ \pi$. Then $(E, \pi, (z, v), (x, u))$ is said to be a local dynamic state feedback if span $\{dx\} \subset$ span $\{dt, dz\}$. The system S with input u and state x is called* open loop system, *and the system E with input v and state z is* called closed loop system.

The last definition assures that x "*is part of the state z of the extension*", as shown in the next proposition. It also establishes the link between Definition 13 and the standard definition of regular feedback using equations. Note that v is the new input of the closed loop system.

{pDynamic}

Proposition 5 *Let $(E, \pi, (z, v), (x, u))$ be a local dynamic state feedback around some point $\nu \in E$. There exists a set w of functions locally defined around $\nu \in E$, such that span $\{dt, dx, dw\}$ = span $\{dt, dz\}$ in a neighborhood of ν. Then*

[30] If is not the case, one may take $\tilde{W} = W_\zeta \bigcap \pi^{-1}(V_\xi)$ and $\tilde{V} = \pi(\tilde{W})$ and to consider.

$(E, \pi, ((x,w),v),(x,u))$ *is also a local dynamic state feedback whose state equations are locally given by* {eDynamic}

$$
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t), \ldots, u^{(\alpha)}) && \text{(14a)} \quad \texttt{\{eStateX\}}\\
\dot{w}(t) &= g(t, x(t), w(t), v(t), \ldots, v^{(\beta)}) && \text{(14b)} \quad \texttt{\{eStateW\}}\\
u(t) &= \phi(t, x(t), w(t), v(t), \ldots, v^{(\gamma)}) && \text{(14c)} \quad \texttt{\{eStateU\}}
\end{aligned}
$$

where (14a) *are the local state equations of* S*. The state feedback is said to be proper if the state representation* $((x,w),v)$ *(of* E*) is proper with proper*[31] *output* u*. In this case the state equations are given by* {eDynamicproper}

$$
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)) && \text{(15a)} \quad \texttt{\{eStateXproper\}}\\
\dot{w}(t) &= g(t, x(t), w(t), v(t)) && \text{(15b)} \quad \texttt{\{eStateWproper\}}\\
u(t) &= \phi(t, w(t), z(t), v(t)) && \text{(15c)} \quad \texttt{\{eStateUproper\}}
\end{aligned}
$$

Remark 8 *The equation* (14c) *(or* (15c)*) is called* control law *and* (14) *(or* (15)*) are the state equations of the* closed loop system*.*

Proof. Since span $\{dx\} \subset$ span $\{dt, dz\}$ and the elements of $\{dt, dx\}$ are independent, one may choose w such that $\{dt, dx, dw\}$ is a local basis of span $\{dt, dz\}$. Now, note that $((x,w),v)$ is a local state representation of E. In fact, the coordinate change map $(t,z) \to (t,x,w)$ is a local diffeomorphism that is a classic state transformation[32].

The fact that (14a) is the local state representation of S is a consequence of the fact that π is Lie-Bäcklund (see the proof of proposition 4 and appendix F). Equations (14b) and (14c) are consequence of the fact that $\{t, x, w, (v^{(k)} : k \in \mathbb{N}\}$ is a local coordinate system for E. □

5.1 A complexity classification of state-feedback

{sCSF}

The last section have considered endogenous and exogenous feedback, which is a classification of complexity of feedback extensions. In particular, a state feedback $(E,(z,v),(x,u))$ may be endogenous or exogenous according the classification of the extension E. For endogenous feedbacks, one may identify E and S. Hence an endogenous state-feedback may be denoted by $((z,v),(x,u))$, since it is completely determined by the choice of two different state representations (z,v) and (x,u) of S.

The following definition is a complexity classification of endogenous state feedbacks (Delaleau & Pereira da Silva 1998a).

{dQuasistatic}

Definition 14 *(quasi-static/static feedback) Let* $((z,v),(x,v))$ *be an endogenous state feedback of a system* S*.*

- *It is said to be a quasi-static state feedback if span* $\{dt, dz\}$ *= span* $\{dt, dx\}$*.*

[31] See Definitions 3 and 4.

[32] See Part 3 of Lemma 1, in the Part I of this survey

- *It is said to be a static state feedback if it is quasi-static and span* $\{dt, dx, du\} =$ *span* $\{dt, dz, dv\}$.

{tQuasiStatic}

Theorem 6 *Let* (x, u) *be a proper local state representation of* S *defined around* $\xi \in S$. *Let* $z = (z_1, \dots, z_s)$ *and* $v = (v_1, \dots, v_p)$ *be sets of smooth functions defined locally around* ξ. *Then* (z, v) *is a local state representation defined around* ξ *and* $((z, v), (x, v))$ *is quasi-static if and only*

- *span* $\{dt, dx\} =$ *span* $\{dt, dz\}$ *locally around* ξ.
- *There exist integers* $\alpha, \beta \in I\!N$ *such that, on an open neighborhood of* ξ, *one has:*
  - *span* $\{du\} \subset$ *span* $\{dt,\ dz, dv, \dots, dv^{(\beta)}\}$,
  - *span* $\{dv\} \subset$ *span* $\{dt,\ dx, du, \dots, du^{(\alpha)}\}$,
  - *The set* $\{dt, dz, dv, \dots, dv^{(\alpha+\beta)}\}$ *is linearly independent at* ξ.

Proof. From Definition 14, and from the fact that $\{t, x, (u^{(k)} : k \in I\!N)\}$ and $\{t, z, (v^{(k)} : k \in I\!N)\}$ are local coordinate systems around ξ, it is clear that the given conditions are necessary. To show the sufficiency, let $\tilde{u} = u^{(\alpha)}$ and $\tilde{x} = (x, u^{(0)}, \dots, u^{(\alpha-1)})$. By construction $(\tilde{x}, \tilde{u})$ is a local state representation around ξ. Note also that span $\{d\dot{z}\} \subset$ span $\{dt, dx, d\dot{x}\} \subset$ span $\{dt, dx, du\} \subset$ span $\left\{dt, dz, dv, \dots, dv^{(\beta)}\right\}$. Hence

- By derivation one shows that span $\{d\tilde{u}\} \subset$ span $\left\{dt, dz, dv^{(0)}, \dots, dv^{(\alpha+\beta)}\right\}$.
- span $\{dz, dv\} \subset$ span $\{dt, d\tilde{x}, d\tilde{u}\}$.
- span $\{d\dot{z}\} \subset$ span $\left\{dt, dz, dv^{(0)}, \dots, dv^{(\alpha+\beta)}\right\}$.

The desired result follows from lemma 2 applied to $(\tilde{x}, \tilde{u})$ and (z, v). ☐

To characterize static-state feedback, one may take $\alpha = \beta 0$ in the conditions of the last theorem.

{cStaticStateFeedback}

Corollary 3 *Let* (x, u) *be a proper local state representation of* S. *Let* $z = (z_1, \dots, z_s)$ *and* $v = (v_1, \dots, v_p)$ *be sets of functions defined locally around some* $\xi \in S$. *Then* (z, v) *is a local state representation of* S *and* $((z, v), (x, v))$ *is a static state feedback around* ξ *if and only, one*

- *span* $\{dt, dx\} =$ *span* $\{dt, dz\}$ *locally around* ξ.
- *span* $\{dt, dx, du\} =$ *span* $\{dt, dz, dv\}$ *locally around* ξ, *and the set* $\{dt, dz, dv\}$ *is linearly independent at* ξ.

5.2 Two important endogenous feedbacks

The endogenous state feedback constructed by the dynamic extension algorithm (see Lemma 5) furnishes solutions for many nonlinear control synthesis problems like disturbance decoupling, input-output decoupling, input-output linearization and dynamic linearization (see (Di Benedetto et al. 1989, Huijberts, Nijmeijer & van der Wegen 1991, Pereira da Silva 1996, Delaleau & Pereira da Silva 1998a, Delaleau & Pereira da Silva 1998b, Delaleau & Rudolph 1998).

Using the notations of Lemma 5, if one takes the state representation (x_{k^*}, u_{k^*}), then the corresponding state feedback $(E, (x_{k^*}, u_{k^*}), (x, u))$ is a *proper endogenous state feedback*. However, the extension of the state is unnecessary, and it will be shown that one may regard the same construction as a quasi-static feedback.

Corollary 4 *(Delaleau & Pereira da Silva 1998a)*[33] *Let (x, u) be a proper state representation of a system S with proper output y. Consider that the regularity assumptions of Lemma 5 hold and one chooses $\bar{y}_k \subset \bar{y}_{k+1}$ and $\widehat{u}_{k+1} \subset \widehat{u}_k$ in the procedure described in that lemma. Let $\tilde{y}_0 = \bar{y}_0$ and define $\tilde{y}_k$ from the equation $\bar{y}_k = (\bar{y}_{k-1}, \tilde{y}_k)$ for $k = 1, \ldots k^*$. Let $z = x$ and $v = (\tilde{y}_0^{(0)}, \ldots, \tilde{y}_{k^*}^{(k^*)}, \widehat{u}_{k^*})$. Then $((z, v), (x, u))$ is a quasi-static feedback.*

Proof. The result is a simple consequence of the statement of Lemma 5 of Appendix C and Theorem 6. The details are left to the reader □

When the regularity conditions of Lemma 5 holds, the last result constructs a quasi-static feedback even in the case when the output rank $\rho(y)$ is less than the number of output components. The regularity assumptions needed in order to construct this feedback are stronger than the ones that are needed to prove the following result (see Appendix H).

Theorem 7 *Let S be a system with local proper state representation (x, u) and proper output y. Let $\mathrm{card}\, y = p$ and let $\alpha \in \mathbb{N}^p$ be a multiindex. Consider the same assumptions of theorem 3. Choose w in a way that $\{dt, dx, dw\}$ is a local basis of span $\left\{dt, dx, dy^{\langle\langle \alpha-1\rangle\rangle}\right\}$ and let $\widehat{u}$ be such that $\{dt, dx, dw, d\widehat{u}\}$ is a local basis of span $\left\{dt, dx, du, dy^{\langle\langle \alpha\rangle\rangle}\right\}$. Let $\widehat{v} = (y_1^{(\alpha_1)}, \ldots, y_p^{(\alpha_p)})$, and let $v = (\widehat{v}, \widehat{u})$. Let $\chi = (x, w)$. Then (χ, v) is a state representation of S. Furthermore $((\chi, v), (x, u))$ is an endogenous state feedback with proper state equations.*

Proof. Let z be such that $\{dt, dz, dy^{\langle\langle \alpha-1\rangle\rangle}\}$ is a local basis of span $\{dt, dx, dy^{\langle\langle \alpha-1\rangle\rangle}\}$. By the proof of Theorem 3, $(\tilde{z}, \tilde{v})$ is a local state representation where $\tilde{z} = (z, \bar{y})$, $\tilde{v} = (\widehat{v}, \widehat{u})$, $\bar{y} = (y_1^{(0)}, \ldots, y_1^{(\alpha_1-1)}, \ldots, y_p^{(0)}, \ldots, y_p^{(\alpha_p-1)})$, and $\widehat{v} = (y_1^{(\alpha_1)}, \ldots, y_p^{(\alpha_p)})$. By construction, $\{dt, dw, dx\}$ is also local basis of span $\{dt, d\tilde{z}\}$ and $\{dt, dw, dx, dv\}$ is a basis of span $\{dt, d\tilde{z}, d\tilde{v}\}$. By Corollary 3,

[33] In (Delaleau & Pereira da Silva 1998a) one may find a differential-algebraic version of this result.

(χ, v) is also a state representation. Since span $\{du, d\dot{x}, d\dot{w}\} \subset$ span $\{dt, dw, dx, dv\}$, the state representation (χ, v) is proper, and so it admits local state equations in the form (15). $\square$

It can be shown that the feedback constructed in the last proof solves several dynamic feedback synthesis problems like disturbance decoupling, input output decoupling, dynamic linearization, and input-output linearization.

6 A notion of regularity for implicit systems

{sRegular}

Let $\Psi(w) = 0$ be the set of n "*algebraic*" equations in the variables $w_1, \dots w_s$ given by

$$\begin{array}{rcl} \Psi_1(w_1, \dots, w_s) & = & 0 \\ \vdots & \vdots & \vdots \\ \Psi_n(w_1, \dots, w_s) & = & 0 \end{array} \qquad (16)$$

{eEstrela2}

and let $F(\dot{w}, w) = 0$ be set of *Differential-Algebraic Equations (DAE)* given by[34]

$$\begin{array}{rcl} F_1(w_1, \dots, w_s, \dot{w}_1, \dots, \dot{w}_s) & = & 0 \\ \vdots & \vdots & \vdots \\ F_n(\dot{w}_1, \dots, \dot{w}_s, w_1, \dots, w_s) & = & 0 \end{array}$$

One may consider that differential-algebraic equations are generalizations of algebraic equations. With some regularity assumptions, an algebraic equation (16) defines an immersed manifold in the space $\mathcal{W} = I\!R^s$ of the possible real values of the variables $w = (w_1, \dots w_s)$. The space $\mathcal{W}$ is the space of free algebraic variables w. Now, in order to extend this point of view in the context of DAE's, one may considere that the trivial diffiety $T^s(w)$ is the space of free differential variables w (see section 2). Hence one may expect that a DAE can define an immersed manifold in $T^s(w)$, or more specifically, a system in the sense of section 2.

When studying finite dimensional differential geometry, the definition of a regular manifold from a set equations (16) can be done using a rank condition. For instance, let $\Psi : I\!R^n \to I\!R^m$ be a smooth map and let $M = \Psi^{-1}(0)$. Let $J\Psi|_w = \partial\Psi/\partial x|_w$. One says that $\bar{w}$ is a regular point of $J\Psi$ if the rank of $J\Psi|_w$ is constant for $w \in V_{\bar{w}}$, where $V_{\bar{w}}$ is an open neighborhood of $\bar{w}$. Assume that all $\bar{w} \in M$ are regular points of $J\Psi$, with rank $J\Psi|_w$ equal to r for all $w \in M$. Then M is an immersed submanifold of $I\!R^n$. In fact, around any point ξ of M, the rank theorem allows to construct *adapted* local coordinates (x, z) in a way that, locally, $\partial\Psi/\partial x = 0$ and $\Psi(x, z) = 0$ if and only if $z = 0$. In this case, a local chart of M is the map that associates $(x, 0) \in M$ to x. The insertion map $\iota : M \to I\!R^n$ in these coordinates reads $\iota(x) = (x, 0)$. The immersion ι is in fact an embedding (see (Warner 1971)).

[34] In appendix J it will be shown that a general implicit system can be converted to this form.

When studying implicit systems, the situation is quite similar. One has a set of differential equations and, in some sense, a good notion of regularity might be related to adapted local coordinates. Now, the "*suitable geometry*" is the geometry of diffieties. In this context, the rank theorem is not available, but a notion of regularity may be stated by choosing special coordinates. For instance, when working with diffieties, the characterization of an immersion[35] ι is based on the existence of a local chart (x, z) in a way that $\iota(x) = (x, 0)$. Following this idea, as one is working with control systems, and their state representations are the "*suitable*" choice of coordinates (see Definition 3), it is quite natural to expect that a regular implicit system is an immersion between two systems for which there exists, in some sense to be formalized, *adapted state representations*.

After this preliminary discussion, the geometric definition of an implicit system is now given. For this, recall the notion of a solution (or integral curve) of a system (see Part I of this paper). Let $I\!R$ be the diffiety of global coordinate s and Cartan field $\frac{d}{ds}$, where $\frac{d}{ds}$ is the standard operation of differentiation of real functions. A solution is a Lie-Backlund mapping between an open interval of $I\!R$ and S. In other words, given a system S with Cartan field ∂_S, *a solution* is a map $\sigma : (a, b) \subset I\!R \to S$ such that $\dot\sigma(t) = \sigma_*|_t(\frac{\partial}{ds}) = \partial_S|_{\sigma(t)}$ for all $t \in (a, b)$.

Definition 15 *(implicit system) An implicit system Δ is a pair (S, y), where S is a control system*[36] *equipped with a set of outputs y, called* constraints*. A solution of the implicit system is a solution*[37] *$\sigma : (a, b) \to S$ of S such that $y(\sigma(t)) = 0$, for all $t \in (a, b)$. An implicit system Δ may be denoted by (S, y), or simply by Δ when S and y are defined by the context.* {dPairImplicit}

Note that an implicit system may be defined by equations. For instance, define S by {eExplicit}

$$\begin{aligned} \dot x(t) &= f(t, x(t), u(t)) && \text{(17a)} \\ y(t) &= h(t, x(t), u(t)) && \text{(17b)} \end{aligned}$$ {eExplicita} {eExplicitb}

then the corresponding implicit system Δ is given by {eImplicit}

$$\begin{aligned} \dot x(t) &= f(t, x(t), u(t)) && \text{(18a)} \\ y(t) &= h(t, x(t), u(t)) = 0 && \text{(18b)} \end{aligned}$$ {eImplicita} {eImplicitb}

Note that (x, u) is a state representation of S, but it is not necessarily a state representation of Δ. For instance, if $y_1 = x_1$, and $y_2 = u_2 + x_3$, then $y \equiv 0$ induces a relation among the components of x and u. For this reason x is called pseudo-state of Δ and u is called pseudo-input of Δ.

Remark 9 *A "*general*" implicit system is of the form*

$$H(t, \dot w(t), w(t)) = 0 \qquad (19)$$ {eImplicitF}

[35] See the Part I of this survey.

[36] The definition of control system is given in the end of section 2.

[37] See the definition of solution , (or integral curves) in the first part of this survey, Definition **??**.

Let $x = w$, and let $u = \dot{w}$. It is clear that the system (19) *is equivalent to the implicit system*

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= H(t,x,u) \equiv 0 \end{aligned}$$

which is in the form (18). *However, a precise notion of equivalence is needed in order to clarify this question (see Appendices* **??** *and J).*

Given an implicit system $\Delta = (S, y)$, define the subset $\tilde{\Delta}$ of S by

$$\tilde{\Delta} = \{\xi \in S \mid y^{(k)}(\xi) = 0, k \in I\!N\} \quad (20)$$ {eDeltay}

The notion of strongly adapted state equations is instrumental for the definition of regular implicit systems. For convenience, the definition 11 is restated here in a more convenient form.

{dStrongly2}

Definition 16 *(Strongly Adapted State Representations) Let S be a system with output y and let Y be a local output subsystem defined around $\xi \in S$. A local state representation (x, u) defined around ξ is said to be* strongly adapted *to the output subsystem Y, if $x = (x_a, x_b)$, $u = (u_a, u_b)$, the local state equations are of the form*

$$\dot{x}_a = f_a(t, x_a, u_a), \quad (21a)$$ {eSubaa}

$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b). \quad (21b)$$ {eSubbb}

where (21a) *is the local state representation of Y. Furthermore*[38] *span* $\left\{dx_a, (du_a^{(k)} : k \in I\!N)\right\} =$ *span* $\left\{dy^{(k)} : k \in I\!N\right\}$ *and the set of functions $\{x_a, u_a^{(k)} : k \in I\!N)\}$ is contained in the set $\{y^{(k)} : k \in I\!N\}$.*

Note that the definitions of *adapted state equations* and *strongly adapted state equations* coincide for time-invariant systems. It must be pointed out that the dynamics (21b) is a generalization of the zero dynamics, since it can be defined even when the decoupling problem for system S with output y is not solvable. It is worth recalling that the connection of the zero dynamics with implicit systems was already considered for instance in (Byrnes & Isidori 1988, Krishnan & McClamroch 1990, Byrnes & Isidori 1991).

{dRegularDelta}

Definition 17 *(regular implicit system) An implicit system $\Delta = (S, y)$ is regular if*

1. *$\tilde{\Delta} \neq \emptyset$.*
2. *There exists a local output subsystem Y for system S with output y around all $\xi \in \tilde{\Delta}$.*

[38]This is stronger than saying that span $\left\{dt, dx_a, (du_a^{(k)} : k \in I\!N)\right\} =$ span $\left\{dt, dy^{(k)} : k \in I\!N\right\}$.

3. *Around all $\xi \in \tilde{\Delta}$, system S admits a local state representation that is strongly adapted to Y (see definition 16).*

The following definition regards an implicit system as an immersed submanifold.

{dEquivalent1}

Definition 18 *Consider an implicit system $\Delta = (S, y)$. An equivalent system is a* control system [39] Γ *such that*

- *There exists an injective Lie-Bäcklund immersion $\iota : \Gamma \to S$.*
- *A smooth curve $\sigma : (a, b) \to S$ is a solution of Δ if and only if there exists a solution $\nu : (a, b) \to \Gamma$ of Γ such that $\sigma(t) = \iota \circ \nu(t)$ for all $t \in (a, b)$.*

The next proposition shows that the last definition implies that $y^{(k)} \circ \iota$ must be identically zero for all $k \in I\!N$.

{pNullY}

Proposition 6 *If Γ is equivalent to an implicit system $\Delta = (S, y)$ according to Def. 18 and $\iota : \Gamma \to S$ is the corresponding Lie-Bäcklund immersion, then $\iota(\Gamma) \subset \tilde{\Delta}$, where $\tilde{\Delta}$ is defined by* (20).

Proof. See Appendix M. □

Another definition of equivalence between an implicit system and a control system Γ is the next definition.

{dEquivalent}

Definition 19 *Consider an implicit system $\Delta = (S, y)$. A* canonically equivalent system *is a* control system Γ *such that there exists an injective Lie-Bäcklund* embedding[40] $\iota : \Gamma \to S$ *such $\iota(\Gamma) = \tilde{\Delta}$.*

Note that one may identify $\iota(\Gamma)$ with $\tilde{\Delta}$. As the subset topology of $\tilde{\Delta}$ coincides with the topology of $\tilde{\Delta}$ induced by ι, one may consider, without great loss of generality, that $\Gamma = \tilde{\Delta}$ and ι is the insertion map.

The next proposition shows that Def. 19 is stronger than Def. 18:

{pEmbedding}

Proposition 7 *If Γ is cannonically equivalent to $\Delta = (S, y)$ according to Def. 19, then it is equivalent to Δ according to Def. 18.*

Proof. See Appendix N. □

The next result shows that one can construct a canonically equivalent system Γ for every regular implicit system and Γ may be identified with $\tilde{\Delta}$.

{tDelta}

Theorem 8 *Let $\Delta = (S, y)$ be a regular implicit system such that S is a control system. Then there exists a canonically equivalent system Γ.*

[39] Remember also that a diffiety Γ is a control system if it admits a state representation around every point $\gamma \in \Gamma$.

[40] Remember that an injective immersion ι is an embedding if, for every open set $U \subset \Gamma$, then $\iota(\Gamma)$ is an open subset of S.

Proof. Let $\Gamma = \tilde{\Delta} \subset S$, where $\tilde{\Delta}$ is defined by (20). Let ι be the insertion map. Let τ be the time-function of system S and let τ_Γ be the map τ restricted to Γ. It will be shown that $\Gamma = \tilde{\Delta} \subset S$ has a canonical structure of immersed (embedded) submanifold of S with the following properties:

1. The map $\tau_\Gamma : \Gamma \to I\!R$ is a submersion.

2. The Cartan field ∂_Γ can be canonically defined for Γ by the rule $\iota_* \partial_\Gamma = \partial_S \circ \iota$.

3. One has $\partial_\Gamma(\tau_\Gamma) = 1$. In particular $(\Gamma, \partial_\Gamma, \tau_\Gamma)$ is a system in the sense of section 2.

4. The insertion $\iota : \Gamma \to S$ is a Lie-Bäcklund immersion.

5. System Γ admits a local classical state representation around every point $\xi \in \Gamma$, that is, Γ is a control system.

6. System Γ is canonically equivalent to Δ according to definition 19.

We show first that Γ is an immersed manifold. For this, consider the topological subspace $\Gamma \subset S$ with the subset topology. For each point $\xi \in \Gamma$, by definition there exists local charts $\phi : \hat{U} \to \tilde{U} \subset I\!R^A$, where $\phi = \{t, z_a, V_a, z_b, V_b\}$, $V_a = \{v_a^{(k)} : k \in I\!N\}$, $V_b = \{v_b^{(k)} : k \in I\!N\}$, and we have span $\{dt, dz_a, dV_a\}$ = span $\{dt, dy^{(k)} : k \in I\!N\}$. This local chart is adapted to a local output subsystem $\pi : \hat{U} \to Y$, and is such that $\pi(t, z_a, V_a, z_b, V_b) = (t, z_a, V_a)$. Furthermore, by definition, $\mathcal{Z} = \{z_a, V_a\} \subset \mathcal{Y} = \{y^{(k)} : k \in I\!N\}$. By construction, if $\nu \in \hat{U} \cap \Gamma$, then $y^{(k)}(\nu) = 0$ for all $k \in I\!N$. This implies that all the components of $\mathcal{Z}$ are also null in ν. If we show that the functions in $\mathcal{W} = \mathcal{Y} - \mathcal{Z}$ are also null in $\nu \in \Gamma \cap \hat{U}$, we will show that a point ν is in $\Gamma \cap \hat{U}$ if and only if $z_a = 0$ and $V_a = 0$ in ν. In fact, note first that, since span $\{d\mathcal{Z}\}$ = span $\{d\mathcal{Y}\}$ and the functions of $\mathcal{Z}$ are part of a coordinate system, all the functions θ in $\mathcal{Y}$ can be locally written in $\hat{U}$ of the form $\theta = \theta(z_a, V_a)$ (see Lemma 1 of Part I of this survey). Now take a point $\xi \in \hat{U} \cap \Gamma$. Then $\theta(\xi) = 0$. Now let $\nu \in \hat{U}$ be such that $(z_a, V_a)|_\nu = (z_a, V_a)|_\xi = (0, 0)$. Then $\theta\left((z_a, V_a)|_\nu\right) = \theta(0, 0) = \theta\left((z_a, V_a)|_\xi\right) = 0$.

Now consider the map $\mu : \Gamma \cap \hat{U} \to \mu(\Gamma \cap \hat{U}) \subset I\!R^B$ such that the expression of μ in the coordinates $(phi, \hat{U})$ is given by $\mu(t, 0, 0, z_b, V_b) = (t, z_b, V_b)$. We shall show that these maps form a smooth atlas of Γ. By construction it is clear that these maps are homeomorphisms. Hence it suffices to show that these charts are C^∞ compatible. For convenience denote the functions of the chart ϕ by $\{t, X, Z\}$ and the functions of the chart μ by $\{t, Z\}$, where $X = \{z_a, V_a\}$ and $Z = \{z_b, V_b\}$.

Now let $\mu_i : \Gamma \cap U_i \to \tilde{V}_i$, $i = 1, 2$, be two local charts constructed in that way, based, respectively, on the local charts of S given by $\phi_i = \{t, X_i, Z_i\}$, $i = 1, 2$. In particular, it follows that $\mu_i \circ \phi_i(t, 0, Z_i) = (t, Z_i)$, $i = 1, 2$. Without loss of generality, assume that $U_1 = U_2$. Consider the local coordinate change $(t, X_1, Z_1) = \phi_1 \circ \phi_2^{-1}(t, X_2, Z_2)$. Note that the map $\rho : \tilde{V}_2 \to \tilde{V}_1$ such that $(t, Z_1) \mapsto (t, Z_2)$ defined by $(t, 0, Z_1) = \phi_1 \circ \phi_2^{-1}(t, 0, Z_2)$ is a local diffeomorphism with inverse defined by $(t,0,Z_2)=\phi_2\circ\phi_1^{-1}(t,0,Z_1)$. Since $\rho=\mu_1\circ\mu_2^{-1}$, we conclude that such charts are C^∞-compatible. Since in the coordinates (t,Z_1) for Γ one has $\tau_\Gamma(t,Z_1)=t$, from this construction is also clear that $\tau_\Gamma:\Gamma\to I\!R$ is a submersion.

Now let $\iota:\Gamma\to S$ be the insertion map. In the coordinates ϕ and μ previously constructed, we have $\iota(t,Z)=(t,0,Z)$. In particular, ι is an immersion between $I\!R^A$-manifolds and so $\iota_*(\zeta)$ is injective for all $\zeta\in\Gamma$. Remember that any function η of the set $X=\{z_a,V_a\}\subset\mathcal{Y}$ is such that $\dot\eta|_\nu=0$ for every $\nu\in\Gamma\cap\hat U$. To be as clear as possible, for the moment one let $\{\tilde t,\tilde x_b,\tilde V_b\}$ stand for the local coordinate functions of Γ[41], where Now note that $\iota(\tilde t,\tilde x_b,(\tilde V_b)=(t,x_a,V_a,x_b,V_b)$ where $t=\tilde t$, $x_a=0$, $V_a=0$, $x_b=\tilde x_b$ and $V_b=\tilde V_b$. It follows that, regarding $\iota_*(\nu)$ as a linear map from $T_\nu\Gamma$ to $T_{\iota(\nu)}S$, one may write

$$\iota_*(\nu)\left(\alpha_0\tfrac{\partial}{\partial\tilde t}+\textstyle\sum_{i=1}^{n_b}\alpha_i\tfrac{\partial}{\partial\tilde x_{b_i}}+\sum_{j=1}^{m_b}\sum_{k\in I\!N}\beta_{jk}\tfrac{\partial}{\partial\tilde u_{b_j}^{(k)}}\right)\Bigg|_\nu= \\ \left(\alpha_0\tfrac{\partial}{\partial t}+\textstyle\sum_{i=1}^{n_b}\tfrac{\partial}{\partial x_{b_i}}+\sum_{j=1}^{m_b}\sum_{k\in I\!N}\beta_{jk}\tfrac{\partial}{\partial\tilde u_{b_j}^{(k)}}\right)\Bigg|_{\iota(\nu)}$$

where the real coeficients α_0, α_i, β_{jk} are arbitrary. Now as $\operatorname{span}\{d\mathcal{Y}\}=\operatorname{span}\left\{dy^{(k)}:k\in I\!N\right\}=\operatorname{span}\{dx_a,dV_a\}$, it is clear that $\operatorname{span}\{d\mathcal{Y}\}^\perp\Big|_{\iota(\nu)}$ coincides with the image of $\iota_*(\nu)$.

In other words, one has shown that the image of $\iota_*(\nu)$ contains every tangent vector $\tau_{\iota(\nu)}$ such that $\tau_{\iota(\nu)}(\eta)=0$ for every function η in the set $\mathcal{Y}$. In particular, we have that the image of $\iota_*(\nu)$ contains $\frac{d}{dt}(\iota(\nu))$ for every $\nu\in\Gamma\cap\hat U$. So we can define ∂_Γ by the rule $\iota_*\partial_\Gamma=\frac{d}{dt}\circ\iota$. By definition, it follows that ι is a Lie–Bäcklund immersion.

Let $\phi=(t,x_a,V_a,x_b,V_b)$ and $\mu=(t,x_b,V_b)$ be, respectively, the coordinates of S and Γ constructed above. In this coordinates we have

$$\partial_\Gamma=\frac{\partial}{\partial t}+\sum_{i=1}^{n_b}f_{b_i}(t,0,0,x_b,V_b)\frac{\partial}{\partial x_{b_i}}+\sum_{i=1}^{m_b}\sum_{j\in I\!N}u_{b_i}^{(j+1)}\frac{\partial}{\partial u_{b_i}^{(j)}},\qquad(22)$$ {eCartanGamma}

where $f_{b_i}=\frac{d}{dt}(x_{b_i})=f_{b_i}(t,x_a,V_a,x_b,V_b)$, $i\in\lfloor n_b\rceil$. In other words, (x_b,u_b) is a state representation of Γ. By (22) it is also clear that $\partial_\Gamma(\tau_\Gamma)=1$. □

The next theorem shows that the notion of equivalence that is stated in Def. 19 is intrinsic.

{tEquivalentImplicit}

Theorem 9 *Let Γ_1 and Γ_2 be two control systems that are* canonically equivalent *to an implicit system (S,y). Then Γ_1 and Γ_2 are Lie-Bäcklund isomorphic.*

Proof. It is a consequence of Theorem 13, given in Appendix I. □

[41]The notation $\{t,x_b,V_b\}$ would be an acceptable abuse of notation, but would create some confusion in the present proof.

Now, one might ask when a given implicit system is regular. This question may be partially answered by the following results. The next theorem is a generalization of a result of (Pereira da Silva & Corrêa Filho 2001).

{t8}

Theorem 10 *Let* $\Delta = (S, y)$ *be an implicit system. Assume that* S *admits a* global *proper state representation* (x, u) *for which* y *is also proper. Let* $\tilde{\Delta}$ *be defined as in* (20)*, where* $\dim x = n$*. Consider the codistributions* $\mathbb{Y}_k$*,* $\mathcal{Y}_k$ *and* Y_k *defined by* (7)*.*

Assume that

- $\tilde{\Delta} \neq \emptyset$*, where* $\tilde{\Delta}$ *is defined by* (20)*.*
- *Every* ν *in* $\tilde{\Delta}$ *is a regular point of* $\mathcal{Y}_k$ *and* Y_k *for* $k = 0, \ldots, n$*.*
- *Every* ν *in* $\tilde{\Delta}$ *is a regular point of* $\mathbb{Y}_{n-1}$ *and* $\mathbb{Y}_n$*.*

Then Δ *is a regular implicit system.*

Proof. Direct Consequence of theorem 2 and Definition 17. □

Theorem 3 is also useful for the characterization of regular implicit systems.

{t9}

Theorem 11 *Let* $\Delta = (S, y)$ *be an implicit system. Assume that* S *admits a* global *proper state representation* (x, u) *for which* y *is also proper. Let* $\tilde{\Delta}$ *be defined as in* (20) *and assume that* $\tilde{\Delta} \neq \emptyset$*. Suppose that the assumptions of theorem 3 holds around every point of* $\tilde{\Delta}$*. Then* Δ *is a regular implicit system.*

Proof. From the Remark 3 just after Theorem 3, around each point $\nu \in \tilde{\Delta}$ there exists an output subsystem that is strongly adapted to Y. Then it is clear that the statement of definition 17 holds. □

The last result may be generalized for the non-invertible case. Note that the assumptions of theorem 10 are stronger than the ones of the next theorem (see appendix H).

{t10}

Theorem 12 *Let* $\Delta = (S, y)$ *be an implicit system. Assume that* S *admits a* global *proper state representation* (x, u) *for which* y *is also proper. Let* $\tilde{\Delta}$ *be defined as in* (20)*. Suppose that the assumptions of theorem 5 holds for some* $\alpha \in I\!N$ *around every* $\nu \in \tilde{\Delta}$*. Then* Δ *is a regular implicit system.*

Proof. Direct consequence of theorem 5. □

7 Concluding remarks

This paper covers some basic aspects of an infinite dimensional differential geometric approach of nonlinear control systems. Recall that the finite dimensional inverse function theorem assures, via the independence of the differentials of a set of functions, that they form a set of local coordinate functions. The point of view of this paper is that a state representation is the "*suitable*" notion of local coordinate system in control theory. In particular, any result that assures the existence of a state representation based on regularity assumptions can be regarded as a generalization of the inverse function theorem. In this context, one may consider that the Dynamic Extension Algorithm (see Lemma 5 of Appendix C) is a version of the inverse functions theorem. It becomes clear that theorems 3 and 5 are generalizations of lemma 5 in the sense that they require less regularity assumptions and what they show is essentially the same thing: the existence of the output subsystem and the corresponding adapted state equations.

A key concept of this article is the notion of subsystem, that is important for the definition of state representation, the definition of feedback, the study of decoupling theory, the notion of flatness etc. The definition of output subsystem and the corresponding adapted state equations particularly important for establishing a geometric notion of regularity of implicit systems. This notion of regularity of implicit systems have been considered in several works, for instance (Pereira da Silva & Corrêa Filho 2001, Pereira da Silva, Veloso Pazzoto & Corrêa Filho 2006, Pereira da Silva & Batista 2010). It seems that this notion is particularly useful for establishing results for an implicit system $\Delta = (S, y)$ that are based only on the geometry of the explicit system S and the map y, and so can be always computed by effective methods. Those results do not rely on a explicit state representation of Δ, which does exist for regular implicit systems, but it cannot be always computed[42]. The relationship of this notion of regularity and other notions of the literature (for instance, see (Rheinboldt 1984, Reich 1990, Rheinboldt 1991)) is a subject of future research.

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[42] In order to obtain a explicit state representation of Δ one may apply existence theorems that are not effective in the general case.

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A Proof of Lemma 1

{aLemmaMenos}

Proof. To show the first claim, assume that there exists $a \in I\!N$ such that

$$\begin{array}{ll} \text{(a)} & \operatorname{span}\{dz\} \subset \operatorname{span}\{dt,\, dx, du^{(0)}, \ldots, du^{(a)}\}, \text{ and} \\ \text{(b)} & \operatorname{span}\{dz\} \not\subset \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(a-1)}\right\}. \end{array} \tag{23}$$

{eEEEE}

Note that

$$dz = \alpha_0 dt + \sum_{i=1}^{n} \alpha_i dx_i + \sum_{j=0}^{a} \sum_{k=1}^{m} \beta_{jk} du_k^{(j)},$$

where α_i, β_{jk} are smooth functions defined on $U \subset S$. Since $\operatorname{span}\{dz\} \not\subset \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(a-1)}\right\}$ some β_{aj} is not the null function for some $j \in \{1, \ldots m\}$. Then,

$$d\dot{z} = \dot{\alpha}_0 dt + \sum_{i=1}^{n} (\dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i) + \sum_{j=0}^{a} \sum_{k=1}^{m} (\dot{\beta}_{jk} du_k^{(j)} + \beta_{jk} du_k^{(j+1)}).$$

As $\operatorname{span}\{d\dot{x}\} \subset \operatorname{span}\{dt, dx, du\}$, it follows that $\operatorname{span}\{d\dot{z}\} \subset \operatorname{span}\{dt, dx, du^{(0)}, \ldots, du^{(a+1)}\}$ but $\operatorname{span}\{d\dot{z}\} \not\subset \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(a)}\right\}$. Since (z, v) is proper, $\operatorname{span}\{d\dot{z}\} \subset \operatorname{span}\{dt, dz, dv\}$. As $\operatorname{span}\{dz\} \subset \operatorname{span}\{dt, dx, du^{(0)}, \ldots, du^{(a)}\}$, it follows that $\operatorname{span}\{dv\} \not\subset \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(a)}\right\}$. Note that the reasoning above holds for $a = 0$. In this case, $\operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(a-1)}\right\}$ stands for $\operatorname{span}\{dt, dx\}$. Now assume, as an absurd, that for some[43] $b \in I\!N$, one has $\operatorname{span}\{dv\} \subset \operatorname{span}\{dt, dx, du^{(0)}, \ldots, du^{(b)}\}$, but $\operatorname{span}\{dz\} \not\subset \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(b-1)}\right\}$. Then there exists a such that (23) holds. Then $b \leq a$. Then, from the reasoning above, $\operatorname{span}\{dv\} \not\subset \operatorname{span}\{dt, dx, du^{(0)}, \ldots, du^{(a)}\}$, which is an absurd. Note that the reasoning above holds for $b = 0$, where $\operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(b-1)}\right\}$ stands for $\operatorname{span}\{dt, dx\}$.

Now one will show the second part of the statement of the lemma for $\gamma = 0$. By the first part of the statement of the lemma, it follows that $\operatorname{span}\{dz\} \subset \operatorname{span}\{dt, dx\}$. Note that one may consider that $y = v$ is a proper output of system S when (x, u) is the corresponding state representation. Now define the codistributions[44] $\mathcal{V}_{-1} = \operatorname{span}\{dt, dx\}$, $V_{-1} = \operatorname{span}\{dt\}$, $\mathcal{V}_k = \operatorname{span}\left\{dt, dx, dv^{(0)}, \ldots, dv^{(k)}\right\}$ and $V_k = \operatorname{span}\left\{dt, dv^{(0)}, \ldots, dv^{(k)}\right\}$, for $k \in I\!N$.

Remember that $\dim u = \dim v = m$ by the uniqueness of the differential dimension (see section 2). Now choose a regular point[45] $\nu \in V$ of the codistributions $\mathcal{V}_k$, for $k = 0, \ldots, n$, where $n = \dim x$. Around this regular point, define $\rho_k = \dim V_k - \dim V_{k-1}$ and $\sigma_k = \dim \mathcal{V}_k - \dim \mathcal{V}_{k-1}$, for $k \in I\!N$. Since $\{t, z, (v^{(k)} : k \in I\!N)\}$, is a local coordinate system, the codistribution V_k are always nonsingular for all $k \in I\!N$, since the differentials of the functions forming a coordinate system are independent. In particular $\rho_k = \dim V_k - \dim V_{k-1} = \operatorname{card} v = m$, for all $k \in I\!N$.

Let k^* be the integer defined in part 5 of Lemma 5. Note that $0 \leq k^* \leq n$, and that $\sigma_k = \rho_k = m = \dim v$ for all $k \geq k^*$. From part 9 of Lemma 5, it follows $\dim(\mathcal{V}_k \cap \operatorname{span}\{du\}) = \sigma_k$. It follows easily from dimensional arguments that $\operatorname{span}\{du\} \subset \mathcal{V}_{k^*}$.

Now, as the set $\{dt, dz, dv^{(0)}, \ldots, dv^{(k^*)}\}$ is independent and $\operatorname{span}\{dz\} \subset \operatorname{span}\{dt, dx\}$, one may choose a subset $\widehat{x} \subset x$ such that $\{dt, d\widehat{x}, dz, dv^{(0)}, \ldots, dv^{(k^*)}\}$ is a basis of $\mathcal{V}_{k^*}$. Now, the proof will be completed by showing that $\operatorname{span}\{d\widehat{x}\} \subset \operatorname{span}\left\{dt, dz, dv, \ldots, dv^{(k^*)}\right\}$, and so, $\operatorname{span}\{du\} \subset \operatorname{span}\{dt, dz,$

[43] Since $\{t, x, (u^{(k)} : k \in I\!N)\}$ is a local coordinate system, there exists an open neighborhood of ξ and $a \in I\!N$ such that, restricted to U one has $\operatorname{span}\{dz\} \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(a)}\right\}$.(see Prop. 1).

[44] Corresponding to the ones of Lemma 5.

[45] Remember that the set of regular points of $\mathcal{V}_k$ is open and dense in V.

$dv, \ldots, dv^{(k^*)}\}$ showing that $\beta \leq n$, at least on an open and dense set. For this, define $\mathcal{V} = \text{span}\,\{dt, dz, dv^{(0)}, \ldots, dv^{(k^*)}\}$. From the fact that $\sigma_k = m = \dim v$ for $k \geq k^*$, it follows easily from dimensional arguments, that $\mathcal{V}_{k^*+k} = \text{span}\,\{d\widehat{x}\} \oplus \mathcal{V} \oplus \text{span}\,\{dv^{(k^*+1)}, \ldots, dv^{(k^*+k)}\} = \text{span}\,\{d\widehat{x}\} \oplus \text{span}\,\{dt, dz, dv^{(0)}, \ldots, dv^{(k^*+k)}\}$, for all $k \in I\!N$.

Assume now that $\text{span}\,\{d\widehat{x}\} \not\subset \text{span}\,\{dt, dz, dv, \ldots, dv^{(k^*-1)}\}$. In particular $\text{span}\,\{d\widehat{x}\} \not\subset \text{span}\,\{dt, dz, dv, \ldots, dv^{(k)}\}$ for any $k \in I\!N$, no matter one try to restrict these codistributions to any open neighborhood W of ξ for which $W \subset V$. But this is a contradiction, since $\{t, z, (v^{(k)}, k \in I\!N)\}$ is a local coordinate system.

One has shown (abusing notation) that the map $\phi_k = \frac{\partial}{\partial u^{(k)}}(\widehat{x})$ is zero on an open and dense set for $k > \beta$. As ϕ_k is smooth, it must be identically zero, showing the second part of the lemma for $\gamma = 0$.

To show the second part of the lemma for $\gamma \geq 1$ it suffices to apply the result for $\gamma = 0$ to the state representations $(\widetilde{x}, \widetilde{u})$ and (z, v), where $\tilde{x} = (x, u^{(0)}, \ldots, u^{(\gamma-1)})$ and $\widetilde{v} = u^{(\gamma)}$. □

B Proof of Lemma 2

{aLemma3}

The proof of Lemma 2 is based on Lemma 1 of Part I of this survey.

Proof. (Of Lemma 2) As the desired result is local, without loss of generality one may assume that S is an open set U of $I\!R^A$ with global coordinates (ϕ, U) where $\phi = \{t, x, (u^{(k)} : k \in I\!N)\}$ and Cartan field

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{24}$$

{eCartanFieldW}

Let $\pi_1 : U \subset I\!R^A \to R \subset I\!R^{n+m+1}$ the map defined by $(t, x, u, u^{(1)}, u^{(2)} \ldots) \mapsto (t, x, u)$, where $u = u^{(0)}$. Note that $R = \pi_1(U)$ is an open subset of $I\!R^{n+m+1}$. Since the state representation is proper, then f_i depend only on (t, x, u), and one may write $f_i = \tilde{f}_i \circ \pi_1$ for convenient smooth functions $\tilde{f}_i : R \to I\!R$. We shall denote $f = (f_1, \ldots, f_n)$ and $\tilde{f}$ is such that $f = \tilde{f} \circ \pi_1$.

Let $W = R \times I\!R^B \subset I\!R^A$. Note that $U \subset W$. Since $\tilde{f}_i, i = 1, \ldots, n$ are well defined in R, one may extend[46] S to a system denoted by W, and defined on the open set W, with global coordinates $(\tilde{\phi}, W)$ and Cartan field (24), where $\tilde{\phi}|U = \phi$.

As the state representation is proper, then $\text{span}\,\{d\dot{x}\} \subset \text{span}\,\{dt, dx, du\}$. So, $\text{span}\,\{du, d\dot{x}, d\dot{z}\} \subset \text{span}\,\{\mathbb{S}\}$, and thus, from Lemma 1 of Part I of this survey, there exists an open neighborhood $V \subset U$ of ξ, an open set[47] $\widehat{V} \subset I\!R^p$, and smooth maps $\chi : \widehat{V} \to I\!R^n$, $\mu : \widehat{V} \to I\!R^m$, $X^{(1)} : \widehat{V} \to I\!R^n$ and $Z^{(1)} : \widehat{V} \to$

[46] Here one is using the fact that S is an open subset of $I\!R^A$.
[47] Note that $p = 1 + s(\alpha + 1)r$, where $r = \text{card}\, v$.

$I\!R^s$ such that, for all $\zeta \in V$ {eDelta}

$$
\begin{aligned}
x(\zeta) &= \chi \circ \delta(\zeta) && (25a) \\
u(\zeta) &= \mu \circ \delta(\zeta) && (25b) \\
\dot{x}(\zeta) &= X^{(1)} \circ \delta(\zeta) && (25c) \\
\dot{z}(\zeta) &= Z^{(1)} \circ \delta(\zeta) && (25d)
\end{aligned}
$$

{eDeltax} {eDeltau} {eDeltaxdot} {eDeltazdot}

where $\delta : V \to \widehat{V}$ is the smooth surjective and open map defined by $\delta(\zeta) = (t(\zeta), z(\zeta), v(\zeta), \ldots, v^{(\alpha)}(\zeta))$. Without loss of generality, one may assume $V = U$, otherwise one may restrict U to V accordingly.

Let $h : \widehat{V} \to\subset I\!R^{n+m+1}$ be defined by

$$h(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) = (\tilde{t}, \chi(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}), \mu(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)})).$$

By construction, it is clear that

$$\pi_1 = h \circ \delta \tag{26}$$

{eHdelta}

and so $h(\widehat{V}) = R$. Regarding $\dot{x}$ as a map defined on U, one may write

$$\dot{x} = \tilde{f} \circ \pi_1 \tag{27}$$

{eFpi1}

Then one concludes that $\tilde{f} \circ h \circ \delta = \tilde{f} \circ \pi_1 = \dot{x} = X^{(1)} \circ \delta$. As δ is surjective, it follows that

$$\tilde{f} \circ h = X^{(1)} \tag{28}$$

{eFh}

is an identity on $\widehat{V}$.

By Lemma 1 of Part I of this survey, the mapping $\chi(t, z, v, \ldots, v^{(\alpha)})$ is the expression of x in local coordinates $\{t, z, v, \ldots, v^{(\alpha)}, w\}$ for some convenient set w. Since span $\{dx\} \subset$ span $\left\{dt, dz, dv, \ldots, dv^{(\alpha-1)}\right\}$, then it is clear that

$$\frac{\partial \chi(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)})}{\partial \tilde{v}^{(\alpha)}} \equiv 0 \text{ in } \widehat{V}. \tag{29}$$

{eAlpham}

{aP}

Proposition 8 *Consider the map $\chi^{(1)} : \widehat{V} \to I\!R^n$ defined by:*

$$\chi^{(1)}(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) = \frac{\partial \chi}{\partial \tilde{t}} + Z^{(1)}(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) \frac{\partial \chi}{\partial \tilde{z}} + \sum_{j=0}^{\alpha-1} \frac{\partial \chi}{\partial \tilde{v}^{(j)}} \tilde{v}^{(j+1)} \tag{30}$$

{eChi1}

Then it follows that[48]

$$\dot{x}(\zeta) = \chi^{(1)} \circ \delta(\zeta) = X^{(1)} \circ \delta(\zeta), \forall \zeta \in V \tag{31}$$

{eChi1xdott}

The proof of the proposition is deferred to the end of this appendix.

Since δ is surjective by (31), it follows that, in the entire $\widehat{V}$, one has

$$\chi^{(1)} = X^{(1)} \tag{32}$$

{eChi1xdot}

[48] Recall that $V = U$.

From (25c), it follows that

$$\dot{x}(\zeta) = \chi^{(1)}(t(\zeta), z(\zeta), v(\zeta), \ldots, v^{(\alpha)}(\zeta)), \forall \zeta \in U \tag{33}$$ {eXdot}

Consider now the system S_1 with global coordinates $\psi = \{\tilde{t}, \tilde{z}, (\tilde{v}^{(k)} : k \in \mathbb{N})\}$ defined on the open set $S_1 = \widehat{V} \times \mathbb{R}^C$, with Cartan field given by

$$\partial_{S_1} = \frac{\partial}{\partial \tilde{t}} + Z^{(1)}(\tilde{t}, \tilde{z}, \ldots, \tilde{v}^{(\alpha)}) \frac{\partial}{\partial \tilde{z}} + \sum \tilde{v}_j^{(k+1)} \frac{\partial}{\partial \tilde{v}_j^{(k)}} \tag{34}$$ {eCartanFieldS1}

Now, using the map μ that appears in 25b, one may define maps $\mu^{(k)} : S_1 \to \mathbb{R}^m$ by the rules: {eMuk}

$$\begin{aligned} \mu^{(0)} &= \mu(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}), \\ &\quad \text{for } (\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) \in \widehat{V} && \text{(35a)} \\ \mu^{(k+1)} &= \frac{\partial \mu^{(k)}}{\partial \tilde{t}} + Z^{(1)} \frac{\partial \mu^{(k)}}{\partial \tilde{z}} + \sum_{j=0}^{\alpha+k} \frac{\partial \mu^{(k)}}{\partial \tilde{v}^{(j)}} \tilde{v}^{(j+1)} \\ &= \mu^{(k+1)}(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha+k)}), \\ &\quad \text{for } (\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha+k)}) \in \widehat{V} \times \mathbb{R}^{kr} && \text{(35b)} \end{aligned}$$

Let $\tilde{\pi} : S_1 \to \widehat{V}$ be the canonical projection $(\tilde{t}, \tilde{z}, \tilde{v}, \tilde{v}^{(1)}, \ldots) \mapsto (\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)})$. By definition it is clear that, in $\widehat{V}$ one has $\mu^{(0)} = \mu \circ \tilde{\pi}$.

Define the map $\Gamma : S_1 \to \tilde{S} = W = R \times \mathbb{R}^B$ such that $(\tilde{t}, \tilde{z}, \tilde{v}^{(0)}, \tilde{v}^{(1)}, \ldots) \to (t, x, u, u^{(0)}, \ldots)$, where {eGamma}

$$\begin{aligned} t &= \tilde{t} \quad (\tilde{t} = t \circ \Gamma) && \text{(36a)} && \text{\{eGammat\}} \\ x &= \chi \circ \tilde{\pi} = \chi(\tilde{t}, \tilde{z}, \ldots, \tilde{v}^{(\alpha)}), \quad (\chi \circ \tilde{\pi} = x \circ \Gamma) && \text{(36b)} && \text{\{eGammax\}} \\ u &= \mu \circ \tilde{\pi} = \mu(\tilde{t}, \tilde{z}, \ldots, \tilde{v}^{(\alpha)}), \quad (\mu \circ \tilde{\pi} = u \circ \Gamma) && \text{(36c)} && \text{\{eGammau\}} \\ u^{(k)} &= \mu^{(k)}(\tilde{t}, \tilde{z}, \ldots, \tilde{v}^{(\alpha+k)}), \quad (\mu^{(k)} = u^{(k)} \circ \Gamma) && \text{(36d)} \end{aligned}$$

By definition it is clear that

$$\pi_1 \circ \Gamma = h \circ \tilde{\pi} \tag{37}$$ {eGammapi1}

is an identity on S_1.

To show that Γ is Lie-Bäcklund, one has to show that, for every coordinate function θ of W, one has $\partial_{S_1}(\theta \circ \Gamma) = \frac{d}{dt}(\theta) \circ \Gamma$. In fact, note from (36a) and (34), that $\partial_{S_1}(t \circ \Gamma) = \partial_{S_1}(\tilde{t}) = \mathbf{1}$. On the other hand, from (24), one obtains $\frac{d}{dt}(t) \circ \Gamma = \mathbf{1} \circ \Gamma = \mathbf{1}$. Now, from (36b) and (30), it follows that $\partial_{S_1}(x \circ \Gamma) = \partial_{S_1}(\chi(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)})) = \chi^{(1)}(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) = \chi^{(1)} \circ \tilde{\pi}$. On the other hand, from (27), (37), (28) and (32), one obtains $\frac{d}{dt}(x) \circ \Gamma = \tilde{f} \circ \pi_1 \circ \Gamma = \tilde{f} \circ h \circ \tilde{\pi} = X^{(1)} \circ \tilde{\pi} = \chi^{(1)} \circ \tilde{\pi}$. Now, $\partial_{S_1}(u^{(k)} \circ \Gamma) = \partial_{S_1}(\mu^{(k)}) = \mu^{(k+1)}$. On the other hand, $\frac{d}{dt}(u^{(k)}) \circ \Gamma = u^{(k+1)} \circ \Gamma = \mu^{(k+1)}$, showing that Γ is Lie-Backlund.

Now, one will define a map Λ that will be shown to be the inverse of Γ. Since span $\{dz, dv\} \subset$ span $\{dt, dx, du\}$, note that there exist a map $\mathcal{Z} : R \to I\!R^s$, which is the expression of z in the original coordinate system (ϕ, U), such that

$$z(\zeta) = \mathcal{Z} \circ \pi_1(\zeta) \tag{38a}$$ {eZpi1}

for all $\zeta \in U$. The same reasoning leads to the existence of a map $\mathcal{V} : R \to I\!R^r$ such that

$$v(\zeta) = \mathcal{V} \circ \pi_1(\zeta) \tag{38b}$$ {eVpi1}

for all $\zeta \in U$. Define the map $\Lambda : U \to S_1$ such that $(t, x, u, u^{(1)}, \ldots) \mapsto (\tilde{t}, \tilde{z}, \tilde{v}, \tilde{v}^{(1)}, \ldots)$, where {eLambda}

$$\begin{aligned}
\tilde{t} &= t \quad (t = \tilde{t} \circ \Lambda) && (39a) \\
\tilde{z} &= \mathcal{Z} \circ \pi_1 = z \quad (z = \mathcal{Z} \circ \pi_1 = \tilde{z} \circ \Lambda) && (39b) \\
\tilde{v} &= \mathcal{V} \circ \pi_1 = v \quad (v = \mathcal{V} \circ \pi_1 = \tilde{v} \circ \Lambda) && (39c) \\
\tilde{v}^{(k)} &= v^{(k)} \quad (v^{(k)} = \tilde{v}^{(k)} \circ \Lambda) && (39d)
\end{aligned}$$

{eLambdat} {eLambdaz} {eLambdav} {eLambdavk}

Note that $\Lambda(\zeta) = (\delta(\zeta), v^{(\alpha+1)}(\zeta), v^{(\alpha+2)}(\zeta), v^{(\alpha+3)}(\zeta), \ldots)$. In particular, it follows that $\Lambda(U) \subset S_1 = \widehat{V} \times I\!R^C$. Recall that $\tilde{\pi} : S_1 \to \widehat{V}$ is the canonical projection $(\tilde{t}, \tilde{z}, \tilde{v}, \tilde{v}^{(1)}, \ldots) \mapsto (\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)})$. Then it is clear that

$$\tilde{\pi} \circ \Lambda = \delta \tag{40}$$ {eLambdapitil}

To show that Λ is Lie-Bäcklund, note from (39a) that $\frac{d}{dt}(\tilde{t} \circ \Lambda) = \frac{d}{dt}(t) = \mathbf{1}$. On the other hand, from (34), one has $\partial_{S_1}(\tilde{t}) \circ \Lambda = \mathbf{1} \circ \Lambda = \mathbf{1}$. Note from (39b) that $\frac{d}{dt}(\tilde{z} \circ \Lambda) = \frac{d}{dt} z = \dot{z}$. On the other hand, from (34), (40) and (25d), one has $\partial_{S_1}(\tilde{z}) \circ \Lambda = Z^{(1)} \circ \tilde{\pi} \circ \Lambda = Z^{(1)} \circ \delta = \dot{z}$. Now, from (39d), (34) and (24), one has $\partial_{S_1}(\tilde{v}^{(k)}) \circ \Lambda = \tilde{v}^{(k+1)} \circ \Lambda = v^{(k+1)} = \frac{d}{dt}(\tilde{v}^{(k)} \circ \Lambda)$.

Now it will be shown that $\iota = \Gamma \circ \Lambda$ is the identity map on U. In fact, note from (36a) and (39a), that (restricted to U), $t \circ \iota = t \circ \Gamma \circ \Lambda = \tilde{t} \circ \Lambda = t$. Furthermore, from (36b), (40) and (25a), it follows that $x \circ \iota = x \circ \Gamma \circ \Lambda = \chi \circ \tilde{\pi} \circ \Lambda = \chi \circ \delta = x$. Now, from (36c), (40) and (25b), it follows that $u \circ \iota = u \circ \Gamma \circ \Lambda = \mu \circ \tilde{\pi} \circ \Lambda = \mu \circ \delta = u$. Since, both Γ and Λ are Lie-Bäcklund, it follows that ι is so. Assume by induction, that $u^{(k)} \circ \iota = u^{(k)}$. Then, $u^{(k+1)} \circ \iota = \langle du^{(k)}; \frac{d}{dt} \rangle \circ \iota = \langle du^{(k)}; \frac{d}{dt} \circ \iota \rangle = \langle du^{(k)}; \iota_* \frac{d}{dt} \rangle = \langle \iota^*(du^{(k)}); \frac{d}{dt} \rangle = \langle d(u^{(k)} \circ \iota); \frac{d}{dt} \rangle = \langle du^{(k)}; \frac{d}{dt} \rangle = u^{(k+1)}$, showing that $\iota = \Gamma \circ \Lambda$ is the identity map on U.

Since $\iota : U \to U$ is the identity map, it follows that $\Gamma(\mathrm{im}\Lambda) = U$. Let V_1 be the open set given by $V_1 = \Gamma^{-1}(U) \subset S_1$. Then $U = \iota^{-1}(U) = (\Gamma \circ \Lambda)^{-1}(U) = \Lambda^{-1}(\Gamma^{-1}(U)) = \Lambda^{-1}(V_1)$. So $\Lambda(\Lambda^{-1}(V_1)) = \Lambda(U) = \mathrm{im}\Lambda$. It follows that $\mathrm{im}\Lambda \subset V_1$. Then $\iota = \Gamma \circ \Lambda = (\Gamma|V_1) \circ \Lambda$ is the identity map on U.

It will be shown now that $\jmath = \Lambda \circ (\Gamma|V_1)$ is the identity map on V_1. For this, let $\tilde{Z} : \widehat{V} \to I\!R^s$ and $\tilde{V} : \widehat{V} \to I\!R^r$ be the canonical projections defined respectively by $\tilde{Z}(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) = \tilde{z}$ and $\tilde{V}(\tilde{t}, \tilde{z}, \tilde{v}, \ldots, \tilde{v}^{(\alpha)}) = \tilde{v}$. Then it is clear that, on $V_1 \subset S_1$ one may write

$$\tilde{z} = \tilde{Z} \circ \tilde{\pi} \tag{41a}$$ {eZtilpitil}

$$z = \tilde{Z} \circ \delta \tag{41b}$$ {eZtildelta}

$$\tilde{v} = \tilde{V} \circ \tilde{\pi} \tag{41c}$$ {eVtilpitil}

$$v = \tilde{V} \circ \delta \tag{41d}$$ {eVtildelta}

From (26), (38a) and (41b), it follows that $\mathcal{Z} \circ h \circ \delta = \mathcal{Z} \circ \pi_1 = z = \tilde{Z} \circ \delta$. Since δ is surjective, then it follows that

$$\tilde{Z} = \mathcal{Z} \circ h \tag{42}$$ {eZtilh}

is an identity in $\widehat{V}$. Analogously, From (26), (38b) and (41d), it follows that

$$\tilde{V} = \mathcal{V} \circ h \tag{43}$$ {eVtilh}

Now note from (39a) and (36a) that (restricted to V_1), $\tilde{t} \circ \jmath = \tilde{t} \circ \Lambda \circ (\Gamma | V_1) = t \circ (\Gamma | V_1) = \tilde{t}$. Note also from (39b), (37), (42) and (41a) that $\tilde{z} \circ \jmath = \tilde{z} \circ \Lambda \circ (\Gamma | V_1) = \mathcal{Z} \circ \pi_1 \circ (\Gamma | V_1) = \mathcal{Z} \circ h \circ \tilde{\pi} = \tilde{Z} \circ \tilde{\pi} = \tilde{z}$. Analogously, by (39c),(37), (43) and (41c), one shows that $\tilde{v} \circ \jmath = \tilde{v}$. Since, both Γ and Λ are Lie-Bäcklund, it follows that $\jmath$ is so. Assume by induction that $\tilde{v}^{(k)} \circ \jmath = \tilde{v}^{(k)}$. Then, $\tilde{v}^{(k+1)} \circ \jmath = \langle d\tilde{v}^{(k)}; \partial_{S_1} \circ \jmath \rangle = \langle d\tilde{v}^{(k)}; \jmath_* \partial_{S_1} \rangle = \langle \jmath^*(d\tilde{v}^{(k)}); \partial_{S_1} \rangle = \langle d(\tilde{v}^{(k)} \circ \jmath); \partial_{S_1} \rangle = \langle d\tilde{v}^{(k)}; \partial_{S_1} \rangle = \tilde{v}^{(k+1)}$. Hence, $\jmath$ is the identity map on V_1. It follows that $\Gamma | V_1 : V_1 \to U$ and $\Lambda : U \to V_1$ are Lie-Bäcklund isomorphisms. In particular (z, v) is a local state representation around ξ. $\square$

Proof of Proposition 8 The proposition is essentially the chain rule. The proof is presented here for completeness.

Let $\zeta \in V$ and define $\tau = \delta_*(\frac{d}{dt}|_\zeta)$. For simplicity, one lets $\frac{d}{dt}$ stand for $\frac{d}{dt}|_\zeta$. Then

$$\tau = \alpha \frac{\partial}{\partial \tilde{t}} + \sum_{h=1}^{s} \beta_h \frac{\partial}{\partial \tilde{z}_h} + \sum_{j=0}^{\alpha} \sum_{k=0}^{r} \gamma_{j_k} \frac{\partial}{\partial \tilde{v}_k^{(j)}}$$

for convenient real coefficients α, β_h and γ_{j_k}. As $z = \tilde{z} \circ \delta$ and $v^{(j)} = \tilde{v}^{(j)} \circ \delta$ for $j = 0, \dots, \alpha$, then

$$\begin{aligned}
\alpha &= \delta_*(\frac{d}{dt})(\tilde{t}) = \frac{d}{dt}(\tilde{t} \circ \delta) = \frac{d}{dt}(t) = 1 \\
\beta_h &= \delta_*(\frac{d}{dt})(\tilde{z}_h) = \frac{d}{dt}(\tilde{z}_h \circ \delta) = \frac{d}{dt} z_h = \dot{z}_h \\
\gamma_{j_k} = &= \delta_*(\frac{d}{dt})(\tilde{v}_k^{(j)}) = \frac{d}{dt}(\tilde{v}_k^{(j)} \circ \delta) = \frac{d}{dt} v_k^{(j)} = v_k^{(j+1)}
\end{aligned}$$

It follows that $\tau = \delta_*(\frac{d}{dt}) = \frac{\partial}{\partial \tilde{t}} + \sum_{h=1}^{s} \dot{z} \frac{\partial}{\partial \tilde{z}_h} + \sum_{j=0}^{\alpha} \sum_{k=0}^{r} v_k^{(j)} \frac{\partial}{\partial \tilde{v}_k^{(j)}}$.

Now note that, by (25c) and (25a), one may write

$$X^{(1)} \circ \delta = \dot{x} = \frac{d}{dt}(\chi \circ \delta) = \delta_*(\frac{d}{dt})\chi = \tau(\chi)$$

Now, by (25d) and (29) it follows that, if one defines $\chi^{(1)} : \widehat{V} \to I\!R^n$ by (30), then one may write.

$$\dot{x} = X^{(1)} \circ \delta = \chi^{(1)} \circ \delta$$

This completes the proof of proposition 8. $\square$

C The dynamic extension algorithm.

{sDEA}

The Dynamic Extension Algorithm (DEA), a well known algorithm in nonlinear control theory, is essentially a tool for computing system right-inverses and the output rank (Fliess 1989, Descusse & Moog 1987, Nijmeijer & Respondek 1988, Pereira da Silva 1996, Delaleau & Pereira da Silva 1998b). The information computed by the DEA is essentially the same information computed by the inversion algorithm (Silverman 1969, Singh 1980, Singh 1981)[49]. The DEA has an intrinsic interpretation (Di Benedetto et al. 1989, Delaleau & Pereira da Silva 1998a). Recall that the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators. According to the ideas of the end of Section 2, this algorithm can be regardeded as the choice of a new local state representation of system S. Let S be a system with local classical state representation (x, u) and classical output y, with local state equations given by

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) \qquad (44a) \\ y(t) &= h(t, x(t), u(t)) \qquad (44b) \end{aligned}$$

Let $\xi \in S$. Let $(x_{-1}, u_{-1}) = (x, u)$ be the original state representation of S, with output $y^{(0)} = y$. In step k of this algorithm $(k = 0, 1, 2, \ldots)$ one will construct a local classical state representation (x_k, u_k) with output $y^{(k+1)} = h_k(x_k, u_k)$ defined on an open neighborhood U_k of $\xi \in S$. The following regularity condition is needed in order to perform the kth step of the algorithm:

$$\text{The codistribution span}\left\{dt, dx_{k-1}, dy^{(k)}\right\} \text{ is nonsingular at } \xi. \qquad (45)$$ {eRegularityK}

Each step of the algorithm may be interpreted three sub-steps as follows[50]

- (S1) Choose $\bar{y}_k$ among the components of y by completing $\{dt, dx_{k-1}\}$ into a local basis $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}\}$ of span $\left\{dt, dx_{k-1}, dy^{(k)}\right\}$ around ξ;
- (S2) Choose $\hat{u}_k$ among the components of u_{k-1} by completing $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}\}$ into a local basis $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}, d\hat{u}_k\}$ of span $\{dt, dx_{k-1}, du_{k-1}\}$ around ξ.
- (S3) Define $x_k = (x_{k-1}, \overline{y}_k^{(k)})$ and $u_k = (\overline{y}_k^{(k+1)}, \hat{u}_k)$.

{r9}

Remark 10 *Let $\hat{v}_k = \hat{u}_k$, $\bar{v}_k = \bar{y}_k^{(k)}$ and $v_k = (\bar{v}_k, \hat{v}_k)$. Since $\mathrm{card}\, u_{k-1} = \mathrm{card}\, v_k$, and span $\{dt, dx_{k-1}, du_{k-1}\} =$ span $\{dt, dx_{k-1}, dv_k\}$, then $(S1)$ and $(S2)$ defines a (local) static-state feedback (see corollary 3). Hence, $(S3)$ is an extension of state by putting some integrators in series with some inputs (namely, the components of $\bar{v}_k$). In particular, if the regularity conditions* (45) *are met for $k = 0, 1, \ldots, r$, then $\mathrm{card}\, u_k = \mathrm{card}\, u = m$ for $k = 0, 1, \ldots, r$.*

[49] The book (Kotta 1995) considers the inversion algorithm for discrete systems.

[50] The sub-steps (S1), (S2) and (S3) may be performed even when the system is not affine, but in this case, one must apply nonconstructive results for the computation of the feedbacks, like the inverse function theorem. When the system is affine, the algorithm is effective (see (Di Benedetto et al. 1989, Pereira da Silva 2000)).

The following lemma summarizes the main geometric properties of the DEA for time-varying systems (see (Pereira da Silva 2000)).

{lDynExtAl}

Lemma 5 *Let S be a system with Cartan field $\frac{d}{dt}$. Let (x,u) be a local state representation of S defined on $U_{-1} \subset S$, with classical output y and state equations* (17)*. Let $n = \mathrm{card}\, x$ and $m = \mathrm{card}\, u$. Let $V_k \subset U_{-1}$ be the open and dense set of regular points of the codistributions Y_i and $\mathcal{Y}_i$ for $i = 0, \ldots, k$ defined by* (7b)-(7c)*. Let $\xi \in V_n$. In the k*th *step of the* Dynamic Extension Algorithm*, one may construct*[51] *a new local classical state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$, input $u_k = (\bar{y}_k^{(k+1)}, \widehat{u}_k)$ and output $y^{(k+1)} = h_k(t, x_k, u_k)$ defined in an open neighborhood $U_k \subset U_{k-1} \subset V_n$ of ξ, such that*

{i1}

1. *The set $\{dt, dx_k\}$ is a basis of span $\left\{dt, dx, dy, \ldots, dy^{(k)}\right\} = \mathcal{Y}_k$ (on $U_k \subset V_n$).* {i2}
2. *The set $\{dt, dx_k, du_k\}$ is a basis of span $\left\{dt, dx, dy, \ldots, dy^{(k+1)}, du\right\} = \mathcal{Y}_{k+1} +$ span $\{du\}$ (on U_k).* {i3}
3. *It is always possible to choose $\bar{y}_k^{(k)}$ in a way that $\bar{y}_{k-1}^{(k)} \subset \bar{y}_k^{(k)}$ (on U_k).* {i3l}
4. *It is always possible to choose $\widehat{u}_k \subset \widehat{u}_{k-1}$ (on U_k).* {i4}
5. *Let $k \geq 0$. The sequence $\sigma_k = \dim(\mathcal{Y}_k|_\xi) - \dim(\mathcal{Y}_{k-1}|_\xi)$ is nondecreasing, the sequence $\rho_k = \dim(Y_k|_\xi) - \dim(Y_{k-1}|_\xi)$ is nonincreasing, and both sequences converge to the same integer ρ, called the* output rank at ξ*, for some $k^* \leq n = \dim x$.* {i5}
6. *The DEA may be performed for arbitrary $k \in I\!N$, and one may choose $U_k, k \in I\!N$ in a way that the dimensions of Y_i and $\mathcal{Y}_i$ are constant on U_k for $i \in I\!N$. Furthermore, one may choose $U_k = U_{k^*}$ for $k \geq k^*$. In particular $V_k = V_n$ for all $k \geq n$.* {i5l}
7. *$Y_k \cap$ span $\{dx\}|_\nu = Y_{k^*-1} \cap$ span $\{dx\}|_\nu$ for every $\nu \in U_{k^*}$ and $k \geq k^*$.* {i6}
8. *For $k \geq k^*$, one may choose $\bar{y}_k = \bar{y}_{k^*}$ (in U_{k^*}). Furthermore, for $r \geq 0$, one may write $Y_{r+k*} = Y_{r+k^*-1} +$ span $\left\{d\bar{y}_{k^*}^{(r+k^*)}\right\}$ (in U_{k^*}).* {i9}
9. $\dim \mathcal{Y}_k \cap$ *span* $\{du\}|_\nu = \sigma_k, k = 0, 1, 2, \ldots,$ *for every $\nu \in U_{k^*}$.* {i10}
10. *There exists a nonnegative integer δ, with $\delta \leq n$, such that $\sigma_k = 1 + \delta + \sigma_{k^*}(k+1), k \geq k^*$. When the system is right invertible, that is, when $\sigma_{k^*} = \mathrm{card}\, y$, the integer δ is called the* defect *of the output y, and it coincides locally with* $\dim \mathcal{Y}_k - \dim Y_k$ *for all $k \geq k^*$.*

[51] Note that $\mathrm{card}\, u_k = \mathrm{card}\, u_{k-1}$.

Remark 11 *The Dynamic extension algorithm can be performed around every point ξ of V_n. However, in the step k the state representation (x_k, u_k) that is computed by the DEA is defined only on $U_k \subset U_{k-1} \subset V_n$. In the proof it will be clear that this is due to the fact that in every step one is applying Corollary 3 of section 5.1, which holds only locally. It is important that k^* and U_{k^*} do depend on ξ. However, since $k^* \leq n$, then 5 implies the convergence of σ_k and ρ_k for $k \geq n$ for any $\xi \in V_n$ (independently of the choice of ξ). By 6 and 7 it follows easily that $Y_k \cap \mathrm{span}\,\{dx\}\,|_\nu = Y_n \cap \mathrm{span}\,\{dx\}\,|_\nu$ for every $k \geq n$ and every $\nu \in V_n$. In fact, to see this it suffices to take into account that is always such $k^* \leq n$ for all choices of working point ξ.*

Proof. We shall show that 1 and 2 are true for $\xi \in V_k$. Then, after showing 6, this will imply that 1 and 2 are true for all $\xi \in V_n$ for all $k \in I\!N$.

1 and 2. Let $\xi \in V_k$. Assume by induction that one has constructed a proper state representation (x_{k-1}, u_{k-1}) on U_{k-1} for which 1 and 2 hold. It will be shown that, following the steps (S1), (S2) and (S3), one may construct a proper state representation (x_{k-1}, u_{k-1}) on $U_k \subset U_{k-1}$ such that 1 and 2 holds.

One will show 1 and 2 by induction. Note first that if one assumes that 1 holds for some $k \in I\!N$ and that $\xi \in V_k$, then one has that ξ is a regular point of $\mathcal{Y}_k = \mathrm{span}\left\{dt, dx, dy^{(0)}, \ldots, dy^{(k)}\right\} = \mathrm{span}\left\{dt, dx_{k-1}, dy^{(k)}\right\}$, and so the regularity condition (45) holds. Hence, (S1), (S2) and (S3) may be performed on some open neighborhood of ξ. By (S2) and Corollary 3, it is clear from Remark 10 that (x_k, u_k) is a state representation around some open neighborhood U_k of ξ.

Assume by induction that the state representation (x_{k-1}, u_{k-1}) with output $y^{(k)}$ is proper, *i. e.*, $\mathrm{span}\,\{d\dot{x}_{k-1}\} \subset \mathrm{span}\,\{dt, dx_{k-1}, du_{k-1}\}$ and $\mathrm{span}\left\{dy^{(k)}\right\} \subset \mathrm{span}\,\{dt, dx_{k-1}, du_{k-1}\}$. We show first that $\mathrm{span}\,\{d\dot{x}_k\} \subset \mathrm{span}\,\{dt, dx_k, du_k\}$.

This property holds for $k = -1$ (that is (x, u) with output $y^{(0)}$). Then from (S1), (S2) and (S3) we have $\mathrm{span}\,\{d\dot{x}_k\} \subset \mathrm{span}\left\{dt, dx_{k-1}, d\dot{x}_{k-1}, d\bar{y}_k^{(k)}, d\bar{y}_k^{(k+1)}\right\}$ $\subset \mathrm{span}\,\{dt, dx_k, du_k\}$.

In step $k = 0$, we choose a partition $y^{(0)} = (\bar{y}_0^{(0)}, \hat{y}_0^{(0)})$ in a way that (S1) is satisfied for $k = 0$ and construct $\hat{u}_0$ satisfying (S2). Then $d\hat{y}_0^{(0)} \in \mathrm{span}\{dt, dx, d\bar{y}_0^{(0)}\}$. Thus, $d\dot{\hat{y}}_0^{(0)} \in \mathrm{span}\{dt, dx, d\dot{x}, d\bar{y}_0^{(0)}, d\bar{y}_0^{(1)}\} \subset \mathrm{span}\{dt, dx, du, d\bar{y}_0^{(0)}, d\bar{y}_0^{(1)}\}$. So, by (S3), $d\dot{y} \in \mathrm{span}\{dt, dx_0, du_0\}$. Then it is easy to see that 1 and 2 are satisfied for $k = 0$ and the output $y^{(0)}$. Now assume that, in the step $k-1$ we have a local state representation (x_{k-1}, u_{k-1}) satisfying 1 and 2 that is defined on some open neighborhood U_{k-1} of ξ. Choose a partition $y^{(k)} = (\bar{y}_k^{(k)}, \hat{y}_k^{(k)})$ in a way that (S1) is satisfied and construct $\hat{u}_k$ satisfying (S2). By 1 for $k-1$ and (S1) it follows that, $\mathrm{span}\{dx_k\} = \mathrm{span}\{dt, dx, dy, \ldots, dy^{(k)}\}$. By construction, notice that $d\dot{\hat{y}}_k^{(k+1)} \in \mathrm{span}\{dt, dx_{k-1}, d\dot{x}_{k-1}, d\bar{y}_k^{(k)}, d\bar{y}_k^{(k+1)}\} \subset \mathrm{span}\{dt, dx_{k-1}\; du_{k-1}, d\bar{y}_k^{(k)}, d\bar{y}_k^{(k+1)}\}$. So, $dy^{(k+1)} \in \mathrm{span}\{dt, dx_k,\; du_k\}$. We show now that if 2 holds for $k-1$, then $\mathrm{span}\{dt, dx_k,\; du_k\} = \mathrm{span}\{dt, dx, dy, \ldots, dy^{(k+1)}, du\}$, completing the induction. In fact, note that $\mathrm{span}\{dt, dx_k,$

$du_k\} = \text{span}\{dx_{k-1},\, d\bar{y}_k^{(k)},\, d\hat{u}_k\} + \text{ span}\left\{d\bar{y}_k^{(k+1)}\right\}$. By (S2) and the induction hypothesis it follows that $\text{span}\{dt, dx_k,\, du_k\} = \text{span}\{dt, dx, du, dy, \ldots,\, dy^{(k)}\} +$ span $\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. Since $dy^{(k+1)} \in \text{span}\{dt, dx_k,\, du_k\}$, then 2 holds for k and the fact that (x_k, u_k) with output $y^{(k+1)}$ is proper. This shows 1 and 2.

4. Easy consequence of 1, 2 and (S2).

3, 5, 6. Since $\xi \in V_n$, ξ is a regular point of $\mathcal{Y}_j$ and Y_j for $j = 0, 1, \ldots, n$. From the reasoning above, it follows that 1 and 2 holds for $k = 0, \ldots n$. From Definition 3, as $\{t, x_k, u_k\}$ are subsets of a set of coordinate functions in U_k, from 1 it follows that the dimensions $\mathcal{Y}_j$, $j = 0, \ldots, k$ are constant in U_k for $k = 0, \ldots n$. Without loss of generality, one may assume that one may restrict U_k in a way that the dimension of Y_k is also constant in U_k or $k = 0, \ldots, n$. Since $U_k \subset U_{k-1}$ one may assume that

$$\begin{array}{c}\text{The dimensions of } Y_i \text{ and } \mathcal{Y}_i \text{ are constant in } U_k \\ \text{for } i = 0, 1, \ldots, k \text{ and } k \in \lfloor n \rceil.\end{array} \tag{46}$$ {eConstantN}

We show first that

$$\begin{array}{c}\dim Y_k(\nu) - \dim Y_{k-1}(\nu) \geq \dim Y_{k+s+1}(\nu) - \dim Y_{k+s}(\nu) \\ \text{for every } \nu \in U_k, k \in \lfloor n \rceil, \text{and } s \in I\!N\end{array} \tag{47}$$ {eSIGMA}

For this note that, if the 1-forms $\{\eta_1, \ldots, \eta_s\} \subset Y_k$ are linearly dependent mod Y_{k-1}, *i. e.*, if $\sum_{i=1}^{s} \alpha_i \eta_i + \alpha_0 dt + \sum_{i=1}^{p} \sum_{j=0}^{k-1} \beta_{ij} dy_i^{(j)} = 0$ then, differentiation in time[52] gives

$$\sum_{i=1}^{s} (\dot{\alpha}_i \eta_i + \dot{\alpha}_0 dt + \alpha_i \dot{\eta}_i) + \sum_{i=1}^{p} \sum_{j=0}^{k-1} (\dot{\beta}_{ij} dy_i^{(j)} + \beta_{ij} dy_i^{(j+1)}) = 0. \tag{48}$$ {eZZero}

In other words, $\dot{\eta}_1, \ldots, \dot{\eta}_s$ are linearly dependent mod Y_{k+1}. Let $k \in \lfloor n \rceil$. Note that there exists $r \in I\!N$ such that $\dim Y_k - \dim Y_{k-1} = r$ on every $\nu \in U_k$. Then we may choose a partition $y = (\bar{y}^T, \hat{y}^T)$ such that $\bar{y}$ has r components and we locally have $Y_k = \text{span}\left\{d\bar{y}^{(k)}\right\} + Y_{k-1}$. Let $\hat{y}_j$ be an arbitrary component of $\hat{y}$ for $j \in \lfloor p - r \rceil$. By construction we have that $\{d\hat{y}_j^{(k)}, d\bar{y}^{(k)}\}$ is linearly dependent mod Y_{k-1} for every $j \in \lfloor p - r \rceil$. From the remark above it follows that the set $\{d\hat{y}_j^{(k+1)}, d\bar{y}^{(k+1)}\}$ is (locally) dependent mod Y_k for every $j \in \lfloor p - r \rceil$, showing (47) for $s = 0$. In particular the sequence ρ_k is nonincreasing for $k = 0, 1, \ldots n$. Now further differentiations of (48) shows easily that $\{d\hat{y}_j^{(k+s)}, d\bar{y}^{(k+s)}\}$ is linearly dependent mod Y_{k+s-1}, showing (47) for $s \in I\!N$.

We show now that

$$\begin{array}{c}\dim \mathcal{Y}_k(\nu) - \dim \mathcal{Y}_{k-1}(\nu) \leq \dim \mathcal{Y}_{k+s+1}(\nu) - \dim \mathcal{Y}_{k+s}(\nu) \\ \text{for every } \nu \in U_k, k \in \lfloor n \rceil, s \in I\!N\end{array} \tag{49}$$ {eRHO}

[52] That is, the computation of the Lie-derivative with respect to $\frac{d}{dt}$.

Assume that (x_k, u_k) is a state representation constructed around a neighborhood U_k of a point ξ and satisfying (S1), (S2), (S3), 1 and 2. Since $\mathrm{span}\{dt, dx_k\} = \mathcal{Y}_k$ and $\bar{y}_k^{(k+1)} \subset u_k$, it follows that

$$\text{the components of } d\bar{y}_k^{(k+1)} \text{ are independent mod } \mathcal{Y}_k \tag{50}$$ {eIndependentYbar}

since the components of $\bar{y}_k^{(k+1)}$ are also components of the input u_k, the set $\{dt, dx_k, du_k\}$ is linearly independent[53] and furthermore span $\{dt, dx_k\} = \mathcal{Y}_k$. Hence $\bar{y}_{k+1}^{(k+1)}$ may be chosen satisfying 3. In particular, $\sigma_{k+1} \geq \sigma_k$ for $k = 0, \ldots n$. Now, since $\{dt, dx_k, (du_k^{(k)} : k \in I\!\!N)\}$ it a set differentials of a subset of coordinate functions, then it is a set locally linearly independent covector fields. As the state representation (x_k, u_k) with output $y^{(k)}$ is proper, by 1, it follows after some differentiations that $\mathcal{Y}_{k+s-1} \subset \mathrm{span}\left\{dt, dx_k, (du_k^{(j)} : j = 0, \ldots, s-1)\right\}$. Now, by (S3), $\bar{y}_k^{(k+s+1)}$ is a subset of $u_k^{(s)}$. It is then clear that

$$\text{the components of } d\bar{y}_k^{(k+s+1)} \text{ are independent mod } \mathcal{Y}_{k+s} \tag{51}$$ {eIndependentYbarS}

showing (49).

To show the convergence of sequences ρ_k and σ_k for some $k^* \leq n$, assume that $\nu \in U_k$. Denote $\mathrm{span}\{dx\}$ by X. Then $\mathcal{Y}_k = X + Y_k$ and thus

$$\dim \mathcal{Y}_k(\nu) = \dim X(\nu) + \dim Y_k(\nu) - \dim(Y_k(\nu) \cap X(\nu)).$$

Denote for $k \in I\!\!N$:

$$\begin{array}{rcl} s_k(\nu) & = & \dim \mathcal{Y}_k(\nu) - \dim \mathcal{Y}_{k-1}(\nu) \\ p_k(\nu) & = & \dim Y_k(\nu) - \dim Y_{k-1}(\nu) \end{array}$$

Note that $\rho_k = p_k(\nu)$ and $\sigma_k = s_k(\nu)$ are constant for every $\nu \in U_k$. We also have

$$s_k(\nu) = p_k(\nu) - \dim(Y_k(\nu) \cap X(\nu)) + \dim(Y_{k-1}(\nu) \cap X(\nu)) \tag{52}$$ {eDim}

To be consistent with the notation of statement 5, note that

$$\sigma_k = s_k(\xi) \text{ and } \rho_k = p_k(\xi). \tag{53}$$ {eSigmaRhoE}

Now let $k^* \leq n$. Assume that

$$\left\{ \begin{array}{l} \xi \in V_{k^*} \subset V_n \\ \sigma_{k^*} = \rho_{k^*} = \rho \end{array} \right. \tag{54}$$ {eDuasEstrelas}

Then one will show that, for all $s \in I\!\!N$ one may execute the DEA for the step $k^* + s$, constructing a state representation that is defined in U_{k^*+s} and furthermore

$$\left\{ \begin{array}{l} \xi \in V_{k^*+s} \\ \sigma_{k^*+s} = \rho_{k^*+s} = \rho \\ Y_{k^*+s} \cap X = Y_{k^*-1} \cap X \text{ on } U_{k^*+s} \end{array} \right. \tag{55}$$ {eTresEstrelas}

[53] It is part of a local coordinate system

From (54) and (47), it follows that

$$\rho = \rho_{k^*} \geq \rho_{k^*+s+1}, \text{for all } s \in I\!N \tag{56}$$ {eALPHA}

From (46), (53), (54) and (49), it follows that

$$\rho = \sigma_{k^*} \leq \sigma_{k^*+s+1}, \text{for all } s \in I\!N \tag{57}$$ {eBETA}

Note from (52) that

$$s_{k^*}(\nu) - p_{k^*}(\nu) = -\dim(Y_{k^*} \cap X|_\nu) + \dim(Y_{k^*-1} \cap X|_\nu)$$

As $s_{k^*}(\nu)$ and $p_{k^*}(\nu)$ are constant in U_{k^*}, this shows that (54) implies that (55) holds for $s = 0$. Assume now that (55) holds for some $s \in I\!N$ and one may apply the DEA for $k = 0, \ldots, k^* + s$, constructing a state representation defined in U_{k^*+s}. From the same reasoning that one has applied to obtain (52), one may write

$$s_{k^*+s+1}(\nu) - p_{k^*+s+1}(\nu) = -\dim(Y_{k^*+s+1} \cap X|_\nu) + \dim(Y_{k^*+s} \cap X|_\nu) \tag{58}$$ {eGAMMA}

From (56) and (57), one may say that the left side of (58) is greater than or equal to zero. Now, as $\dim(Y_k \cap X|_\nu)$ is nondecrasing as k increases with a fixed ν, then it follows that the right hand side of (58) is nonpositive. Hence the only possibility is to have

$$\dim(Y_{k^*+s+1} \cap X|_\nu) = \dim(Y_{k^*+s} \cap X|_\nu), \nu \in U_{k^*+s}$$

and

$$s_{k^*+s+1}(\nu) = p_{k^*+s+1}(\nu) = \rho, \text{for all } \nu \in U_{k^*+s} \tag{59}$$ {eCONSTANT}

Now, as $\xi \in V_{k^*+s}$, it is a regular point of $\mathcal{Y}_k$ and of Y_k for $k = 0, \ldots, k^* + s$. From (59), it is clear that $\xi \in V_{k^*+s+1}$. This shows that (55) holds when replacing s by $s + 1$. Recall that the only regularity assumption that is needed in order to execute the step k of the DEA is (45). In particular, by 1, one may execute the DEA for $k = k^* + s + 1$ constructing a state representation that is defined on U_{k^*+s+1} and furthermore (55) holds. This completes the proof of (55) by induction.

Now note that, since $\dim(Y_k \cap X|_\xi)$ increases with k, and $0 \leq \dim(Y_k \cap X|_\xi) \leq n$, it is clear that there exists some $k^* \leq n$ such that (54) holds. Recall that on has show that 1 and 2 holds for $\xi = V_k$ for all $k \in I\!N$. Now, this means that 1 and 2 holds for an arbitrary $\xi \in V_n$ and $k \in I\!N$, since it is easy to verify that one has also shown that $V_k = V_n$ for all $k \geq n$.

To complete the proof of 5, note that (47) shows that ρ_k is nonincreasing for $k \leq n$ and that σ_k is nondecreasing for $k \leq n$, and one has shown that $\sigma_k = \rho_k = \rho$ for all $k \geq k^*$.

Now one will show that it is possible to choose $U_k = U_{k^*}$ for $k \geq k^*$, Recall that $\operatorname{card} \bar{y}_k = \sigma_k$. Since $u_k = (\bar{y}_k^{(k+1)}, \hat{u}_k)$, it is clear from Remark 10 that $\operatorname{card} \hat{u}_k = m - \sigma_k$. As $\sigma_k = \sigma_{k^*}$ for $k \geq k^*$, it follows from 4 that one may choose $\hat{u}_k = \hat{u}_{k^*}$ for $k \geq k^*$. Analogously, from 3 and the fact that $\sigma_k = \sigma_{k^*}$ for $k \geq k^*$, it follows that one may take $\bar{y}_k = \bar{y}_{k-1}$ for $k \geq k^*$. Now, as $u_k = (\bar{u}_k, \hat{u}_k)$, where $\bar{u}_k = \bar{y}_k^{(k+1)}$, note that, with these choices, in the step $k > k^*$, (S3) reduces to $x_k = (x_{k-1}, \bar{u}_{k-1})$ and $u_k = (\bar{u}_{k-1}, \hat{u}_{k-1})$, which is the operation of putting integrators in series with the first σ_k inputs. This feedback (which is essentially a coordinate change) is well defined in the entire U_{k-1}. Hence one may take $U_k = U_{k^*}$ for all $k > k^*$, showing 6. Note that the first affirmation of 5 is implied by (46), (59) and from the fact $U_k = U_{k^*}$ for all $k > k^*$.

7. Easy consequence of (55) that was shown above.

8. The first part of 8 follows easily from 3 from the fact that card $\bar{y}_k = \sigma_k$ and from 5. The second part of 8 follows easily from the equality card $\bar{y}_k = \sigma_k$, from the fact that the components of $d\bar{y}_k^{(k+1)}$ are independent mod Y_k (see 50) and from the fact that $\sigma_k = \rho_k = \rho$ for $k \geq k^*$.

9. Note from 5 that $\dim(\mathcal{Y}_k) = 1 + n + \sum_{j=0}^{k} \sigma_j$. As (x_k, u_k) is a state representation, then $\dim \operatorname{span}\{dt, dx_k, du_k\} = 1 + \operatorname{card} x_k + \operatorname{card} u_k = 1 + n + \sum_{j=0}^{k} \sigma_j + m$. As $\dim(\mathcal{Y}_{k+1} + \operatorname{span}\{du\}) = \dim \mathcal{Y}_{k+1} + \dim \operatorname{span}\{du\} - \dim(\mathcal{Y}_{k+1} \cap \operatorname{span}\{du\})$. Hence the desired result follows from 2.

10. Let $k \geq k^*$. By 1 it follows that $\dim \mathcal{Y}_k = 1 + n + \sigma_0 + \ldots + \sigma_k = 1 + n + \sum_{j=0}^{k-1}(\sigma_j - \sigma_k) + (k+1)\sigma_k$. Now define $\delta = n + \sum_{j=0}^{k^*-1}(\sigma_j - \sigma_{k^*})$. As σ_k is non decreasing, then $\delta \leq n$. As $\sigma_k = \sigma_{k^*}$ for $k \geq k^*$, it follows that $\delta = n + \sum_{j=0}^{k}(\sigma_j - \sigma_k)$ for all $k \geq k^*$. It is clear that $\dim \mathcal{Y}_k = 1 + \delta + (k+1)\sigma_{k^*}$ for $k \geq k^*$. Now, as $\dim Y_{k^*}|_\xi = 1 + \sum_{k=0}^{k^*} \rho_k$ and $\dim \mathcal{Y}_{k^*}|_\xi = n + \dim Y_{k^*} - \dim Y_{k^*} \bigcap \operatorname{span}\{dx\}$. Let $\alpha = \dim Y_{k^*} \bigcap \operatorname{span}\{dx\}$. It follows that $\delta + (k^*+1)\sigma_{k^*} = n - \alpha + \sum_{k=0}^{k^*} \rho_k$, where $0 \leq \alpha \leq n$. Hence $\delta = n - \alpha + \sum_{k=0}^{k^*}(\rho_k - \sigma_{k^*})$. By 5, it follows that $\rho_k \geq \sigma_{k^*}$ and so $\delta \geq 0$. Now assume that the system is right invertible, that is, $\sigma_{k^*} = \operatorname{card} y$. By 5 it follows that $\rho_k \geq \rho_{k^*} \geq \sigma k^* = \sigma_k$. As $\rho_k \leq \operatorname{card} y$, it follows that $\rho_k = \operatorname{card} y$ for all $k \in I\!N$. In this case, $\delta = n - \alpha = \dim \mathcal{Y}_{k^*} - \dim Y_{k^*}$, which coincides with the dimension of the zero dynamics.

□

D A basis of the module H is independent pointwise

{sIndependent}

Proposition 9 *(Pomet 1995) Let S_1 be a diffiety , and let $\{x_i, i \in A\}$ be local coordinates defined on $U \subset S_1$. Let B_2 be a basis of the $C^\infty(U)$-module given by $H = span\{dx_i : i \in A\}$. Then the set B_2 is pointwise independent.* {pIndependent}

Proof. Let $B_1 = \{dx_i, i \in A\}$. Hence, B_1 is another basis of H. When written in this basis, a 1-form may be regarded as an infinite column vector of functions in $C^\infty(U)$, with only a finite number of nonzero elements. As B_1 and B_2 are basis of T^*S_1, there exists basis transformation matrices α, from B_1 to B_2, and

β, from B_2 to B_1. Note that the columns of β are the expression of the forms of B_2 in the basis $\{dx_i, i \in A\}$. These matrices have an infinite number of rows and columns, but each column have only a finite number of nonzero elements. Hence the multiplication αw of a matrix α by a 1-form w and the products $\alpha\beta$ and $\beta\alpha$ are well defined. By construction, one must have that $\alpha\beta$ and $\beta\alpha$ are equal to the identity matrix. In particular, when computing the matrix β at a point $\nu \in S_1$, one gets a invertible matrix $\beta(\nu)$ of real numbers. So, the columns of $\beta(\nu)$ must be $I\!R$-independent. In fact, if there exist an infinite column vector w, with only a finite number of nonzero elements, such that $\beta(\nu)w = 0$, then $\alpha(\nu)\beta(\nu)w = 0$ implies $w = 0$. $\square$

E Proof of theorem 2

{atY}

In this proof we use the results and the notations of Lemma 5. Let $n = \dim x$. By lemma 5, around $\xi \in U$, there exists a local state representation (x_{n-1}, u_{n-1}) defined in V_ξ such that {eAux}

$$\begin{aligned} \operatorname{span}\{dt, dx_{n-1}\} &= \operatorname{span}\left\{dt, dx, dy, \ldots, dy^{(n-1)}\right\}, && (60a)\\ \operatorname{span}\{dt, dx_{n-1}, du_{n-1}\} &= \operatorname{span}\left\{dt, dx, du, dy, \ldots, dy^{(n)}\right\}, && (60b) \end{aligned}$$

and where $u_{n-1} = (\bar{y}^{(n)}_{n-1}, \hat{u}_{n-1})$. Now choose a subset z_a of $\{y, \ldots, y^{(n-1)}\}$ in a way that $\{dt, dz_a\}$ is a local basis of $\operatorname{span}\left\{dt, dy, \ldots, dy^{(n)}\right\}$ and choose z_b in a way that $\{dt, dz_a, dz_b\}$ is a local basis of $\operatorname{span}\left\{dt, dx, dy, \ldots, dy^{(n)}\right\}$ around ξ. Let $v_a = \bar{y}^{(n)}_{n-1}$ and $v_b = \hat{u}_n$. By construction, $((z_a, z_b), (v_a, v_b))$ is a local state-representation of S defined in an open neighborhood V_ξ of ξ. In fact, by corollary 3, it is linked to (x_{n-1}, u_{n-1}) by local static-state feedback. Since (x_{n-1}, u_{n-1}) is proper, then $((z_a, z_b), (v_a, v_b))$ is also proper and (8b) holds.

By Lemma 5 part 8, we have that $\operatorname{span}\{dt, dz_a, dv_a\} = \operatorname{span}\{dt, dy, \ldots, dy^{(n)}\}$. It follows that $\operatorname{span}\{d\dot{z}_a\} \subset \operatorname{span}\{dt, dz_a, dv_a\}$. Hence (8a) holds. By derivation it is easy to show that $\operatorname{span}\left\{dt, dz_a, (dv_a^{(k)} : k \in I\!N)\right\} = \operatorname{span}\{dt, dy^{(k)} : k \in I\!N\}$.

Let Y be the diffiety of global coordinates $\{t, z_a, (v_a^{(k)} : k \in I\!N)\}$ with Cartan field

$$\partial_Y = \frac{\partial}{\partial t} + \sum_{i=1}^{n_a} f_{a_i}(t, z_a, v_a)\frac{\partial}{\partial x_i} + \sum_{j=0}^{\infty}\sum_{k=1}^{m_a} v_{a_k}^{(j+1)}\frac{\partial}{\partial v_{a_k}^{(j)}}.$$

where $n_a = \operatorname{card} z_a$ and $m_a = \operatorname{card} v_a$. In particular (z_a, v_a) is a global state representation of Y. Now it clear that the map $\pi : V_\xi \to Y$ such that $(t, z_a, z_b, (v_a^{(k)}, v_b^{(k)} : k \in I\!N) \mapsto (t, z_a, (v_a^{(k)} : k \in I\!N))$ is a Lie-Bäcklund submersion.

To complete the proof, it suffices to show that $\operatorname{span}\left\{dz_a, (dv_a^{(k)} : k \in I\!N)\right\} = \operatorname{span}\left\{dy^{(k)} : k \in I\!N\right\}$. We show first that $\operatorname{span}\{dt\} \cap \mathbb{Y}_k|_\xi = \{0\}$ for every point

ξ of $\tilde{\Delta}$. In fact, let $\xi \in \tilde{\Delta}$ and let $\eta = \sum_{i=0}^{k} \sum_{j=1}^{r} \alpha_{ij} dy_j^{(i)}|_\xi = \beta dt|_\xi$. Then $\beta|_\xi = \langle \eta; \frac{d}{dt} \rangle|_\xi = \sum_{i,j} \langle \alpha_{ij} dy_j^{(i)}; \frac{d}{dt} \rangle|_\xi = \sum_{i,j} \alpha_{ij} \langle dy_j^{(i)}; \frac{d}{dt} \rangle|_\xi = \sum_{i,j} \alpha_{ij} y_j^{(i+1)}|_\xi = 0$.

Using the notation (7a)-(7b), one may write

$$\dim Y_k = \dim (\operatorname{span} \{dt\}) + \dim \mathbb{Y}_k - \dim (\operatorname{span} \{dt\} \cap \mathbb{Y}_k)$$

The nonsingularity of span $\{dt\}$, Y_k and $\mathbb{Y}_k$ for $k = n-1$ and for $k = n$ implies the nonsingularity of span $\{dt\} \cap \mathbb{Y}_{n-1}$ and span $\{dt\} \cap \mathbb{Y}_n$ around any point of $\tilde{\Delta}$. In particular, span $\{dt\} \cap \mathbb{Y}_{n-1} =$ span $\{dt\} \cap \mathbb{Y}_{n-1} = \{0\}$ in a neighborhood of every point $\nu \in \tilde{\Delta}$. We show now that one has span $\{dz_a\} = \mathbb{Y}_{n-1}$ around $\nu \in \tilde{\Delta}$. Since $z_a \subset \{y, \dots, y^{(n-1)}\}$, it is clear that span $\{dz_a\} \subset \mathbb{Y}_{n-1}$. To show the inverse inclusion, take some $\omega \in \mathbb{Y}_{n-1}$. Then $\omega = \sum_{i=1}^{n_a} \alpha_i dz_{a_i}|_x i + \beta dt$ for convenient functions $\alpha_i, i \in \lfloor n_a \rceil$, and β. Let $\xi \in \tilde{\Delta}$ and let V_ξ be an open neighborhood of ξ for which span $\{dt\} \cap \mathbb{Y}_{n-1} = 0$. If for some $\nu \in V_\xi$ one has $\beta|_\nu \neq 0$, then $\beta dt|_\nu$ will be in span $\{dt\} \cap \mathbb{Y}_{n-1}|_\nu$. In particular, ω belongs to span $\{dz_a\}$. By similar arguments, one shows that span $\{dz_a, dv_a\} = \mathbb{Y}_n$. By derivation, it follows easily that span $\left\{dz_a, (dv_a^{(k)} : k \in I\!N)\right\} =$ span $\left\{dy^{(k)} : k \in I\!N\right\}$.

F A common abuse of notation

{aAbuse}

Let $\phi : S \to S_2$ be a smooth mapping between two diffieties with Cartan fields respectively given by ∂_S and ∂_{S_2}. Let $v_2 : S_2 \to I\!R$ be a function and let $v = v_2 \circ \phi$.

{pAbuse}

Proposition 10 *For a given function $v_2 : S_2 \to I\!R$ define $v = v_2 \circ \phi$. The map ϕ is a Lie-Bäcklund mapping if and only for every function $v_2 : S_2 \to I\!R$, then one has $\partial_{S_1}^k v = \{\partial_{S_2}^k v_2\} \circ \phi$ for every $k \in I\!N$.*

Proof. Suppose first that the map ϕ is Lie-Bäcklund. By induction, assume that that $\partial_{S_1}^{k-1} v = \partial_{S_2}^{k-1} v_2 \circ \phi$. Then $\partial_{S_1}^k v = \partial_{S_1}(\partial_{S_1}^{k-1} v) = \partial_{S_1}(\partial_{S_2}^{k-1} v_2 \circ \phi) = \phi_* \partial_{S_1}(\partial_{S_2}^{k-1} v_2) = \{\partial_{S_2}(\partial_{S_2}^{k-1} v_2)\} \circ \phi = \{\partial_{S_2}^k v_2\} \circ \phi$. Now assume that the statement of the proposition holds. Let $(x_i, i \in A)$ be a local coordinate chart for S_2. If the action of two fields on the functions of a coordinate systems coincides then these two fields also coincides. Now note that, for all $i \in A$, one has $\phi_* \partial S_1(x_i) = \partial_{S_1}(x_i \circ \phi) = \{\partial_{S_2} x_i\} \circ \phi$. In particular $\phi_* \partial S_1 = \partial_{S_2} \circ \phi$, and so ϕ is Lie-Bäcklund. □

It is worth to stress that, in many circumstances, one may abuse notation by letting both $\partial_{S_1}^k v$ and $\{\partial_{S_2}^k v_2\}$ simply by $v^{(k)}$ (or $v_2^{(k)}$). Proposition 10 shows that, if $\phi : S \to S_2$ is Lie-Bäcklund, this abuse of notation makes sense.

G Other notions of regular dynamic feedback

The notion of regular dynamic feedback was originally based on equations of the form (15) (Di Benedetto et al. 1989). Now consider system E defined by

(15) with input $\tilde{u} = v$, state $\tilde{x} = (x, w)$ and output $\tilde{y} = u$. Roughly speaking, the definition of (Di Benedetto et al. 1989) says that (15) is a regular dynamic feedback if it is (rigth- and left-) invertible, in the sense that the output rank (see Lemma 5) coincides with the output dimension .

The corresponding output filtrations of lemma 5 for this system are given by {eOutputInput}

$$\begin{aligned}
\widetilde{\mathcal{Y}}_{-1} &= \operatorname{span}\{dt, d\tilde{x}\} = \operatorname{span}\{dt, dx, dw\} \\
\widetilde{\mathcal{Y}}_k &= \operatorname{span}\left\{dt, d\tilde{x}, d\tilde{y}^{(0)}, \dots, d\tilde{y}^{(k)}\right\} \\
&= \operatorname{span}\left\{dt, dx, dw, du^{(0)}, \dots, du^{(k)}\right\}, k \in I\!N \qquad (61a) \\
\widetilde{Y}_{-1} &= \operatorname{span}\{dt\} \\
\widetilde{Y}_k &= \operatorname{span}\left\{dt, d\tilde{y}^{(0)}, \dots, d\tilde{y}^{(k)}\right\} \\
&= \operatorname{span}\left\{dt, du^{(0)}, \dots, du^{(k)}\right\}, k \in I\!N \qquad (61b)
\end{aligned}$$

We shall consider another output $\bar{y} = (x, u)$ of system E. Since $\operatorname{span}\{d\dot{x}\} \subset \operatorname{span}\{dt, dx, du\}$, for the output $\bar{y}$, the corresponding output filtrations are given by {eFFiltrationbarAA}

$$\begin{aligned}
\overline{\mathcal{Y}}_{-1} &= \operatorname{span}\{dt, d\tilde{x}\} = \operatorname{span}\{dt, dx, dw\} \\
\overline{\mathcal{Y}}_k &= \operatorname{span}\left\{dt, d\tilde{x}, d\bar{y}^{(0)}, \dots, d\bar{y}^{(k)}\right\} \\
&= \widetilde{\mathcal{Y}}_k \qquad (62a) \text{ \{eFFiltrationbara\}} \\
\overline{Y}_{-1} &= \operatorname{span}\{dt\} \\
\overline{Y}_k &= \operatorname{span}\left\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(k)}\right\} \\
&= \operatorname{span}\left\{dt, dx, du^{(0)}, \dots, du^{(k)}\right\} \qquad (62b) \text{ \{eFFiltrationbar\}}
\end{aligned}$$

Definition 20 *(Regular dynamic feedback I) Given a system E defined by* (15), *with input $\tilde{u} = v$, state $\tilde{x} = (x, w)$ and outputs $\tilde{y} = u$ and $\bar{y} = (x, u)$. Assume that $\dim x = n$, $\dim u = \dim v = m$, and $\dim w = s$. Let $\alpha = n + s$. Then the feedback is said to be regular around $\nu \in E$ if ν is a regular point of the filtrations* (61) *and* (62) *(for both outputs $\tilde{y}$ and $\bar{y}$) for $k = 0, \dots, \alpha$ and the system* (15) *is invertible, that is, if the rank of u, considered as an output of the system E coincides with $\operatorname{card} u$.* {dEquations}

For analytic systems, it is easy to show that this definition coincides with the definition of (Di Benedetto et al. 1989), at least in an open and dense subset of E.

The next two propositions are instrumental for showing the link between the definitions 13 and 20.

{pAuxiliar}

Proposition 11 *Consider a dynamic state feedback* (15)*. Let ν be a regular point of the feedback according definition 20. Then there exists $\beta \in I\!N$ big enough such that*

{III1}

1. *span* $\{dv\} \subset$ *span* $\left\{dt, dx, dw, du, \dots, du^{(\beta)}\right\}$ *locally around* ν. {III2}
2. *The set* $\{dt, dx, du, \dots, du^{(\beta)}\}$ *is locally linearly independent around* ν. {III3}
3. *The codistribution span* $\left\{dt, dw, dx, du, \dots, du^{(\beta)}\right\}$ *is nonsingular around* ν. {III4}
4. *The codistribution span* $\left\{dt, dw, dx, du, \dots, du^{(\beta-1)}\right\}$ *is nonsingular around* ν. {III5}
5. *span* $\{dw, dx\} \cap$ *span* $\left\{dt, du, \dots, du^{(\beta-1)}\right\}$ $=$ *span* $\{dw, dx\} \cap$ *span* $\{dt, du, \dots, du^{(\beta)}\}$ *around* ν.

Proof. Consider the integer k^* as defined in Part 5 of of Lemma 5. Denote respectively by $k^*_{\tilde{y}}$ and $k^*_{\bar{y}}$ the values of this integer respectively for the output $\tilde{y}$ and $\bar{y}$. Let $\alpha = \operatorname{card} \tilde{x} = \operatorname{card} x + \operatorname{card} w$. Let $\tilde{y} = u$ and $\bar{y} = (x, u)$. Let $\beta = \max\{k^*_{\tilde{y}}, k^*_{\bar{y}}\}$,

Since it is assumed that the rank of the output $\tilde{y}$ is $\operatorname{card} v$, the application of lemma 5 part 9 gives 1. By lemma 5 part 5, and from the fact that $\widetilde{\mathcal{Y}}_k = \bar{\mathcal{Y}}_k, k \in I\!N$, the output rank of both outputs $\tilde{y}$ and $\bar{y}$ coincide. By part 5 of lemma 5, and from the fact that $\{dt, dx\}$ is independent, it follows easily that 2 holds (note that $\bar{Y}_\beta = \operatorname{span}\left\{dt, dx, du, \dots, du^{(\beta)}\right\}$). Since ν is a regular point of the output filtrations (61) for $k = 0, \dots, \alpha$, then that 3 and 4 holds. Finally, 5 is a consequence of part 7 of lemma 5 applied to the output $\tilde{y}$. $\square$

The last proposition suggests the following alternate definition of regular dynamic feedback.

{dRegularAlternate}

Definition 21 *(regular dynamic feedback II) Let E be the system defined by the equations* (15) *and S be the system defined by the equations* (15a)*. Then* (15) *is called a local* regular dynamic feedback *if the system E defined by* (15) *is such that, for every $\nu \in E$ one has:*

{iIII1}

1. *span* $\{dv\} \subset$ *span* $\left\{dt, dx, dw, du, \dots, du^{(\beta)}\right\}$ *locally around* ν. {iIII2}
2. *The set* $\{dt, dx, du, \dots, du^{(\beta)}\}$ *is locally linearly independent around* ν. {iIII3}
3. *The codistribution span* $\left\{dt, dw, dx, du, \dots, du^{(\beta)}\right\}$ *is nonsingular around* ν. {iIII4}
4. *The codistribution span* $\left\{dt, dw, dx, du, \dots, du^{(\beta-1)}\right\}$ *is nonsingular around* ν. {iIII5}
5. *span* $\{dw, dx\} \cap$ *span* $\left\{dt, du, \dots, du^{(\beta-1)}\right\}$ $=$ *span* $\{dw, dx\} \cap$ *span* $\{dt, du, \dots, du^{(\beta)}\}$ *around* ν.

The following result shows that all the previous definitions of regular dynamic feedback are essentially equivalent.

{pp8}

Proposition 12 *Consider the system E defined by* (15) *and the system S defined by* (15a)*. Then*

1. *Let $\tilde{x} = (x, w)$ and $\tilde{u} = v$. If* (15) *defines a regular feedback in the sense of definition 21 then $(E, (\tilde{x}, \tilde{u}), (x, u))$ is a dynamic state feedback in the sense of definition 13.*

2. *If $\nu \in E$ is a regular point of the dynamic feedback* (15) *in the sense of definition 20, Let $\tilde{x} = (x, w)$ and $\tilde{u} = v$. Then $(E, (\tilde{x}, \tilde{u}), (x, u))$ is a dynamic state feedback in the sense of definition 13.*

3. *Let $D \subset E$ be the open and dense subset of regular points of the codistributions* (61) *and* (62) *for $k = 0, \ldots, \alpha$, where $\alpha = \mathrm{card}\, x + \mathrm{card}\, w$. Let $\tilde{x} = (x, w)$ and $\tilde{u} = v$. If $(E, (\tilde{x}, \tilde{u}), (x, u))$ is a dynamic state feedback in the sense of definition 13 with classical state equations around some $\nu \in D$, then ν is a regular point of the dynamic feedback according definition 20.*

4. *Let $\tilde{x} = (x, w)$ and $\tilde{u} = v$. Around points of the set D defined in 3, if $(E, (\tilde{x}, \tilde{v}), (x, u))$ is a dynamic state feedback in the sense of definition 13, then it is regular according definition 21.*

Proof. Consider system (15). To show 1, note that, by definition, (x, u) is a state representation of S and $(\tilde{x}, \tilde{u}) = ((x, w), v)$ is a state representation of E. Denote by $u^{(k)}$ the kth-derivative of u considered as an output of E. It suffices to prove that the map $\pi : E \to S$ defined (around some $\nu \in E$) by $\pi(t, x, w, (v^{(k)} : k \in I\!N)) = (t, x, (u^{(k)} : k \in I\!N))$ is a Lie-Bäcklund submersion. The fact that π is Lie-Bäcklund is an easy consequence of (15) and the remarks of Appendix F. Now, take $\tilde{w}$, with $\tilde{w} \subset w$, in a way that the set $\{dt, d\tilde{w}, dx, du, \ldots, du^{(\beta-1)}\}$ is a local basis of span $\left\{dt, dw, dx, du, \ldots, du^{(\beta-1)}\right\}$. Let $\tilde{z} = (\tilde{w}, x)$ and $\tilde{v} = u$.

By definition 21, the set $\{dt, dx, d\tilde{v}, \ldots, d\tilde{v}^{(\beta)}\}$ is locally linearly independent around ν. By construction, one may write

$$\mathrm{span}\,\{d\tilde{x}\} \subset \mathrm{span}\,\{dt, d\tilde{z}, d\tilde{v}, \ldots, d\tilde{v}^{(\beta-1)}\}. \tag{63}$$ {eC}

By definition 21, it follows that span $\{d\tilde{u}\}$ = span $\{dv\}$ $\subset$ span $\{dt, dx, dw, du^{(0)}, \ldots, du^{(\beta)}\}$ = span $\left\{dt, dx, dw, du^{(0)}, \ldots, du^{(\beta-1)}\right\}$ + span $\left\{du^{(\beta)}\right\}$ = span $\left\{dt, dx, d\tilde{w}, du^{(0)}, \ldots, du^{(\beta-1)}\right\}$ + span $\left\{du^{(\beta)}\right\}$ = span $\{dt, dx, d\tilde{w}, du^{(0)}, \ldots, du^{(\beta)}\}$.

From (15a), one has span $\{d\dot{x}\}$ $\subset$ span $\{dt, dx, du\}$. Since $((x, w), v)$ is a proper state representation of E, then span $\{d\dot{w}\}$ $\subset$ span $\{dt, dx, dw, dv\}$. Since span $\{d\tilde{w}\}$ $\subset$ span $\left\{dt, dx, dw, du, \ldots, du^{(\beta-1)}\right\}$, by derivation one has, span $\left\{d\tilde{w}^{(1)}\right\}$ $\subset$ span $\left\{dt, dx, d\tilde{w}, dv, du, \ldots, du^{(\beta)}\right\}$. By definition 21, span $\{dv\}$ $\subset$ span $\left\{dt, dw, dx, du, \ldots, du^{(\beta)}\right\}$. Hence,

$$\mathrm{span}\,\left\{d\dot{\tilde{z}}, d\tilde{u}\right\} \subset \mathrm{span}\,\{dt, d\tilde{z}, d\tilde{v}, \ldots, d\tilde{v}^{(\beta)}\}. \tag{64}$$ {eD}

By (15c) one may write

$$\operatorname{span}\{d\tilde{z}, d\tilde{v}\} \subset \operatorname{span}\{dt, dx, dw, dv\} = \operatorname{span}\{dt, d\tilde{x}, d\tilde{u}\}\,. \tag{65}$$ {eA}

We show now that $\{dt, d\tilde{z}, d\tilde{v}^{(0)}, \ldots, d\tilde{v}^{(\beta)}\}$ is (locally) linearly independent. In fact, the set $\{d\tilde{w}, dt, dx, du, \ldots, du^{(\beta-1)}\}$ is (locally) linearly independent by construction (it is a basis of $\operatorname{span}\{dw\} + \overline{Y}_{\beta-1}$) (see (62a) for the definition of $\mathcal{Y}_\beta$ (62b) for the definition of $\bar{Y}_\beta$). By part 2 of Def. 21 the sets $\{dt, dx, du^{(0)}, \ldots, du^{(\alpha)}\}$ for $\alpha = \beta$ and $\alpha = \beta - 1$ are basis respectively of $\overline{Y}_\beta$ and $\overline{Y}_{\beta-1}$. In particular, the set $\{dt, du, \ldots, du^{(\alpha)}\}$ for $\alpha = \beta$ and $\alpha = \beta - 1$ are basis respectively of $\tilde{Y}_\beta$ and $\tilde{Y}_{\beta-1}$. Since $\operatorname{span}\{dx, dw\} + \tilde{Y}_k = \tilde{\mathcal{Y}}_k$ for $k \in I\!N$, then one may write

$$\begin{aligned}
\dim \tilde{\mathcal{Y}}_{\beta-1} &= \dim(\operatorname{span}\{dx, dw\}) + \dim \tilde{Y}_{\beta-1} - \dim(\operatorname{span}\{dx, dw\} \cap \tilde{Y}_{\beta-1}) \\
\dim \tilde{\mathcal{Y}}_{\beta} &= \dim(\operatorname{span}\{dx, dw\}) + \dim \tilde{Y}_{\beta} - \dim(\operatorname{span}\{dx, dw\} \cap \tilde{Y}_{\beta})
\end{aligned}$$

By part 2 of definition 21, it follows that $\dim \tilde{Y}_\beta - \dim \tilde{Y}_{\beta-1} = m = \operatorname{card} u$. Subtracting the last two equations and using part 5 of definition 21, one gets $\dim \tilde{Y}_\beta - \dim \tilde{Y}_{\beta-1} = \operatorname{card} u = m$. Since $\tilde{\mathcal{Y}}_\beta = \tilde{\mathcal{Y}}_{\beta-1} + \operatorname{span}\left\{du^{(\beta)}\right\}$ and $\tilde{\mathcal{Y}}_{\beta-1} = \operatorname{span}\{dt, d\tilde{w}, dx, du, \ldots, du^{(\beta-1)}\}$, it is clear that $\{d\tilde{w}, dt, dx, du, \ldots, du^{(\beta)}\}$ generates $\tilde{\mathcal{Y}}_\beta$. By dimensional arguments, it follows that this set must be linearly independent showing that

$$\{dt, d\tilde{z}, d\tilde{v}^{(0)}, \ldots, d\tilde{v}^{(\beta)}\} \text{ is locally linearly independent.} \tag{66}$$ {eB}

Summarizing, from (65), (66), (63) and (64), one has shown that

(A) $\operatorname{span}\{d\tilde{z}, d\tilde{v}\} \subset \operatorname{span}\{dt, d\tilde{x}, d\tilde{u}\}$.

(B) The set $\{dt, d\tilde{z}, d\tilde{v}^{(0)}, \ldots, d\tilde{v}^{(\beta)}\}$ is locally linearly independent pointwise.

(C) $\operatorname{span}\{d\tilde{x}\} \subset \operatorname{span}\left\{dt, d\tilde{z}, d\tilde{v}^{(0)}, \ldots, d\tilde{v}^{(\beta-1)}\right\}$.

(D) $\operatorname{span}\left\{d\dot{\tilde{z}}, d\tilde{u}\right\} \subset \operatorname{span}\left\{dt, d\tilde{z}, d\tilde{v}, \ldots, d\tilde{v}^{(\beta)}\right\}$.

By lemma 2 applied to the state representation $(\tilde{x}, \tilde{u})$ and the set of functions $(\tilde{z}, \tilde{v})$, it follows that $(\tilde{z}, \tilde{v}) = ((\tilde{w}, x), u)$ is a local state representation of E. Let $\gamma = \{t, x, (u^{(k)} : k \in I\!N)\}$. Then in the local coordinates $\{\tilde{w}, \gamma\}$ for E and γ for S, the map π reads $\pi(\tilde{w}, \gamma) = \gamma$. So π is a local submersion. This shows 1.

To prove 2, it suffices to see that it is a straightforward consequence of 1 and proposition 11.

To show 3, note from the proof of proposition 4 that $\{t, x, (u^{(k)} : k \in I\!N)\}$ is part of a local coordinate system of E. Then $\dim \bar{Y}_k - \dim \bar{Y}_{k-1} = m = \operatorname{card} u$ for all $k \in I\!N$. By part 5 of Lemma 5 applied to system E, it follows that, for $k > \alpha$, with α big enough, one has $\dim \tilde{\mathcal{Y}}_k - \dim \tilde{\mathcal{Y}}_{k-1} = m = \operatorname{card} u$. Hence the system is invertible.

Finally, note that 4 is a consequence of 3 and of Proposition 11. □

In figure 1 one finds a summary of the results of this appendix.

$$\begin{tikzcd} \boxed{\text{Def. 13}} \arrow[r, shift left, "4"] \arrow[d, shift left, "3"] & \boxed{\text{Def. 21}} \arrow[l, shift left, "1"] \\ \boxed{\text{Def. 20}} \arrow[u, shift left, "2"] \arrow[ur, "\text{Prop. 11}"'] & \end{tikzcd}$$

Figure 1: The relationship between different notions of dynamic feedback according. The numbers near the arrows refers to the items of Proposition 12. {fFeedback}

H A comparison of regularity assumptions

{aCompare}

One may consider that the main result of this paper is Lemma 2 and its consequences and corollaries, like theorem 3 and theorem 5.

In this appendix, the regularity assumptions of theorems 3 and 5 are compared with the ones of Lemma 5, showing that that this lemma needs stronger regularity assumptions than those theorems. Note that Lemma 5 holds for $\xi \in V_n$, where V_n is the open and dense set of regular points of $\mathcal{Y}_k, Y_k$ for $k = 0, \ldots, n$, where $n = \dim x$ (see (7b)-(7a)). We shall begin with the right-invertible case.

H.1 The right-invertible case

Proposition 13 *Consider a system S with a proper state representation (x, u) and a proper output y, where $\dim x = n$, $\dim u = m$ and $\dim y = l$. Assume that the regularity assumptions of Lema 5 holds around some $\nu \in S$, that is, $\nu \in V_n$. Assume also that the system is right invertible, i. e., there exists $k^* \in I\!N$ big enough such that$\dim \mathcal{Y}_{k^*} - \dim \mathcal{Y}_{k^*-1} = l$. Then the regularity assumptions of theorem 3 holds for $\alpha = (k^*, k^*, \ldots, k^*) \in I\!N^l$.*

Proof. Note that

1. $\operatorname{span}\{dy\} \subset \operatorname{span}\{dt, dx, du\}$ (y is proper).
2. By part 7 of Lemma 5, there exists $k^* \in I\!N$, where $0 \leq k^* \leq n$, such that, $\operatorname{span}\{dx\} \cap \operatorname{span}\left\{dt, dy, \ldots, dy^{(k^*-1)}\right\} = \operatorname{span}\{dx\} \cap \operatorname{span}\left\{dt, dy, \ldots, dy^{(k^*)}\right\}$.
3. $\operatorname{span}\{dt, dx, dy, \ldots, dy^{(k^*)}\}$ is locally nonsingular around ν (by the regularity assumption of $\mathcal{Y}_k$, $k = 0, 1, \ldots, n$).
4. $\operatorname{span}\{dt, dx, du, dy, \ldots, dy^{(k^*)}\}$ is locally nonsingular around ν (by part 2 of lemma 5)[54].

[54]By lema 5, (x_{k^*-1}, u_{k^*-1}) is a state representation around ν and then $\operatorname{span}\{dt, dx_{k^*-1}, du_{k^*-1}\}$ is nonsingular.

5. The set $\{dt, dy, \ldots, dy^{(k^*)}\}$ is locally nonsingular around ν (regularity assumption of Y_k, $k = 0, 1, \ldots, n$.)

Then the regularity assumptions of theorem 3 holds for $\alpha = (k^*, k^*, \ldots, k^*) \in \mathbb{N}^l$. □

Remember now the example of Respondek (12). For this example we have already pointed out that the regularity assumptions of Lemma 5 do not hold, but the ones of theorem 3 holds. So, one may say that the regularity assumptions of Lemma 5 are stronger than the ones of Theorem 3.

H.2 The non-invertible case

Adding an output $y_3 = \phi(y_1, y_2)$ to the example (12), one construct easily a system for which the assumptions of theorem 5 holds, but not the ones of lemma 5. The next proposition shows that the regularity assumptions of lemma 5 implies the ones of theorem 5, even in the case where the system is noninvertible.

{pNoninvertible}

Proposition 14 *Let S be a system with proper state representation (x, u) and proper output y, both defined around some $\nu \in S$. Suppose that the regularity assumptions of Theorem 5 holds around ν. Then there exists a partition $y = (\bar{y}, \widehat{y})$ such that the assumptions 1 to 8 of theorem 5 holds, locally around ν, i. e.,*

{ppyintxu}

1. *span* $\{d\bar{y}\} \subset$ *span* $\{dt, dx, du\}$.

{ppyxnaocresce}

2. *span* $\{dx\} \cap$ *span* $\left\{dt, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha-1)}\right\} =$ *span* $\{dx\} \cap$ *span* $\left\{dt, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha)}\right\}$.

{ppyxnaosing1}

3. *span* $\left\{dt, dx, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha-1)}\right\}$ *is locally nonsingular around* ν.

{ppyxunaosing}

4. *span* $\left\{dt, dx, du, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha)}\right\}$ *is locally nonsingular around* ν.

{pptyindependent}

5. *The set* $\{dt, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha)}\}$ *is pointwise independent in an open neighborhood of* ξ.

6. *span* $\left\{dy^{(0)}, \ldots, dy^{(\alpha-1)}\right\} \subset$ *span* $\left\{dt, dx, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(\alpha-1)}\right\}$

7. *span* $\left\{dt, dy^{(0)}, \ldots, dy^{(k)}\right\}$ *is nonsingular for* $k = \alpha$ *and* $k = \alpha - 1$.

8. *span* $\left\{dy^{(\alpha)}\right\} \subset$ *span* $\left\{dt, dy^{(0)}, dy^{(1)}, \ldots, dy^{(\alpha-1)}, d\bar{y}^{(\alpha)}\right\}$.

where α is the integer k^*, $\bar{y} = \bar{y}_{k^*}$, and $\widehat{y} = \widehat{y}_{k^*}$ that are defined in the statement of lemma 5.

Proof. In this proof we shall consider the notations of lemma 5. Along this proof one considers that $\alpha = k^*$ and $\bar{y} = \bar{y}_{k^*}$ (where k^* and $\bar{y}_{k^*}$ are defined in that Lemma). Now, as y_{k^*} is a subset of y, let $\widehat{y}$ be such that $\widehat{y} \cup \bar{y} = y$ with $\widehat{y} \cap \bar{y} = \emptyset$.

Note first that 1 says that $\bar{y}$ is proper and 7 is a consequence of the regularity assumptions ($\mathcal{Y}_k$ is nonsingular). Note also that 8 is a consequence of part 8 of lemma 5 for $r = 0$.

By part 3 of lemma 5 one may take $\bar{y}_k \subset \bar{y}_{k+1}$. By part 8 of that lemma, one may take $\bar{y}_k = \bar{y}_{k+1}$ for $k \geq k^*$. From this and from part 1 of that lemma, it follows easily that

$$\begin{aligned} \mathcal{Y}_k &= \operatorname{span}\left\{dt, dx, d\bar{y}_{k^*}^{(0)}, \ldots, d\bar{y}_{k^*}^{(k)}\right\} \\ &= \operatorname{span}\left\{dt, dx, d\bar{y}^{(0)}, \ldots, d\bar{y}^{(k)}\right\} \end{aligned} \tag{67}$$ {eYkyk}

for $k \in I\!\!N$. As $\mathcal{Y}_k = \operatorname{span}\left\{dt, dx, dy^{(0)}, \ldots, dy^{(k)}\right\}$, one concludes that 6 follows from (67). The prof of 3 and 4 may be obtained by the application of Lemma 5 parts 1 and 2.

To show 5, note first that, from the regularity assumptions, one may construct a set of one forms $\eta = \{\eta_1, \ldots, \eta_{\rho_k}\}$ that completes a local basis of Y_{k-1} to a basis of Y_k. It will be shown now that, if $\theta = \{\theta_1, \ldots, \theta_r\} \subset Y_k$ is a set of 1-forms such that $\theta|_\xi$ is dependent *modulo* $Y_{k-1}|_\xi$, then $\dot{\theta}|_\xi$ is dependent *modulo* $Y_k|_\xi$.

In fact, assume that $\theta|_\xi$ is dependent *modulo* $Y_{k-1}|_\xi$. This means that there exist $\alpha_i \in I\!\!R$, and a cotangent vector $\gamma_\xi \in Y_{k-1}|_\xi$ such that

$$\sum_{i=1}^{r} \alpha_i \theta_i|_x i = \gamma_\xi$$

By construction, as $\sum_{i=1}^{r} \alpha_i \theta_i \in Y_k$, it follows that there exists $\tilde{\gamma} \in Y_{k-1}$ and smooth functions $\{delta_1, \ldots, \delta_{\rho_k}\}$ such that

$$\sum_{i=1}^{r} \alpha_i \theta_i = \sum_{j=1}^{\rho_k} \delta_j \eta_j + \gamma$$

Note that $\delta_j|_\xi = 0, j = 1, \ldots, \rho_k$ and $\tilde{\gamma}|_x i \in \gamma_\xi$. Then, the Lie-derivation along $\frac{d}{dt}$ gives

$$\sum_{i=1}^{r} \alpha_i \dot{\theta}_i = \sum_{j=1}^{\rho_k} (\delta_j \dot{\eta}_j + \dot{\delta}_j \eta_j) + \dot{\gamma}$$

with $\dot{\gamma} \in Y_k$, $\delta_j|_\xi = 0$ and $[\sum_{j=1}^{\rho_k} \dot{\delta}_j \eta_j] \in Y_k$. In particular, the set

$$\{\dot{\theta}_1|_\xi \bmod Y_k|_\xi, \ldots, \dot{\theta}_r|_\xi \bmod Y_k|_\xi\}$$

is linearly dependent.

Now, from Part 1 of Lemma 5, the set $\{d\bar{y}_{k^*}^{(k)}\}$ is independent $\mod \mathcal{Y}_{k-1}$ for $k \geq k^*$. In particular, $\{d\bar{y}_{k^*}^{(k)}\}$ is independent $\mod Y_{k-1}$ for $k = 0, 1, \ldots, k^* - 1$, and this shows 5.

To show 2, note from (67) that one may write

$$\begin{aligned}\dim(\mathcal{Y}_k) &= \dim(\text{span}\,\{dx\}) + \dim(\text{span}\left\{dt, d\bar{y}_{k^*}^{(0)}, \ldots, d\bar{y}_{k^*}^{(k)}\right\}) \\ &\quad - \dim(\text{span}\,\{dx\}) \cap (\text{span}\left\{dt, d\bar{y}_{k^*}^{(0)}, \ldots, d\bar{y}_{k^*}^{(k)}\right\})\end{aligned}$$

Subtracting the last equation obtained for $k = k^*$ from the same equation for $k = k^* - 1$, using 5, and the fact that $\dim \mathcal{Y}_{k^*} - \dim \mathcal{Y}_{k^*-1} = \text{card}\, \bar{y}_{k^*} = \sigma_{k^*}$ (see parts 1 and 5 of lemma 5), one obtains

$$\begin{aligned}&\dim\left(\text{span}\,\{dx\} \cap \text{span}\left\{dt, d\bar{y}_{k^*}^{(0)}, \ldots, d\bar{y}_{k^*}^{(k^*)}\right\}\right) = \\ &\dim\left(\text{span}\,\{dx\} \cap \text{span}\left\{dt, d\bar{y}_{k^*}^{(0)}, \ldots, d\bar{y}_{k^*-1}^{(k^*-1)}\right\}\right)\end{aligned}$$

showing 2.

□

I Equivalence of implicit systems and splitting diagrams

{aJEquivalence}

The definition of system equivalence is quite natural in our approach.

{D20}

Definition 22 *Two systems* Γ_1 *and* Γ_2 *are* equivalent*, or* Lie-Bäcklund isomorphic*, if there exists a Lie-Bäcklund isomorphism* $\phi : \Gamma_1 \to \Gamma_2$ *(that is, a Lie-Bäcklund diffeomorphism). Let* $(x_i, u_i), i = 1, 2$ *be local state representations respectively of of* $\Gamma_i, i = 1, 2$ *defined respectively on* $U_i, i = 1, 2$*. The equivalence map* ϕ preserves the state representations *if* $\phi(U_1) = U_2$*,* $x_1 = x_2 \circ \phi$ *and* $u_1 = u_2 \circ \phi$*. The map* ϕ *preserves the inputs if* $\phi(U_1) = U_2$ *and* $u_1 = u_2 \circ \phi$*.If* y_i *is an output for* Γ_i*, i=1,2, one says that* ϕ preserves the outputs*, if* y_1 *and* y_2 *are such that* $y_1 = y_2 \circ \phi$*.*

Now let $\Delta = (S, y)$ be an implicit system. Let $y = (y_1, \ldots, y_s)$. Let $\mathcal{T}^s(y)$ be the trivial time-invariant diffiety with flat output[55] y . Let $\bar{y} : S \to \mathcal{T}^s(y)$ be the map defined by

$$\left\{\begin{array}{l}\bar{y} : S \to \mathcal{T}^s(y) \\ \xi \mapsto (y^{(0)}, y^{(1)}, y^{(2)}, \ldots)|_\xi.\end{array}\right. \tag{68}$$ {eYBar}

By simplicity, one will let 0 stand for the point $(0, 0, 0, \ldots) \in \mathcal{T}^s(y)$. It easy to show that that $\bar{y}$ is Lie-Bäcklund. It is also clear that $\tilde{\Delta} = \bar{y}^{-1}(0)$, where $\tilde{\Delta}$ is defined by (20).

The following definition is useful in the study of equivalence of implicit systems.

[55](see the definition of $\mathcal{T}^s(y)$ in the beginning of section 2.

Definition 23 *Let S and N be systems. Let $\bar{y} : S \to N$ be a Lie-Bäcklund map, denoted by $S \rightharpoonup N$. Let some $\eta \in N$ be fixed. We say that a given system Γ furnishes a* splitting diagram *for $S \rightharpoonup N$ (with* center *$\eta \in N$) if there exists a Lie-Bäcklund embedding*[56] *$\iota : \Gamma \to S$ such that $\iota(\Gamma) = \bar{y}^{-1}(\eta)$. A splitting diagram will be denoted by*

$$\Gamma \hookrightarrow S \rightharpoonup N$$

Let $\Delta = (S, y)$ is an implicit system and Γ a given control system. It is clear that Γ *canonically equivalent* to Δ according Definition 19, if and only if one may construct a splitting diagram $\Gamma \overset{\iota}{\hookrightarrow} S \overset{\bar{y}}{\rightharpoonup} N$, where the map $\bar{y}$ is defined by (68). In particular the next Theorem gives a proof of Theorem 9.

Theorem 13 *Assume that Γ_1 and Γ_2 furnish splitting diagrams for $S \rightharpoonup N$. Then Γ_1 and Γ_2 are equivalent*[57]. {tSplitting}

Proof. Remember that $\iota : \Gamma \to S$ is an embedding implies that, for every point $\xi \in \tilde{\Delta} = \iota(\Gamma)$, there exists an open neighborhood U of ξ, a local chart (ϕ, U) of S, and a local chart (V, ψ) of Γ, such that, $V = \iota^{-1}(U)$ and in these coordinates one has $\iota(x) = (x, 0)$, and furthermore $(x, z) \in \tilde{\Delta} \cap U$ if and only if $z = 0$.

The immersions corresponding to the splitting diagrams will be denoted by $\iota_1 : \Gamma_1 \to S$ and $\iota_2 : \Gamma_2 \to S$. Let $\tilde{\Delta} = \iota_1(\Gamma_1) = \iota_2(\Gamma_2)$. As the map ι_1 is injective, the map $\iota_1 : \Gamma_1 \to \tilde{\Delta}$ is a bijection. We shall show that the map $\iota_1^{-1} \circ \iota_2 : \Gamma_2 \to \Gamma_1$ is a Lie-Backlund isomorphism. For, let $\xi \in \tilde{\Delta}$. By definition there exists local charts (U_1, ϕ_1) and (U_2, ϕ_2) of S, (V_1, ψ_1) of Γ_1 and (V_2, ψ_2) of Γ_2 given by $\phi_1 = (x_1, z_1)$, $\phi_2 = (x_2, z_2)$, $\psi_1 = x_1$, $\psi_2 = x_2$, for which, in these coordinates one has $\iota_1(x_1) = (x_1, 0)$ and $\iota_2(x_2) = (x_2, 0)$. Without loss of generality, assume that $U_1 = U_2 = U$. By definition, the coordinate change map $H : \phi_2(U) \to \phi_1(U)$ given by $(x_2, z_2) \mapsto (x_1, z_1)$, where $x_1 = H_x(x_2, z_2)$ and $z_1 = H_z(x_2, z_2)$ is such that $(x_2, 0) \mapsto (x_1, 0)$, where $x_1 = H_x(x_2, 0)$. Note that $\iota_1^{-1}(x_1, 0) = x_1$. In particular, $\iota_1^{-1} \circ \iota_2(x_2) = \iota_1^{-1}(H_x(x_2), 0) = H_x(x_2)$. In particular the map $\iota_1^{-1} \circ \iota_2 : \Gamma_2 \to \Gamma_1$ is smooth. Analogously, the map $\iota_2^{-1} \circ \iota_1 : \Gamma_1 \to \Gamma_2$ is also smooth. It is clear that the two maps are the inverse of each other. This shows that the map $\theta = \iota_1^{-1} \circ \iota_2 : \Gamma_2 \to \Gamma_1$ is a diffeomorphism. Now, to show that θ is Lie-Bäcklund, note that, by construction, $\iota_1 \circ \theta = \iota_2$. Let ∂_1 and ∂_2 be respectively the Cartan fields of Γ_1 and Γ_2, and let ∂_S be the cartan field of S. As $\iota_i, i = 1, 2$ are Lie-Bäcklund maps, one has $\partial_S \circ \iota_i = (\iota_i)_* \partial_i, i = 1, 2$. As $(\iota_1)_* \theta_* = (\iota_2)_*$, one has $\partial_S \circ \iota_2 = (\iota_2)_* \partial_2 = (\iota_1)_* \theta_* \partial_2$. Let $\xi = \iota(\gamma_i), i = 1, 2$. Then $\partial_S|_\xi = (\iota_1)_*(\gamma_1) \tau_{\gamma_1}$, where $\tau_{\gamma_1} = \theta_*(\gamma_2) \partial_2|_{\gamma_2}$. Note that $(\iota_1)_*$ is a (pointwise) injective linear map. The only possibility is $\tau_{\gamma_1} = \partial_1(\gamma_1)$ showing that θ is Lie-Bäcklund. To show that θ is time-respecting, it suffices to see that ι_1 and ι_2 are injective time-respecting maps and $\iota_1 = \theta \circ \iota_2$. $\square$

The following result is useful for establishing system equivalence in some cases.

[56] Remember that the time is respected.

[57] One could write a proof of Theorem 13 using Lemma 6 of Appendix N. Here one will give a constructive proof

{pEquivalent1}

Proposition 15 *Let $\bar{y}_1 : S_1 \to N_1$ and $\bar{y}_2 : S_1 \to N_2$ be two Lie-Bäcklund mappings denoted respectively by $S_1 \rightharpoonup N_1$ and $S_2 \rightharpoonup N_2$. Let $\theta : S_2 \to S_1$ be an injective embedding such that $\theta(\bar{y}_2^{-1}(\eta_2)) = (\bar{y}_1^{-1}(\eta_1))$. Let $\eta_1 \in N_1$ and $\eta_2 \in N_2$. Let $\iota_2 : \Gamma_2 \hookrightarrow S_2$ be an injective embedding such that $\Gamma_2 \hookrightarrow S_2 \rightharpoonup N_2$ is a splitting diagram for $S_2 \rightharpoonup N_2$. Then if one defines $\iota_1 : \Gamma_2 \to S_1$ by $\iota_1 = \theta \circ \iota_2$, then $\Gamma_2 \hookrightarrow S_1 \rightharpoonup N_1$ is a splitting diagram for $S_1 \rightharpoonup N_1$. In particular, for every splitting diagram $\Gamma_1 \hookrightarrow S_1 \rightharpoonup N_1$, then Γ_1 is equivalent to Γ_2.*

$$\begin{array}{ccccc} \Gamma_2 & \overset{\iota_2}{\hookrightarrow} & S_2 & \overset{\iota_2}{\rightharpoonup} & N_2 \\ & & \downarrow \theta & & \\ \Gamma_1 & \overset{\iota_1}{\hookrightarrow} & S_1 & \overset{\iota_1}{\rightharpoonup} & N_1 \end{array} \tag{69}$$ {eDiagram}

Proof. As the composition of two injective embeddings is an injective embedding (see the Part I of this survey), it is easy to see that $\Gamma_2 \hookrightarrow S_1 \rightharpoonup N_1$ is a splitting diagram for $S_1 \rightharpoonup N_1$. Then the result follows from theorem 13. □

A first application of theorem 13 was already considered in the proof of Theorem 9, that shows that the notion of equivalence of implicit systems that is studied in Definition 19 is compatible with the Definition 22. One will now present some other applications of Theorem 13 and Proposition 15.

{t21}

Theorem 14 *Let $w = (w_1, \ldots, w_r)$ and consider the trivial diffiety $T^r(w)$ (see section 2). Let $\beta = (\beta_1, \ldots, \beta_r)$ be a multiindex. Let*

$$x = w^{\langle\langle\beta\rangle\rangle} = (w_1^{(0)}, \ldots, w_1^{(\beta_1)}, \ldots, w_r^{(0)}, \ldots, w_r^{(\beta_r)}).$$

Consider the implicit system Σ given by two distinct blocks of equations $\phi(x) = 0$ and $\psi(x) = 0$, given by

$$\begin{array}{rcl} \phi_1(w_1^{(0)}, \ldots, w_1^{(\beta_1)}, \ldots, w_r^{(0)}, \ldots, w_r^{(\beta_r)}) = \phi_1(x) & = & 0 \\ \vdots & \vdots & \vdots \\ \phi_p(w_1^{(0)}, \ldots, w_1^{(\beta_1)}, \ldots, w_r^{(0)}, \ldots, w_r^{(\beta_r)}) = \phi_p(x) & = & 0 \\ \psi_1(w_1^{(0)}, \ldots, w_1^{(\beta_1)}, \ldots, w_r^{(0)}, \ldots, w_r^{(\beta_r)}) = \psi_1(x) & = & 0 \\ \vdots & \vdots & \vdots \\ \psi_p(w_1^{(0)}, \ldots, w_1^{(\beta_1)}, \ldots, w_r^{(0)}, \ldots, w_r^{(\beta_r)}) = \psi_q(x) & = & 0 \end{array} \tag{70}$$ {eEstrelaImplicicito}

Define the map $H : T^r(w) \to I\!R^{p+q}$ be defined by $H(x) = (\phi(x), \psi(x))$. Using the notation of definition 15, system Σ may be represented by the pair $\Sigma = (T^r(w), H)$. Analogously the system Δ defined by the first bloc of equations, namely $\phi(x) = 0$, may be denoted by $\Delta = (T^r(w), \phi)$. Assume that Δ is canonically equivalent to a control system S according to Definition 19. Denote the map $\bar{\phi} : \mathcal{T}^r(w) \to \mathcal{T}^p(y)$ defined in the same way that one has defined (68). Let $N = \mathcal{T}^p(y)$. Then there exists a splitting diagram *(see Definition 13) for the map $\bar{\phi}$, denoted by $S \overset{\bar{\phi}}{\rightharpoonup} N$ (with center $0 = (0, 0, 0, \ldots) \in N$) given by*

$$S \overset{\iota}{\hookrightarrow} T^r(w) \overset{\bar{\phi}}{\rightharpoonup} N \tag{71}$$ {eDiagram1}

Assume that the implicit system $(S, \tilde{\psi})$, where $\tilde{\psi} = \psi \circ \iota$, is canonically equivalent to Γ. Then the implicit system $\Sigma = (T^r(w), H)$ is canonically equivalent to Γ.

Proof. Note first that, as ι is Lie-Bäcklund, then using the notation that is used in equation (68), one may write $\overline{\psi \circ \iota} = \bar{\psi} \circ \iota$. In fact, to prove this it suffices to show by induction that $\psi^{(k)} \circ \iota = (\psi \circ \iota)^{(k)}$.

As $(S, \tilde{\psi})$ is canonically equivalent to Γ, by definition, there exists a splitting diagram

$$\Gamma \overset{\iota_2}{\hookrightarrow} S \overset{\bar{\phi}\circ\iota}{\rightharpoonup} N_2 \tag{72}$$ {eDiagram2}

where $N_2 = \mathcal{T}^q(z)$. Define the Lie Bäcklund map $\bar{H} : T^r(w) \to N_1$, where $N_1 = \mathcal{T}^p(w) \times \mathcal{T}^q(w)$ by $\bar{H} = (\bar{\phi}, \bar{\psi})$. It is clear that $\bar{H}^{-1}(0,0) = \bar{\phi}^{-1}(0) \cap \bar{\psi}^{-1}(0)$. Since (72) is a splitting diagram, then $\iota_2(\Gamma) = (\bar{\phi} \circ \iota)^{-1}(0) = \iota^{-1}(\bar{\psi}^{-1}(0))$. Then

$$\iota(\iota_2(\Gamma)) = \iota(\iota^{-1}(\bar{\psi}^{-1}(0)) = \bar{\psi}^{-1}(0) \cap \iota(S)$$

Now, as (71) is also a splitting diagram, then $\iota(S) = \bar{\phi}^{-1}(0)$. Hence

$$\iota(\iota_2(\Gamma)) = \bar{\psi}^{-1}(0) \cap \bar{\phi}^{-1}(0) = \bar{H}^{-1}(0)$$

Define $\iota_1 : \Gamma \to T^r(w)$ by $\iota_1 = \iota \circ \iota_1$. Since the composition of two injective embeddings is an injective embedding, it follows easily that

$$\Gamma \overset{\iota_1}{\hookrightarrow} T^r(w) \overset{\bar{H}}{\rightharpoonup} N_1$$

is a splitting diagram for $T^r(w) \overset{\bar{H}}{\rightharpoonup} N_1$ with center $(0,0) \in N_1$. In particular, Γ is canonically equivalent to $\Sigma = (T^r(w), H)$. □

J General implicit systems and DAEs of first order

{aGeneralForm}

In the literature one considers that a *general* first order DAE is a system of the form

$$H(t, w, \dot{w}) = 0.$$

The transformation of a system of the form (18) into the general form, and the equivalence between this different representations is discussed in this appendix. The conversion of general DAEs of arbitrary order into the form (18) is also presented in the end of this appendix.

J.1 Converting (18) into the general form

Now consider a implicit system $\Delta = (S, y)$ given in the standard form (18) that is repeated here for convenience {eImplicitN}

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{73a}$$ {eImplicitNa}

$$y(t) = h(t, x(t), u(t)) = 0 \tag{73b}$$ {eImplicitNb}

In this work one has considered this implicit system as an explicit system S defined by (73a) along with constraints $y \equiv 0$ given by (73b). However this is not the only form to define a implicit system from the equations (73).

Another point of view is to consider $w = (x, u)$ and to regard (73) as an implicit system $\Sigma = (T^{n+m}(w), H)$, where $T^{n+m}(w)$ is the trivial diffiety with flat output w (see Section 2). In this case our implicit system will be defined by the equation

$$H(t, w, \dot{w}) = 0$$

To see this, let $\phi(t, w, \dot{w}) = \dot{x} - f(t, x, u)$ and let $\psi(t, w, \dot{w}) = h(t, x, u)$. Define $H(t, w, \dot{w}) = (\phi(t, w, \dot{w}), \psi(t, w, \dot{w}))$. As an application of Theorem 14, one may show the equivalence between $\Delta = (S, y)$ and $\Sigma = (T^{n+m}(w), H)$.

Proposition 16 ?? *If Γ is cannonically equivalent to $\Delta = (S, y)$, then Γ is cannonically equivalent to Σ. In particular, if Σ is canonically equivalent to a control system Γ_1 then Γ_1 is Lie-Bäcklund isomorphic to Γ.*

Proof. Let S be the explicit system defined by (73a) with global coordinates $\{t, x, (u^{(k)} : k \in I\!N)\}$. Let $\{t, (x^{(k)}, u^{(k)} : k \in I\!N)\}$ be global coordinates for $\mathbb{T} = \mathcal{T}^r(w)$. Let $\frac{d}{dt}$ be the Cartan field of S. Recall that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{74}$$ {eCartanS}

By induction define the functions:

$$\begin{aligned} \xi^{(0)}(t, x, u) &= x \\ \xi^{(k)}(t, x, u^{(0}, \ldots, u^{(k)}) &= \frac{d}{dt}(\xi^{(k-1)}), k \geq 1 \end{aligned}$$

Let $\partial_{\mathbb{T}}$ be the Cartan field of $\mathbb{T}$. note that

$$\partial_{\mathbb{T}} = \frac{\partial}{\partial t} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor n \rceil}} x_j^{(k+1)} \frac{\partial}{\partial x_j^{(k)}} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{75}$$ {eCartanT}

Define

$$\begin{aligned} \phi^{(0)}(t, w, \dot{w}) &= x^{(1)} - f(t, x^{(0)}, u^{(0)}) \\ \phi^{(k)}(t, w, \ldots, w^{(k)}) &= \partial_{\mathbb{T}}(\phi^{(k-1)}), k \geq 1 \end{aligned}$$

Define the map $\iota : S \to \mathbb{T}$ such that

$$(t, x, (u^{(k)} : k \in I\!N)) \mapsto (t, (\xi^{(k)}, u^{(k)} : k \in I\!N))$$

Remember that a map $\iota : S \to \mathbb{T}$ is Lie-Bäcklund if and only if, for all (coordinate) function $\theta : \mathbb{T} \to I\!R$ one has $\frac{d}{dt}(\theta \circ \iota) = \partial_{\mathbb{T}} \theta \circ \iota$ (see the proof of Proposition

10 in the Appendix F). Note first, that, by definition, (abusing notation), $t\iota = t$, $u^{(k)} \circ \iota = u^{(k)} k \in I\!N$ and $x^{(k)} \circ \iota = \xi^{(k)}$. Now, by (74) and (75) it is clear that[58]:

$$
\begin{aligned}
\frac{d}{dt}(t \circ \iota) &= \frac{d}{dt}(t) = 1 = 1 \circ \iota = \partial_{\mathbb{T}}(t) \circ \iota \\
\frac{d}{dt}(u^{(k)} \circ \iota) &= \frac{d}{dt} u^{(k)} = u^{(k+1)} = u^{(k+1)} \circ \iota = \partial_{\mathbb{T}}(u^{(k)} \circ \iota) \\
\frac{d}{dt}(x^{(k)} \circ \iota) &= \frac{d}{dt}(\xi^{(k)}) = \xi^{(k+1)} = x^{(k+1)} \circ \iota = \partial_{\mathbb{T}}(x^{(k)}) \circ \iota
\end{aligned}
$$

This shows that ι is Lie-Bäcklund. Now note that $\phi : \mathbb{T} \to I\!R^n$ is such that

$$
\phi \circ \iota = \left(x^{(1)} - f(t, x^{(0)}, u^{(0)}) \right) \Big|_{x^{(k)} = \xi^{(k)}, k \in I\!N} = \xi^{(1)} - f(t, \xi^{(0)}, u^{(0)}) = 0.
$$

As ι is Lie-Bäcklund, $(\phi^{(k+1)}) \circ \iota = \partial_{\mathbb{T}}(\phi^{(k)}) \circ \iota = \frac{d}{dt}(\phi^{(k)} \circ \iota)$. By induction, one can show easily that

$$
\phi^{(k)} = \phi^{(k)} \circ \iota = 0, \text{ for all } k \in I\!N.
$$

for a convenient $f^{(k)}$. By construction, note that $\phi^{(k)}$ is of the form $x^{(k+1)} - f^{(k)}(t, x^{(0)}, \ldots, x^{(k)}, \ldots, u^{(0)}, \ldots, u^{(k)})$. Now denote $U = \{u^{(k)}, k \in I\!N\}$ and $\Phi = \{\phi^{(k)}, k \in I\!N\}$. It is easy to show that $\Xi = \{t, x^{(0)}, U, \Phi\}$ is a global coordinate system for $\mathbb{T}$. In the coordinates Ξ for $\mathbb{T}$ and $\{t, x, U\}$ for S, the map ι reads

$$
(t, x, U) \to (t, x, U, 0)
$$

Hence the map ι is a global immersion. As the image of ι is the a slice of a local chart, it is easy to show that ι is an embedding. Now one may define a map $\bar{\phi} \to N$, where $N = \mathcal{T}^n(z)$, where $z = (z_1, \ldots, z_n)$ in the same way that one has defined the map (68). By definition, it is clear that $\bar{\phi}^{-1}(0) = \iota(S)$. In particular one has constructed a splitting diagram

$$
S \overset{\iota}{\hookrightarrow} \mathbb{T} \overset{\bar{\phi}}{\rightharpoonup} N \tag{76}
$$
{eDiagram1a}

Now let $y = h(t, x, u^{(0)})$ be considered as a function $y : S \to I\!R$. If $\psi = h(t, x^{(0)}, u^{(0)})$ is considered as a function $\psi : \mathbb{T} \to I\!R$, then $y = \psi \circ \iota$. Now, one of the assumptions of the theorem is that $(S, y) = (S, \psi \circ \iota)$ is equivalent to Γ. Hence the desired result follows from the application of Theorem 14. $\square$

J.2 Converting DAEs of general form into the form (18)

Now assume that

$$
H(t, w(t), \dot{w}(t)) = 0 \tag{77}
$$
{eHHH}

is a DAE, where $w = (w_1, \ldots, w_r)$ and H is a smooth map. This system may be regarded as an implicit system $\Sigma = (T^r(w), H)$. Note that system (77) may

[58] Here, 1 denotes a function that is identically equal to one.

be put in the form (18) with the aid of the following trick. Considere the state space representation of $T^r(w)$ given by:

$$\dot{w} = v$$

Note that (77) may converted into the form {eConversionH}

$$\begin{aligned} \dot{w} &= v && \text{(78a)} \\ H(t,w,v) &= 0 && \text{(78b)} \end{aligned}$$

{eCA} {eCB}

This system is of the form (18) for $x = w$, $u = v$, $h = H(t,x,u)$ and $f(t,x,u) = u$. Here there is no necessity of the application of any theorem, one only need to note that (78a) is a state representation of $T^r(w)$ and (78b) represents the same restriction (77) that is adjoined to the trivial diffiety $T^r(w)$. Hence (78) is only an alternate form of denoting the same system $\Sigma = (T^r(w), H)$.

J.3 Converting DAEs of arbitrary order into the form (18)

Another common situation is the case of differential equations of arbitrary order. To consider this situation, let $S = T^r(w)$ be the trivial diffiety, where $w = (w_1, \ldots, w_r)$. Remember that $T^r(w)$ is the globally flat system, with global flat output w. Consider the implicit system (S, ϕ) defined by the system obtained by adjoining to $T^r(w)$ the following differential equations[59]

$$\phi_i(t, w_1, \ldots, w_1^{(\alpha_{1i})}, \ldots, w_r, \ldots, w_r^{(\alpha_{ri})}) = 0,\ i \in \{1, \ldots, p\} \qquad (79)$$

{eArOr}

Let $\beta_j = \max_i\{\alpha_{ji}\},\ i \in \{1, \ldots, p\}, j \in \{1, \ldots, r\}$.

Let $x = (w_1, \ldots, w_1^{(\beta_1 - 1)}, \ldots, w_r, \ldots, w_r^{(\beta_r - 1)})$, and $u = (u_1, \ldots, u_r)$, where $u_j = w_j^{(\beta_j)}$, and let $y = \phi(x,u)$. It is clear that (x,u) is a global state representation of $T^r(w)$. The corresponding state equations are given by

$$\begin{aligned} &\left\{ \begin{array}{rcl} \dot{w}_j^{(0)} &=& w_j^{(1)} \\ \dot{w}_j^{(1)} &=& w_j^{(2)} \\ &\vdots& \qquad j \in \{1, \ldots, r\} \\ \dot{w}_j^{(\beta_j)} &=& u_j \end{array} \right. \qquad (80) \\ &0 = y_i = \phi_i(t, w_1, \ldots, w_1^{(\alpha_{1i})}, \ldots, w_r, \ldots, w_r^{(\alpha_{ri})}),\ i \in \{1, \ldots, p\}. \end{aligned}$$

{eArOrM}

It is clear that (80) is in the form (18a)-(18b).

Now let $x = (w_1^{(0)}, \ldots, w_1^{(\beta_1 - 1)}, \ldots, w_r, \ldots, w_r^{(\beta_r - 1)})$, and let $\beta = \sum_{j=1}^r \beta_j$. Consider the implicit system $\Delta_2 = (T^\beta(x), F)$ of the form

$$F(t, x(t), \dot{x}(t)) = 0 \qquad (81)$$

{eFFF}

[59] As in the behavioral approach of Willems (Willems 1992), we do not distinguish input, state and outputs among the variables $w_i, i = 1, \ldots r$ in the differential equations (79).

where $F(t,x,\dot{x}) = (F_1(t,x,\dot{x}), F_2(x,\dot{x}))$, and

$$\begin{aligned} F_1 &= (w_1^{(1)}, w_1^{(2)}, \ldots, w_1^{(\beta_1)}, \ldots, w_r^{(1)}, w_r^{(2)}, \ldots, w_r^{(\beta_r)}) \\ F_2 &= (\phi_1, \ldots, \phi_p). \end{aligned}$$

In the first subsection of this appendix one has shown that, if Γ is canonically equivalent to system S defined by (80), then Γ is canonically equivalent to system Σ defined by (81). This means means the the conversion of (79) into the form (80) (which may be regarded as being in the form (18)) is a convenient approach in order to study the implicit system (79).

K Proof of Proposition 3

{aEquivalenceFlatness}

Proof. 1. Let $\xi \in S$. Assume a state-representation (x,u) is given. For convenience consider that this state representation is defined on a neighborhood $V of \xi$. Suppose that the system is locally flat around ξ according definition 6.

In other words, $(\emptyset, y)$ is another state representation defined[60] for which the Cartan field is given by (3). Then, as $\{t, (y^{(k)} : k \in I\!N)\}$ is a local coordinate system, one may write $x = \chi(t, y^{(0)}, \ldots, y^{(\alpha)})$ and $u = \mu(t, y^{(0)}, \ldots, y^{(\beta)})$. Essentially the result is proved. It suffices to show now that the maps χ and μ may be extended respectively to the spaces $I\!R \times (I\!R^s)^{\alpha+1}$ and $I\!R \times (I\!R^s)^{\beta+1}$.

Note that the map $\pi : V \to I\!R \times (I\!R^s)^{\alpha+1}$ defined $\xi \mapsto (t(\xi), y^{(0)}(\xi), \ldots, y^{(\alpha)}(\xi))$ is open. Let $\hat{V} = \pi(V)$. The map $\chi : \hat{V} \to I\!R^n$ is smooth. One may restrict χ to some open neighborhood of $\hat{U}_\chi$ of $\delta(\xi)$ in such a way that $\chi|_{\hat{U}_\chi}$ may be extended to the entire space[61] $I\!R \times (I\!R^s)^{\alpha+1}$. After that one may construct U_χ by taking $U_\chi = \pi^{-1}(\hat{U})$. A similar construction can be made for μ, and one may take $U = U_\chi \bigcap U_\mu$, showing that the Definition 7 is satisfied on U.

2. Assume that the definition 7 is satisfied on V. Let (ϕ, V) be the local chart of S with coordinate functions $\{t, x, (u^{(k)} : k \in I\!N)\}$ associated to the state representation (x,u). Let $\tilde{U} = \phi(V)$.

Define the map $\tilde{\Lambda} : T^s(y) \to R$, where $T^s(y)$ is the trivial diffiety with flat output y, R is the $I\!R^A$-manifold of global coordinates[62] $\{t, x, (u^{(k)} : k \in I\!N)\}$, defined by $t \mapsto t$, $x = \chi(t, y^{(0)}, \ldots, y^{(\alpha)})$, $u = \chi(t, y^{(0)}, \ldots, y^{(\alpha)})$, and by the usual rules of differentiation, define $u^{(k)} = \chi^{(k)}$, where $\chi^{(0)} = \chi$ and $\chi^{(k)} = \sum_{j=0}^{\alpha+k-1} \sum_{i=1}^{s} \frac{y_i^{(j+1)} \partial \chi^{(k-1)}}{\partial y_i^{(j)}}$. After restricting $\tilde{\Lambda}$ to $\Lambda^{-1}(\tilde{U})$, one may define $\Lambda = \phi^{-1} \circ \tilde{\Lambda}$. Our assumptions implies easily that $\Lambda \circ \Gamma$ is the identity map on V. In fact, at least this is true (in coordinates) for the components t, x and u of $\Lambda(\Gamma(\nu))$, that is, for the map $\Lambda \circ \Gamma$ one has $t \mapsto t$, $x \mapsto x$ and $u \mapsto u$. Now

[60] Without loss of generality one may consider both state representations defined on V.

[61] For instance, by multiplying the components of χ by a function that appears on the construction of partitions of unity(Warner 1971)

[62] One abuse notation an use the same names of u and x.

the fact that $u^{(k)} \mapsto u^{(k)}, k \in \mathbb{N}$ is easily seen from the definition, by derivation using the chain rule. Now note that, on V, $\Gamma = \Gamma \circ (\Lambda \circ \Gamma) = (\Gamma \circ \Lambda) \circ \Gamma$. In particular $(\Gamma \circ \Lambda)$ is the identity map on the image $K = \Gamma(U)$ of Γ. By assumption, it exists $U_1 \subset K$, where U_1 is an open neighborhood of $\Gamma(\xi)$. Restricting Γ to $U = \Gamma^{-1}(U_1)$ and Λ to U_1, one gets $\Gamma|_U = \Gamma|_U = (\Lambda|_{U_1})^{-1}$. Now, by definition, it is clear that Γ is Lie-Bäcklund (see the Part I of this survey). This shows that Definition 7 implies definition 6.

3. From the same arguments of the proof of 1 one may show that Definition 6 implies the existence of the maps χ and μ. Now, by Def. 6, the flat output y is an input. Hence, by the uniqueness of Differential Dimension (see (Fliess et al. 1999, Pereira da Silva 2000)), the cardinal of an input is a system invariant in every connected component of S. Hence $\operatorname{card} y = \operatorname{card} u$ show that definition 6 holds.

4. Assume that Definition 8 holds. Without loss of generality, consider that y is a classic output, that is, $\operatorname{span}\{dy\} = \operatorname{span}\{dt, dx, dy\}$ (otherwise one may consider a convenient extension of the state). Since $u = \mu(y, \ldots, y^{(\beta)})$ it is clear that $\operatorname{span}\{du\} \subset \operatorname{span}\left\{dy, \ldots, dy^{(\beta)}\right\}$. Let ξ be a regular point of the filtrations (7b) and (7c). By Part 9 of Lemma 5, it follows that $\sigma_{k^*} = \operatorname{card} u$. By Part 5 of Lemma 5, as $\rho_k \geq \sigma_{k^*}$ and $\operatorname{card} y = \operatorname{card} u$, from simple dimensional arguments it follows that the set $\{dy^{(0)}, \ldots, dy^{(k)}\}$ is independent for all $k \in \mathbb{N}$. In particular, a convenient application of Corollary 2 gives the desired result. $\square$

L Proof of Proposition 2 – fiber dimension invariance

{aFiber}

Let $\pi : V \to T$ be a submersion. Assume that there exists local coordinates $\tilde{\phi} = (\tilde{x}, \tilde{z})$ of T and $\tilde{\psi} = \tilde{x}$ of V such that π locally reads $(\tilde{x}, \tilde{z}) \mapsto \tilde{z}$. Assume also that there exists local coordinates $\hat{\phi} = (\hat{x}, \hat{z})$ of T and $\hat{\psi} = \hat{x}$ of V such that π locally reads $(\hat{x}, \hat{z}) \mapsto \hat{z}$. Without loss of generality, assume that the domains such charts coincides, that is $\tilde{\phi} : U \to \tilde{U}$, $\hat{\phi} : U \to \hat{U}$, $\tilde{\psi} : W \to \tilde{W}$, $\hat{\psi} : W \to \hat{W}$, where $\nu \in U$ and $\tau = \pi(\nu) \in W$. Furthermore, one may assume that $\pi(U) = W$.

In fact, if it is not the case, that is, if for instance $\tilde{\psi} : W_1 \to \tilde{W}$ and $\hat{\psi} : W_2 \to \tilde{W}$, with $W_1 \neq W_2$, one may take $W = W_1 \cap W_2$ and consider the restrictions of $\tilde{\psi}$ and $\hat{\psi}$ to W. Now let $U = \pi^{-1}(W) \cap U_1 \cap U_2$ where U_1 and U_2 are the domains of $\tilde{\phi}$ an $\hat{\phi}$. The $\pi(U) = W$. As π is an open map, one may replace W by $\pi(U)$, and one may take the correspondent restrictions of $\tilde{\psi}$ and $\hat{\psi}$. By construction, $\pi(U) = W$.

Let $\tilde{z}_0 = \tilde{\phi}(\tau)$ and $\hat{z}_0 = \hat{\phi}(\tau) =$. By construction $\hat{z}_0 = \hat{\phi} \circ \tilde{\psi}^{-1}(\tilde{z}_0)$.

Define the maps $\tilde{\pi} : \tilde{U} \to \tilde{W}$ and $\hat{\pi} : \hat{U} \to \hat{W}$ by $\tilde{\pi} = \tilde{\psi} \circ \pi \circ \tilde{\phi}^{-1}$ and $\hat{\pi} = \hat{\psi} \circ \pi \circ \hat{\phi}^{-1}$. Since the charts are locally adapted, one may has $\tilde{\pi}(\tilde{x}, \tilde{z}) = \hat{z}$ and $\hat{\pi}(\hat{x}, \hat{z}) = \hat{z}$. Since a set of coordinate functions is countable, it suffices to show the result in the case where $\tilde{x}$ is finite.

Define the fiber

$$\tilde{H}_{\tilde{z}_0} = \{\tilde{x} \in I\!R^B \mid (\tilde{x}, \tilde{z}_0) \in \tilde{U}\} \quad (82) \quad \text{\{eBFiber\}}$$

This set is an open set of $I\!R^B$. In fact, for each basic open set $H = (\tilde{H}_1 \times \tilde{H}_2) \subset \tilde{U}$ with $\tilde{H}_2$ containing $\tilde{z}_0$, then $\tilde{H}_{\tilde{z}_0} \cap H = \tilde{H}_1$, which is an open subset of $I\!R^B$. In particular, $\tilde{H}_{\tilde{z}_0}$ is the union of open sets that are formed in this way, showing that $\tilde{H}_{\tilde{z}_0}$ is open. Similarly, one may show that

$$\hat{H}_{\hat{z}_0} = \{\tilde{x} \in I\!R^C \mid (\hat{x}, \hat{z}_0) \in \hat{U}\} \quad (83) \quad \text{\{eBFiber2\}}$$

is an open subset of $I\!R^C$. Now note that

$$\begin{aligned} \tilde{H}_{\tilde{z}_0} &= \tilde{\pi}^{-1}(\tilde{z}_0) = \left(\tilde{\psi} \circ \pi \circ \tilde{\phi}^{-1}\right)^{-1} (\tilde{z}_0) \\ &= \left(\pi \circ \tilde{\phi}^{-1}\right)^{-1} \left(\tilde{\psi}^{-1}(\tilde{z}_0)\right) \\ &= \left(\pi \circ \tilde{\phi}^{-1}\right)^{-1} (\tau) \\ &= \left(\tilde{\phi}^{-1}\right)^{-1} (\pi^{-1}(\tau)) \\ &= \tilde{\phi}\left(\pi^{-1}(\tau)\right) \end{aligned}$$

a similar construction shows that $\hat{H}_{\hat{z}_0} = \hat{\phi}(\pi^{-1}(\tau))$. Hence $\tilde{\phi} \circ \hat{\phi}^{-1}(\hat{H}_{\hat{z}_0}) = \tilde{\phi} \circ \hat{\phi}^{-1}\left(\hat{\phi}(\pi^{-1}(\tau))\right) = \tilde{\phi}\left(\pi^{-1}(\tau)\right) = \tilde{H}_{\tilde{z}_0}$. A similar reasoning shows that $\hat{\phi} \circ \tilde{\phi}^{-1}(\tilde{H}_{\tilde{z}_0}) = \hat{H}_{\hat{z}_0}$. In particular $\tilde{\phi} \circ \hat{\phi}^{-1}(\hat{x}, \hat{z}_0) = (\tilde{x}, \tilde{z}_0)$ for all $\hat{x} \in \hat{H}_{\hat{z}_0}$ and $\hat{\phi} \circ \tilde{\phi}^{-1}(\tilde{x}, \tilde{z}_0) = (\hat{x}, \hat{z}_0)$ for all $\tilde{x} \in \tilde{H}_{\tilde{z}_0}$. So define the map $\alpha : \tilde{H}_{\tilde{z}_0} \to \tilde{H}_{\tilde{z}_0}$ such that $\tilde{x} \mapsto \hat{x}$ is induced by the map $\hat{\phi} \circ \tilde{\phi}^{-1}(\tilde{x}, \tilde{z}_0) = (\hat{x}, \hat{z}_0)$. It is clear that this map is a diffeomorphism with inverse β that is induced by the map $\tilde{\phi} \circ \hat{\phi}^{-1}(\hat{x}, \hat{z}_0) = (\tilde{x}, \tilde{z}_0)$. In particular, from the Brouwer invariance theorem (Brouwer 1912), one has $\operatorname{card} \tilde{x} = \operatorname{card} \hat{x}$.

M Proof of Proposition 6

{aNullY}

Proof. If σ is a solution of Δ then successive differentiation of $y(\sigma(t)) \equiv 0$ with respect to t at t_0 shows easily that $\xi \in \tilde{\Delta}$. In fact, to say that $\sigma(t)$ is a solution of S is equivalent to have $\sigma_*|_t(\frac{d}{dt}) = \partial_S \circ \sigma(t)$, where ∂_S is the Cartan field of S, and $\frac{d}{dt}$ is the standard operation of derivation with respect to t. Then

$$\frac{d}{dt}(y \circ \sigma(t)) = (y \circ \sigma)_*(\frac{d}{dt}) = y_*|_{\sigma(t)}(\sigma_*|_t \frac{d}{dt}) = y_*|_{\sigma(t)}(\partial_S \circ \sigma(t) = y^{(1)} \circ \sigma(t) \equiv 0$$

Continuing this process one obtains

$$y^{(k)} \circ \sigma(t) = 0, \text{ for all } t \in (a, b) \quad (84) \quad \text{\{eNullY\}}$$

This implies that $\sigma(t) \in \tilde{\Delta}$ for all $t \in (a,b)$. Now given a control system Γ, it can be shown[63] that for every point $\gamma \in \Gamma$ there exist a solution $\nu : (a,b) \to \Gamma$ such that $\nu(t_0) = \gamma$ for $t_0 \in (a,b)$. As γ is arbitrary, (84) implies that $\iota(\Gamma) \subset \tilde{\Delta}$. $\square$

N Proof of Proposition 7

{aEmbedding}

The proof of the Proposition 7 is based on the following Lemma, which proof is an easy adaptation of similar results of finite dimensional differential geometry (Warner 1971, Theo. 1.23, p. 26).

{lEmbedding}

Lemma 6 *Let I be a $I\!R^A$-manifold and $\iota : \Gamma \to S$ be an injective embedding between $I\!R^A$ manifolds. Let $\sigma : I \to S$ be a smooth map such that $\sigma(I) \subset \iota(\Gamma)$. Then there exists a unique smooth map $\eta : I \to \Gamma$ such that $\sigma = \iota \circ \eta$, that is, the following diagram is commutative*

$$\begin{tikzcd} \Gamma \arrow[r, "\iota"] & S \\ I \arrow[u, "\eta"] \arrow[ur, "\sigma"] & \end{tikzcd}$$

Proof. (of Prop. 7) Let $I = (a,b) \subset I\!R$. It suffices to show that every solution $\sigma : I \to S$ of Δ is of the form $\sigma = \iota \circ \nu$, where $\nu : I \to \Gamma$ is a solution of Γ. Let $\sigma : (a,b) \to S$ be a solution of Δ, that is, $\sigma(t)$ is a solution of S such that $y(\sigma(t)) \equiv 0$ for all $t \in (a,b)$. In particular, by successive differentiations with respect to the Cartan field ∂_S of S, one shows that $\sigma(I) \subset \tilde{\Delta} = \iota(I)$ (see (84)). By Lemma 6 there exists a unique smooth map $\eta : I \to S$ such that $\sigma = \iota \circ \eta$. Now, let ∂_Γ be the Cartan field of Γ. As ι is Lie Bäcklund, one may write

$$\iota_*(\partial_\Gamma) = \partial_S \circ \iota.$$

Evaluation the last identity at a point $\eta(t) \in \Gamma$, for some $t \in I$, one gets

$$\iota_*|_{\eta(t)} (\partial_\Gamma) = \partial_S \circ \iota \circ \eta(t)$$

Now, as $\sigma = \iota \circ \eta$ is also Lie Bäcklund, one may write

$$\iota_*(\eta_*(\frac{d}{dt})) = \partial_S \circ \iota \circ \eta$$

Where $\frac{d}{dt}$ is the Cartan field of $I\!R$, that is, the standard operation of differentiation of smooth functions. Now, since ι_* is injective at every point γ of Γ, it follows that one may write

$$\eta_*(\frac{d}{dt}) = \partial_\Gamma \circ \eta$$

[63] First consider a local state representation. After that, Borel's theorem (Borel 1895) assures that there exists some input with a prescribed infinite jet. After that, the result can be shown by standard theorems of existence of solutions of ordinary differential equations.

The last equation is equivalent to say that η is a solution of Γ.

Now let $\eta : I \to \Gamma$ be a solution of Γ. Then, as ι is Lie Bäcklund, $(\iota \circ \eta)_*(\frac{d}{dt}) = \iota_*(\eta_*(\frac{d}{dt})) = \iota_*(\partial_\Gamma \circ \eta) = \partial_S \circ \iota \circ \eta$. In particular $\sigma = \iota \circ \eta$ is a solution of S. Now as $\iota(S) = \tilde{\Delta}$, then $y \circ \sigma \equiv 0$. In particular, σ is a solution of the implicit system $\tilde{\Delta}$. $\square$