Relative structure at infinity and nonlinear semi-implicit DAEs*

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Abstract

This paper considers explicit systems with an output $\eta = (z, y)$, where z and y are subvectors of η. One may be interested in controlling the output z independently of the behavior of y. Following this idea, the *problem of relative-decoupling* is introduced. The *Relative Dynamic Extension Algorithm (RDEA)* is presented, and it is shown that it computes some geometric invariants, namely, the *relative structure at infinity* of z, which governs the solvability of the relative-decoupling problem. A generalization of the notion of zero dynamics arises and, when this zero dynamics is absent, the output z is said to be a *relatively flat output.* It is shown that the dimension of the state of the generalized zero dynamics can be easily computed from the (generalized) structure at infinity. In particular, this furnishes a test for verifying if z is a relatively flat output. When one adds the constraint $y \equiv 0$, then the system becomes a *Differential Algebraic System (DAE).* In this context, relative-decoupling and relative-flatness of the explicit system implies respectively, decoupling and flatness of the corresponding DAE. In particular, the RDEA may be used for computing dynamic feedback for decoupling and/or linearizing implicit systems.

Keywords. Structure at infinity; nonlinear control systems; implicit systems; DAEs; differential geometric approach; diffieties; decoupling; flatness; feedback linearization.

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1 Introduction

The aim of this paper is the generalization of the concept of the structure at infinity for a class of nonlinear systems, introducing the *relative structure at infinity*. The approach considered here is mainly based on the approach of (Fliess *et al.*, 1999) and on the results of (Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva and Watanabe, 2002). This work is strongly related to some ideas of (Fliess *et al.*, 1995; Liu and Čelikovský, 1997).

In order to motivate the problems that are studied in this work, consider the following example.

Example 1 *Let S be an explicit system with state $x(t) \in I\!R^6$, input $u(t) \in I\!R^4$ and output $(z(t), y(t))$ with $z(t), y(t) \in I\!R^2$ given by:*

$$\begin{array}{ll} \dot{x}_1(t) = u_1(t) + u_3(t) & \dot{x}_2(t) = x_3(t)u_1(t) + u_4(t) \\ \dot{x}_3(t) = u_2(t) & \dot{x}_4(t) = u_3(t) \\ \dot{x}_5(t) = x_6(t)u_3(t) & \dot{x}_6(t) = u_4 \\ z_1(t) = x_1(t) & z_2(t) = x_2(t) \\ y_1(t) = x_4(t) - t & y_2(t) = x_5(t) \end{array}$$

Assume that one wants to control the components of z independently of the components of y. It seems reasonable to use the Dynamic Extension Algorithm (DEA)[1] for the output y, in order to decompose the input in two subvectors such that, the first one controls the components of y, and the remaining components may be used to control z. Computing the step 1 of this algorithm, one gets, $\dot{y}_1 = u_3 - 1 = \bar{y}_1^{(1)}$ and $\dot{y}_2 = x_6 u_3 = x_6(1 + \bar{y}_1^{(1)})$, which gives the static feedback $u_3 = 1 + \bar{y}_1^{(1)}$ and the dynamic extension $\dot{\bar{y}}_1^{(1)} = \bar{y}_1^{(2)}$. The second step furnishes $y_2^{(2)} = u_4(1 + \bar{y}_1^{(1)}) + x_6 \bar{y}_1^{(2)})$, which gives the static feedback $u_4 = \frac{1}{1+\bar{y}_1^{(1)}}(-x_6 \bar{y}_1^{(2)} + \bar{y}_2^{(2)})$ and the dynamic extension $\dot{\bar{y}}_1^{(2)} = \bar{y}_1^{(3)}$ and $\dot{\bar{y}}_2^{(2)} = \bar{y}_2^{(3)}$. This gives the closed loop system (for simplicity the dynamic extensions are omitted)

$$\begin{array}{ll} \dot{x}_1(t) = u_1(t) + \phi_1 & \dot{x}_2(t) = x_3(t)u_1(t) + \phi_2 \\ \dot{x}_3(t) = u_2(t) & \dot{x}_4(t) = u_3(t) \\ \dot{x}_5(t) = x_6(t)\phi_1 & \dot{x}_6(t) = \phi_2 \end{array}$$

where $\phi_1 = 1 + \bar{y}_1^{(1)}$ and $\phi_2 = \frac{1}{1+\bar{y}_1^{(1)}}(-x_6 \bar{y}_1^{(2)} + \bar{y}_2^{(2)})$. Now note that the inputs u_1 and u_2 can be used to control z independently of y. Differentiating z_1 and z_2 once, one obtains $\dot{z}_1 = u_1 + \phi_1 = \bar{v}_1$ and $\dot{z}_2 = x_3 u_1 + \phi_2$. Consider the static feedback $u_1 = -\phi_1 + \bar{v}_1$ and the dynamic extension $\dot{\bar{v}}_1 = \bar{v}_1^{(1)}$. Differentiation of $\dot{z}_2$ gives $z_2^{(2)} = u_2(-\phi_1 + \bar{v}_1) + x_3(\bar{v}_1^{(1)} - \dot{\phi}_1) + \dot{\phi}_2 = \bar{v}_2$. Defining the static feedback $u_2 = \frac{1}{-\phi_1 + \bar{v}_1}(-x_3(\bar{v}_1^{(1)} - \dot{\phi}_1) - \dot{\phi}_2 + \bar{v}_2)$, one obtains, in closed loop with the composite dynamic feedback, $z_1^{(2)} = \bar{v}_1^{(1)}$ and $z_2^{(2)} = \bar{v}_2$. Disregarding the singularity $\bar{v}_1 = \phi_1$, it seems that the components z_1 and z_2 can be decoupled

[1] This important algorithm is recalled in Section 2.2.

independently of the behavior of $y(t)$. The procedure above is the idea of the *Relative Dynamic Extension Algorithm (RDEA)* and the results of the paper will give a precise meaning and a geometric interpretation for this procedure. Adding the constraint $y \equiv 0$, it will be shown that the corresponding feedback solves both the decoupling problem and the dynamic linearization problem for the corresponding implicit system.

The paper is organized as follows. Section 2 presents some preliminary results and the class of systems considered in this work. Section 3 introduces the RDEA and its geometric properties. Section 4 considers the *relative-decoupling problem*. Section 5 characterizes relatively flat outputs. In Section 6 we study the assertion: "relative-decoupling and relative-flatness implies decoupling and flatness of implicit systems, respectively." In Section 8 we state some conclusions and some auxiliary results are proved in the appendices.

2 Preliminaries and notation

The field of real numbers will be denoted by $\mathbb{R}$. The matrix (or vector) M^T stands for the transpose of M. The set of natural numbers $0, 1, 2, \ldots$ is denoted by $\mathbb{N}$ and the subset $\{1, \ldots, k\} \subset \mathbb{N}$ will be denoted by $\lfloor k \rceil$. We will use the standard notations of differential geometry in the finite and infinite dimensional case. A brief overview of the infinite dimensional approach of (Fliess *et al.*, 1999) is presented in Section 2.1. Some notations and definitions of Section 2.1 are used along the paper (*e. g.* the definition of system S as a *diffiety*, and the definition of *state representation* as a local coordinate system). The cardinal of a set Z is denoted by $\operatorname{card} Z$.

For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x = (x_1, \ldots, x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$. In the same vein, if $x^i = (x_1^i, \ldots, x_{p_i}^i)$ for $i = 1, 2, \ldots$, are sets of functions, then $\{dx^1, dx^2, \ldots\}$ stands for the set $\{dx_1^1, \ldots, dx_{p_1}^1, dx_1^2, \ldots, dx_{p_2}^2, \ldots\}$.

2.1 Diffieties and Systems

The aim of this Section is to introduce a brief overview of the approach of (Fliess *et al.*, 1999). The presentation will follow the lines of (Pereira da Silva and Corrêa Filho, 2001).

$\mathbb{R}^A$-Manifolds. Let A be a countable set. Denote by $\mathbb{R}^A$ the set of functions from A to $\mathbb{R}$. One may define the coordinate function $x_i : \mathbb{R}^A \to \mathbb{R}$ by $x_i(\xi) = \xi(i), i \in A$. This set can be endowed with the Fréchet topology (see (Fliess *et al.*, 1999)). A function $\phi : \mathbb{R}^A \to \mathbb{R}$ is smooth if we locally have $\phi = \psi(x_{i_1}, \ldots, x_{i_s})$, where $\psi : \mathbb{R}^s \to \mathbb{R}$ is a smooth function. Only the dependence on a finite number of coordinates is allowed.

From this notion of smoothness, one can easily introduce the notions of vector fields and differential forms on $\mathbb{R}^A$ and smooth mappings from $\mathbb{R}^A$ to

$I\!R^B$. The notion of $I\!R^A$-manifold can be also established easily as in the finitely dimensional case.

Given an $I\!R^A$-manifold $\mathcal{P}$, $C^\infty(\mathcal{P})$ denotes the set of smooth maps from $\mathcal{P}$ to $I\!R$. Let $\mathcal{Q}$ be an $I\!R^B$-manifold and let $\phi : \mathcal{P} \to \mathcal{Q}$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_p\mathcal{P} \to T_{\phi(p)}\mathcal{Q}$ and $\phi^* : T^*_{\phi(p)}\mathcal{Q} \to T^*_p\mathcal{P}$. The inverse function, implicit function and rank theorems do not hold in this context and the notions of immersions and submersions are defined based on adapted coordinate systems (Fliess *et al.*, 1999; Zharinov, 1992).

Diffieties. A *diffiety* M is a $I\!R^A$ manifold equipped with a distribution Δ of finite dimension r, called *Cartan distribution.* A section of the Cartan distribution is called a *Cartan field.* An *ordinary diffiety* is a diffiety for which $\dim \Delta = 1$ and a Cartan field ∂_M is distinguished and called *the Cartan field.* In this paper we will only consider ordinary Diffieties that will be called simply by *Diffieties.*

A Lie-Bäcklund mapping $\phi : M \mapsto N$ between Diffieties is a smooth mapping that is compatible with the Cartan fields, *i. e.*, $\phi_*\partial_M = \partial_N \circ \phi$. A *Lie-Bäcklund immersion* (respectively, *submersion*) is a Lie-Bäcklund mapping that is an immersion (resp., submersion). A Lie-Bäcklund isomorphism between two diffieties is a diffeomorphism that is a Lie-Bäcklund mapping. Context permitting, we will denote the Cartan field of an ordinary diffiety M simply by $\frac{d}{dt}$. Given a smooth object ϕ defined on M (a smooth function, field or form), then $\dot{\phi}$ stands for $L_{\frac{d}{dt}}\phi$ and $L^n_{\frac{d}{dt}}\phi = \phi^{(n)}, n \in I\!N$.

Systems. The set of real numbers $I\!R$ have a trivial structure of diffiety with the Cartan field $\frac{d}{dt}$ given by the operation of derivation of smooth functions. A *system* is a triple $(S, I\!R, \tau)$ where S is a diffiety equipped with Cartan field ∂_S and $\tau : S \mapsto I\!R$ is a Lie-Bäcklund submersion called *time-function.* The global coordinate function t of $I\!R$ represents *time*, that is chosen once and for all. A Lie-Bäcklund mapping between two systems $(S, I\!R, \tau)$ and $(S', I\!R, \tau')$ is a time-respecting Lie-Bäcklund mapping $\phi : S \mapsto S'$, *i. e.*, $\tau = \tau' \circ \phi$. Context permitting, the system $(S, I\!R, \tau)$ is denoted simply by S.

State Representation and Outputs. A local state representation (x, u) of a system $(S, I\!R, \tau)$ is a local coordinate system, $\psi = \{t, x, U\}$ defined on an open set W, where $t = \tau|W$, $x = \{x_i, i \in \lfloor n\rceil\}$, $U = \{u_j^{(k)} \mid j \in \lfloor m\rceil, k \in I\!N\}$, and $u^{(k+1)} = L_{\frac{d}{dt}}u^{(k)}, k \in I\!N$. The set of functions $x = (x_1, \ldots, x_n)$ is called *state* and $u = (u_1, \ldots, u_m)$ is called *input.* As a consequence of the last definition, in these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in I\!N, \\ j \in \lfloor m\rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{1}$$

A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by (x, u). An *output* y of a system S is a set of functions defined on S. The state representation (x, u) is said to be *classic* if the functions f_i depend only on (t, x, u) for $i = 1, \ldots, n$. The output y is said to be *classic* if y depends only on (t, x, u).

Flatness. A system is (locally) flat if there exists a (local) state representation (x, u) with $x = \emptyset$. In this case, the Cartan field is locally given by:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\substack{k \in \mathbb{N}, \\ j \in \lfloor m \rceil}} y_j^{(k+1)} \frac{\partial}{\partial y_j^{(k)}}, \tag{2}$$

a particular case of (1) in which $y = u$ is called flat output.

Differential Dimension. The number of components of the input of a (local) state representation (x, u) of a system S is called (local) *differential dimension*. The local differential dimension of a connected system is a global invariant called simply *differential dimension* (Fliess *et al.*, 1993; Pereira da Silva, 2000).

Because a flat output y is also an input, the number of components y always coincides with the (local) differential dimension of the system(Fliess *et al.*, 1993; Pereira da Silva, 2000).

System associated to differential equations. Now assume that a control system is given by a set of equations

$$\begin{array}{rcl} \dot{t} & = & 1 \\ \dot{x}_i & = & f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n \rceil \\ y_j & = & \eta_j(x, u, \ldots, u^{(\beta_j)}),\ j \in \lfloor p \rceil \end{array} \tag{3}$$

One can associate to these equations a diffiety S of global coordinates $\psi = \{t, x, U\}$, with U as defined above, and Cartan field $\frac{d}{dt}$ given by (1). *In particular, in this work, a system* (3) *is always interpreted as a manifold* S *in this way.*

Endogenous feedback. In this Section we introduce a simplified notion of endogenous feedback based on coordinate changes. This definition is convenient for our purposes, but it is not suitable for studying feedback equivalence (see (Fliess *et al.*, 1999) for a notion of endogenous feedback that is an equivalence relation between systems).

Two local state representations (x, u) and (z, v) of S induce a local coordinate change map called *endogenous feedback*. If we have span $\{dt, dx\}$ = span $\{dt, dz\}$ and span $\{dt, dx, du\}$ = span $\{dt, dz, dv\}$, then we locally have diffeomorphisms $(t, x) \mapsto (t, z)$ and $(t, x, u) \mapsto (t, z, v)$ called static-state feedback. The extension of state by integrators is another particular example of endogenous feedback. For instance, putting integrators in series with the first k inputs of the state representation (x, u) one obtains a new state $z = (x, u_1, \ldots, u_k)$ and a new input $v = (\dot{u}_1, \ldots, \dot{u}_k, u_{k+1}, \ldots, u_m)$. Note that the local coordinate functions of S related to these state representations in this case are the same, but they are joined together in a different way, giving rise to (x, u) and (z, v), which are related by an endogenous feedback.

Subsystems. A (local) subsystem S_a of a system S is a pair (S_a, π), where S_a is a system with a time notion τ_a and Cartan field ∂_a, and π is a Lie-Bäcklund submersion $\pi : U \subset S \to S_a$ between the open subset $U \subset S$ and S_a.

A local state representation $x = (x_a, x_b)$, $u = (u_a, u_b)$ is said to be adapted to a subsystem S_a if we locally have

$$\dot{x}_a = f_a(t, x_a, u_a) \quad (4a)$$
$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \quad (4b)$$

and (x_a, u_a) is a local state representation of S_a with state equations[2] (4a). Under mild assumptions, it is shown in (Pereira da Silva and Corrêa Filho, 2001) that a subsystem always possesses adapted state representation.

Output Subsystem. Given a system S with output y, a (local) output subsystem is a subsystem Y with corresponding submersion $\pi : U \subset S \to Y$ such that $\pi^* T^* Y = \text{span}\,\{dt,\, (dy^{(k)} : k \in I\!N)\}$.

2.2 Dynamic extension algorithm

The Dynamic Extension Algorithm (DEA), well known algorithm in nonlinear control theory, is essentially a tool for computing system right-inverses and the output rank (Fliess, 1989; Descusse and Moog, 1987; Nijmeijer and Respondek, 1988; Pereira da Silva, 1996; Delaleau and Pereira da Silva, 1998*b*). The DEA has an intrinsic interpretation (Di Benedetto *et al.*, 1989; Delaleau and Pereira da Silva, 1998*a*). Recall that the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators. According to the ideas of the end of Section 2.1, this algorithm can be regarded as the choice of a new local state representation of system S. Now we state a slightly different version of DEA that is useful for our purposes. Let S be the system (14) with Cartan field $\frac{d}{dt}$ defined by (1), classical state representation (x, u) and classical output y. Assume that $y^{(0)} = y = a_0(t, x) + b_0(t, x)u$ and denote $x_{-1} = x$, $u_{-1} = u$, $f_{-1}(t, x) = f(t, x)$, $g_{-1}(t, x) = g(t, x)$. The step k of this algorithm ($k = 0, 1, \ldots$) is described below:

Algorithm 1 *(DEA)* **Step k.** *In the step $k - 1$ we have constructed state equations*

$$\dot{x}_{k-1} = f_{k-1}(t, x_{k-1}) + g_{k-1}(t, x_{k-1})u_{k-1} \quad (5)$$
$$y^{(k)} = a_k(t, x_{k-1}) + b_k(t, x_{k-1})u_{k-1} \quad (6)$$

where $x_{k-1} = (x, \bar{v}_0, \ldots, \bar{v}_{k-1})$. Assume that $(\bar{t}, \bar{x}_{k-1})$ is a regular point for the matrix $b_k(t, x_{k-1})$ and let σ_k be the rank of b_k around $(\bar{t}, \bar{x}_{k-1})$. There exists a partition[3] *$y^{(k)} = (\bar{y}_k^{(k)}, \widehat{y}_k^{(k)})$ of $y^{(k)}$ such that $\dim \bar{y}_k^{(k)} = \sigma_k$ and we may define a (locally) regular static-state feedback*

$$u_{k-1} = \alpha_k(t, x_{k-1}) + \beta_k(t, x_{k-1})v_k \quad (7)$$

[2]Here we abuse notation and we do not distinguish x_a (functions defined on S_a) from $x_a \circ \pi$ (functions defined on S). The same remark applies to u_a.

[3]Including possibly a reordering of its elements.

where $v_k = (\bar{v}_k, \widehat{v}_k)$ *is such that*

$$\begin{array}{rcl} \bar{y}_k^{(k)} & = & \bar{v}_k \\ \widehat{y}_k^{(k)} & = & \widehat{y}_k^{(k)}(t, x_{k-1}, \bar{v}_k) \end{array} \tag{8}$$

Add the dynamic extension:

$$\begin{array}{l} \bar{u}_k = \dot{\bar{v}}_k \\ \widehat{u}_k = \widehat{v}_k \end{array} \tag{9}$$

and define $u_k = (\bar{u}_k, \widehat{u}_k)$. *This defines a new set of state equations:*

$$\dot{x}_k = f_k(t, x_k) + g_k(t, x_k)u_k \tag{10}$$

where $x_k = (x_{k-1}, \bar{y}_k^{(k)})$ *and* $u_k = (\bar{y}_k^{(k+1)}, \widehat{u}_k)$. *By construction we have* $y^{(k)} = y^{(k)}(t, x_k)$. *Hence we may compute*

$$\begin{array}{rcl} y^{(k+1)} & = & \frac{\partial y^{(k)}}{\partial t} + \frac{\partial y^{(k)}}{\partial x_k}(f_k + g_k u_k) \\ & = & a_{k+1}(t, x_k) + b_{k+1}(t, x_k)u_k \end{array} \tag{11}$$

The following lemma summarizes the main geometric properties of the DEA for time-varying nonlinear systems.

Lemma 1 *(Pereira da Silva and Corrêa Filho, 2001) Let* S *be the system* (14) *with Cartan field* $\frac{d}{dt}$ *defined by* (1)*, classical state representation* (x, u) *and classical output* y*. Let* V_k *be the open and dense set of regular points of the codistributions* Y_i *and* $\mathcal{Y}_i$ *for* $i = 0, \ldots, k$ *defined on* (15a) *and* (15b)*. Let* $\xi \in V_k$*. In the* kth *step of the dynamic extension algorithm, one may construct a new local classical state representation* (x_k, u_k) *of the system* S *with state* $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$*, input* $u_k = (\dot{\bar{y}}_k^{(k)}, \widehat{u}_k)$ *and output* $y^{(k+1)} = h_k(t, x_k, u_k)$ *defined in an open neighborhood* U_k *of* ξ*, such that*

1. $\mathrm{span}\,\{dt, dx_k\} = \mathrm{span}\,\{dt, dx, dy, \ldots, dy^{(k)}\} = \mathcal{Y}_k$.
2. $\mathrm{span}\,\{dt, dx_k, du_k\} = \mathrm{span}\,\{dt, dx, dy, \ldots, dy^{(k+1)}, du\} = \mathcal{Y}_k + \mathrm{span}\,\{du\}$.
3. *It is always possible to choose* $\bar{y}_{k+1}^{(k+1)}$ *in a way that* $\bar{y}_k^{(k+1)} \subset \bar{y}_{k+1}^{(k+1)}$.
4. *It is always possible to choose* $\widehat{u}_{k+1} \subset \widehat{u}_k$.
5. *Let* $\xi \in V_n$*. Let* S_k *be the greater open neighborhood of* ξ *such that the dimensions of* $Y_j, \mathcal{Y}_j$ $j \in \{0, \ldots, k\}$ *are constant inside* S_k*. The sequence* $\sigma_k = \dim(\mathcal{Y}_k|_\xi) - \dim(\mathcal{Y}_{k-1}|_\xi)$ *is nondecreasing, the sequence* $\rho_k = \dim(Y_k|_\xi) - \dim(Y_{k-1}|_\xi)$ *is nonincreasing, and both sequences converge to the same integer* ρ*, called the* output rank at ξ*, for some* $k^* \leq n = \dim x$.
6. $S_k = S_{k^*}$ *for* $k \geq k^*$.
7. $Y_k \cap \mathrm{span}\,\{dx\}|_\nu = Y_{k^*-1} \cap \mathrm{span}\,\{dx\}|_\nu$ *for every* $\nu \in S_{k^*}$ *and* $k \geq k^*$.

8. *For $k \geq k^*$, one may choose $\bar{y}_k = \bar{y}_{k^*}$ around every point in U_{k^*}. Furthermore, $Y_{k+k*} = Y_{k+k^*-1} + \operatorname{span}\left\{d\bar{y}_{k^*}^{(k^*+1)}\right\}$ for $k \geq k^*$.*

Proof. See (Pereira da Silva and Watanabe, 2002; Pereira da Silva, 2000). □

Remark 1 *Note that $\dim \mathcal{Y}_k = 1 + \dim x_k = 1 + n + \sum_{i=0}^{k} \sigma_i$, $\dim \bar{y}_k = \sigma_k$, $\dim u_k = m$ and $\dim \widehat{u}_k = m - \sigma_k$, where $n = \dim x$, and $m = \dim u$.*

Remark 2 *In the step k^* of the dynamic extension algorithm one obtains a state representation (x_{k^*}, u_{k^*}) with input $u_{k^*} = (\bar{y}_{k^*}^{(k^*+1)}, \widehat{u}_{k^*})$. Note that the dynamic extension (9) is completely unnecessary in the step k^*. Without doing it, after the procedure (8), one obtains a state representation (x_{k^*-1}, v_{k^*}) where $v_{k^*} = (\bar{y}_{k^*}^{(k^*)}, \widehat{v}_{k^*})$. Recall that $\widehat{u}_{k^*} = \widehat{v}_{k^*}$ and $x_{k^*} = (x_{k^*-1}, \bar{y}_{k^*}^{(k^*)})$. Hence, similar properties to the ones of Lemma 1 holds for the state representation (x_{k^*-1}, v_{k^*}). By construction we have $\operatorname{span}\{dt, dx_{k^*-1}, dv_{k^*}\} = \operatorname{span}\{dt, dx_{k^*-1}, du_{k^*-1}\}$. By part 2 of Lemma 1, both codistributions coincide with $\mathcal{Y}_{k^*-1} + \operatorname{span}\{du\}$.*

Remark 3 *In the step 0 of the DEA one has constructed*

$$u = \underbrace{\alpha_0(t,x) + \beta_0^1(t,x)\bar{y}_0^{(0)}}_{\eta_0(t,x_0)} + \beta_0^2(t,x)\widehat{u}_0$$

In the subsequent steps, one constructs

$$\widehat{u}_{k-1} = \underbrace{\widehat{\alpha}_k(t,x_{k-1}) + \widehat{\beta}_k^1(t,x_{k-1})\bar{y}_k^{(k)}}_{\eta_k(t,x_k)} + \widehat{\beta}_0^2(t,x_{k-1})\widehat{u}_k$$

If one executes the DEA as in remark 2 one may show that

$$u = \gamma(t, x_{k^*-1}) + \delta(t, x_{k^*-1})\bar{y}_{k^*}^{(k^*)} + \epsilon(t, x_{k^*-1})\widehat{u}_{k^*} \tag{12}$$

2.3 Regular DAE's

This section is devoted to establishing the class of systems considered in this work. In this paper we consider a semi-implicit DAE, *i. e.*, a system Γ of the form

$$\begin{aligned}
\dot{x}(t) &= f(t,x(t)) + g(t,x(t))u(t) && \text{(13a)}\\
y(t) &= a(t,x(t)) + b(t,x(t))u(t) = 0 && \text{(13b)}\\
z(t) &= \phi(x(t)) + \psi(x(t))u(t) && \text{(13c)}
\end{aligned}$$

where $x(t) \in I\!R^n$ is the pseudo-state of the system, $u(t) \in I\!R^m$ is the pseudo-input[4], $z(t) \in I\!R^p$ is the output and $y_i(t), i = 1, \ldots, r$ are the constraints.

[4]Note that u is not a differentially independent input for Γ, since the constraints $y \equiv 0$ induce differential relations linking the components of u. For the same reasons, x is not necessarily a state of Γ.

One can associate to Γ an explicit system S with outputs y and z given by

$$\begin{aligned} \dot{x}(t) &= f(t,x(t)) + g(t,x(t))u(t) && \text{(14a)} \\ y(t) &= a(t,x(t)) + b(t,x(t))u(t) && \text{(14b)} \\ z(t) &= \phi(x(t)) + \psi(x(t))u(t) && \text{(14c)} \end{aligned}$$

Throughout this paper we denote by S the explicit system with output[5] y defined by (14), in the framework of (Fliess *et al.*, 1999) (see Section 2.1). Then $y^{(k)}$ stands for the function $\frac{d^k}{dt^k}y$ defined on S, which may depend $t, x, u^{(0)}, u^{(1)}, \ldots$.

Definition 1 *In the sequel we shall consider the following sequences of codistributions defined on* S

$$\mathcal{Y}_{-1} = \text{span}\{dt, dx\}, \qquad \mathcal{Y}_k = \text{span}\left\{dt, dx, dy, \ldots, dy^{(k)}\right\}, \ k \in I\!N \quad \text{(15a)}$$

$$Y_{-1} = \text{span}\{dt\}, \qquad Y_k = \text{span}\left\{dt, dy, \ldots, dy^{(k)}\right\}, \ k \in I\!N \quad \text{(15b)}$$

$$\mathbb{Y}_{-1} = \{0\}, \qquad \mathbb{Y}_k = \text{span}\left\{dy, \ldots, dy^{(k)}\right\}, \ k \in I\!N \quad \text{(15c)}$$

$$\mathbb{Z}_{-1} = \{0\}, \qquad \mathbb{Z}_k = \text{span}\left\{dz, \ldots, dz^{(k)}\right\}, \ k \in I\!N \quad \text{(15d)}$$

Let $\xi \in S$ be a regular point of the codistributions Y_k and $\mathcal{Y}_k$ for $k = 0, \ldots, n$, where $n = \dim x(t)$. According to (Di Benedetto *et al.*, 1989) (see also (Delaleau and Pereira da Silva, 1998*a*; Pereira da Silva, 2000; Pereira da Silva and Watanabe, 2002)), the sequence of integers $\{\sigma_0, \ldots, \sigma_n\}$, where $\sigma_k = \dim \mathcal{Y}_k|_\xi - \mathcal{Y}_{k-1}|_\xi$ is called the *algebraic structure at infinity* at ξ. It can be shown that the sequence σ_k is nondecreasing and converges for $k^* \leq n$ to the integer $\rho(y) = \sigma_{k^*} = \max\{\sigma_0, \ldots, \sigma_n\}$. One calls $\rho(y)$ by *output rank* at ξ (Fliess, 1989) and k^* by the *convergence index*[6]. Note that the dynamic extension algorithm is a tool for computing the *algebraic structure at infinity* and it constructs a dynamic feedback that may furnish a solution for various synthesis problems.

By the results of (Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva and Watanabe, 2002), one may identify the semi-implicit system Γ given by (13) with the subset of S defined by (see also Prop. 1 of Section 6).

$$\Gamma = \{\xi \in S \mid y^{(k)} = 0, k \in I\!N\} \quad \text{(16)}$$

Definition 2 *Let S be the explicit system defined by* (14) *and consider the codistributions defined by* (15)*. A point $\xi \in S$ is said to be regular if*

(i) *The codistributions $\mathbb{Y}_k$, Y_k and $\mathcal{Y}_k$, defined on S by* (15) *are nonsingular around ξ for $k = 0, \ldots, n$.*

(ii) *The codistributions $L_k = Y_{k^*+k} + \mathbb{Z}_k$ and $\mathcal{L}_k = \mathcal{Y}_{k^*+k} + \mathbb{Z}_k$ are nonsingular around ξ for $k = 0, \ldots, n$, where k^* is the convergence index of output y of system S around ξ..*

[5] We regard y as an output instead of being a constraint.

[6] In (Pereira da Silva and Watanabe, 2002) it is shown that k^* is the *differential index* of the DAE (13).

The semi-implicit system Γ given by (13) *is said to be regular if every point ξ of $\Gamma \subset S$, is regular, where $\Gamma \subset S$ is defined by* (16).

3 Relative Dynamic Extension Algorithm

The following algorithm is instrumental for studying relative-flatness and relative-decoupling:

Algorithm 2 (RDEA)
Preparation Process. *Execute k^* steps of the dynamic extension algorithm for the explicit system S with output y, as described in remark 2, obtaining the state representation $(\widetilde{x}_{-1}, \widetilde{u}_{-1})$, where $\widetilde{x}_{-1} = x_{k^*-1}$, $\widetilde{u}_{-1} = (\omega_0, \mu_{-1})$, $\omega_0 = \omega = \bar{y}_{k^*}^{(k^*)}$ and $\mu_{-1} = \widehat{u}_{k^*}$, with state equations given by*

$$\begin{aligned} \dot{\widetilde{x}}_{-1} &= f_{-1}(t, \widetilde{x}_{-1}) + \bar{g}_{-1}(t, \widetilde{x}_{-1})\omega_0 + \widehat{g}_{-1}(t, \widetilde{x}_{-1})\mu_{-1} && \text{(17a)} \\ z^{(0)} &= a_0(t, \widetilde{x}_{-1}) + b_0(t, \widetilde{x}_{-1})\omega_0 + c_0(t, \widetilde{x}_{-1})\mu_{-1} && \text{(17b)} \end{aligned}$$

Note that equation (17b) *is obtained by substitution of* (12) *in* (14c).
Then execute the steps $k = 0, 1, 2, \ldots$:
Step k. *In step $k-1$ we have constructed a state representation*

$$\begin{aligned} \dot{\widetilde{x}}_{k-1} &= f_{k-1}(t, \widetilde{x}_{k-1}) + \bar{g}_{k-1}(t, \widetilde{x}_{k-1})\omega_k + \widehat{g}_{k-1}(t, \widetilde{x}_{k-1})\mu_{k-1} && \text{(18a)} \\ z^{(k)} &= a_k(t, \widetilde{x}_{k-1}) + b_k(t, \widetilde{x}_{k-1})\omega_k + c_k(t, \widetilde{x}_{k-1})\mu_{k-1} && \text{(18b)} \end{aligned}$$

where $\tilde{x}_{k-1} = (\tilde{x}_{-1}, \omega_0, \omega_1, \ldots, \omega_{k-1}, \bar{z}_0^{(0)}, \ldots, \bar{z}_{k-1}^{(k-1)})$. Let $\widetilde{\sigma}_k = \operatorname{rank} c_k(t, \widetilde{x}_{k-1})$ and assume that this rank is locally constant around some $(t, \widetilde{x}_{k-1})$. Up to a reordering of the components of z, we may assume that the first $\widetilde{\sigma}_k$ rows of $c_k(t, \widetilde{x}_{k-1})$ are locally independent. Then there exists a partition $z = (\bar{z}_k, \hat{z}_k)$, where $\dim \bar{z}_k = \widetilde{\sigma}_k$, and a regular static-state feedback with new input (ω_k, v_k) defined by (see appendix A):

$$\mu_{k-1} = \bar{\alpha}_k(t, \widetilde{x}_{k-1}) + \widehat{\alpha}_k(t, \widetilde{x}_{k-1})\omega_k + \beta_k(t, \widetilde{x}_{k-1})v_k$$

where $v_k = (\bar{v}_k, \hat{v}_k)$ is such that[7]

$$\begin{aligned} \bar{z}_k^{(k)} &= \bar{v}_k \\ \hat{z}_k^{(k)} &= \hat{z}_k^{(k)}(t, \widetilde{x}_{k-1}, \omega_k, \bar{v}_k) \end{aligned} \qquad (19)$$

Add the following dynamic extension

$$\begin{aligned} \dot{\omega}_k &= \omega_{k+1} \\ \dot{\bar{v}}_k &= \bar{\mu}_k \end{aligned} \qquad (20)$$

and let $\hat{\mu}_k = \hat{v}_k$. Hence, one has constructed a new state representation $(\widetilde{x}_k, \widetilde{u}_k)$, with $\widetilde{x}_k = (\widetilde{x}_{k-1}, \omega_k, \bar{z}_k^{(k)})$, $\widetilde{u}_k = (\omega_{k+1}, \mu_k)$, $\mu_k = (\bar{z}_k^{(k+1)}, \widehat{\mu}_k)$ and output $z^{(k)}$

[7] We stress that $\beta_k(t, \widetilde{x}_{k-1})$ is locally nonsingular.

given by:

$$\begin{aligned}
\dot{\widetilde{x}}_k &= f_k(t, \widetilde{x}_k) + \bar{g}_k(t, \widetilde{x}_k)\omega_{k+1} + \widehat{g}_k(t, \widetilde{x}_k)\mu_k & \text{(21a)}\\
z^{(k)} &= \phi_k(t, \widetilde{x}_k) & \text{(21b)}
\end{aligned}$$

Compute

$$z^{(k+1)} = a_{k+1}(t, \widetilde{x}_k) + b_{k+1}(t, \widetilde{x}_k)\omega_{k+1} + c_{k+1}(t, \widetilde{x}_k)\mu_k \qquad \text{(21c)}$$

The following result summarizes the main geometric properties of the *Relative Dynamic Extension Algorithm* for time-varying nonlinear systems.

Lemma 2 *Let S be the system given by* (13a) *with classical state representation (x, u) and classical output y. Let $V_k \subset S$ be the open and dense set of regular points of the codistributions Y_i, $\mathcal{Y}_i$, for $i = 0, \ldots, n$ and of $\mathcal{L}_j, L_j$ for $j \in \{0, \ldots, k\}$, where $L_k = Y_{k^*+k} + \mathbb{Z}_k$ and $\mathcal{L}_k = \mathcal{Y}_{k^*+k} + \mathbb{Z}_k$ for $k = -1, 0, 1, 2, \ldots$ (see* (15)*). Assume that the output rank of the explicit system S is given by $\rho(y)$. Let $\omega = y_{k^*}^{(k^*)}$. In the* kth *step of the relative dynamic extension algorithm, one may construct around $\xi \in V_k$, a new local classical state representation $(\tilde{x}_k, \tilde{u}_k)$ of the system S with state $\tilde{x}_k = (\tilde{x}_{k-1}, \omega^{(k)}, \bar{z}_k^{(k)}) = (\tilde{x}_{-1}, \omega, \ldots, \omega^{(k)}, \bar{z}_1^{(1)}, \ldots, \bar{z}_k^{(k)})$, input $\tilde{u}_k = (\omega^{(k+1)}, \mu_k)$, where $\mu_k = (\bar{z}_k^{(k+1)}, \hat{\mu}_k)$, and output $z^{(k)} = \phi_k(t, \tilde{x}_k)$ defined in an open neighborhood U_k of ξ, such that*

1. $\operatorname{span}\{d\tilde{x}_k\} = \mathcal{L}_k, k = -1, 0, 1, 2, \ldots$*.*
2. $\operatorname{span}\{d\tilde{x}_k, d\tilde{u}_k\} = \mathcal{L}_{k+1} + \operatorname{span}\{du\}, k = -1, 0, 1, 2, \ldots$*.*
3. *It is always possible to choose $\bar{z}_{k+1}^{(k+1)}$ in a way that $\bar{z}_k^{(k+1)} \subset \bar{z}_{k+1}^{(k+1)}$*
4. *When $\bar{z}_k^{(k+1)} \subset \bar{z}_{k+1}^{(k+1)}$, it is always possible to choose $\hat{\mu}_{k+1} \subset \hat{\mu}_k$.*
5. *Let $\xi \in V_n$. The sequence $\widetilde{\sigma}_k = \dim(\mathcal{L}_k|_\xi)) - \dim(\mathcal{L}_{k-1}|_\xi) - \rho(y)$ is nondecreasing, the sequence $\widetilde{\rho}_k = \dim(L_k|_\xi) - \dim(L_{k-1}|_\xi) - \rho(y)$ is nonincreasing, and both sequences converge to the same integer $\widetilde{\rho}(z)$, called the* relative output rank at ξ*, for some $\widetilde{k}^* \leq n = \dim x$.*
6. *Let $S_k \subset V_n$ be the open neighborhood of a given $\xi \in V_n$, such that the dimensions of $\mathcal{L}_j, L_j$ $j \in \{0, \ldots, k\}$ are constant inside S_k. We have $S_k = S_{\widetilde{k}^*}$ for $k \geq \widetilde{k}^*$.*
7. $L_k \cap \operatorname{span}\{dx\}|_\nu = L_{\widetilde{k}^*-1} \cap \operatorname{span}\{dx\}\Big|_\nu$ *for every $\nu \in S_{\widetilde{k}^*}$ and $k \geq \widetilde{k}^*$.*
8. *For $k \geq \widetilde{k}^*$, one may choose $\bar{z}_k = \bar{z}_{\widetilde{k}^*}$ in $U_{\widetilde{k}^*}$. Furthermore, $L_{k+1} = L_k + \operatorname{span}\left\{d\omega^{(k+1)}, d\bar{z}_k^{(k+1)}\right\}$ for $k \geq \widetilde{k}^*$.*
9. *Let $\mathcal{Y} = \operatorname{span}\left\{dt, dy^{(k)} | k \in I\!N\right\}$. Then $\widetilde{\sigma}_k = \dim \frac{\mathcal{L}_k + \mathcal{Y}}{\mathcal{L}_{k-1} + \mathcal{Y}}$. In particular we have $\widetilde{\rho}(z) = \dim \frac{\mathcal{L}_n + \mathcal{Y}}{\mathcal{L}_{n-1} + \mathcal{Y}} = \dim \frac{\mathcal{L}_{\widetilde{k}^*} + \mathcal{Y}}{\mathcal{L}_{\widetilde{k}^*-1} + \mathcal{Y}}$.*

Proof. The proof is deffered to the appendix B. □

The last theorem motivates the following definition.

Definition 3 *Let S be an explicit system with outputs y and z given by* (14). *Assume that the codistributions defined by* (15) *are nonsingular around $\xi \in S$. Let $\rho(y)$ be the output rank of S. The sequence of integers $\{\widetilde{\sigma}_0, \ldots, \widetilde{\sigma}_n\}$, where $\widetilde{\sigma}_k = \dim \mathcal{L}_k - \dim \mathcal{L}_{k-1} - \rho(y)$, computed around $\xi \in S$ is called local* relative structure at infinity *(of output z) at ξ with respect to the output subsystem Y. The integer $\widetilde{\rho}(z) = \widetilde{\sigma}_n$ is called* relative output rank.

4 Relative-decoupling

In this Section we define and solve the problem of relative-decoupling. We show that this problem is solvable if and only if the relative output rank $\widetilde{\rho}(z)$ is equal to the number of components of z.

4.1 System decompositions

Now we introduce some notions of system decompositions that are useful for studying decoupling in our setting.

In (Pereira da Silva and Corrêa Filho, 2001, Theo. 4.3), given a classic state representation (x, u) and a classic output y of S, then the nonsingularity[8] of codistributions (15a) and (15b) for $k = 0, \ldots, n$, where $n = \dim x$, assures the existence and uniqueness[9] of a local output subsystem Y.

Recall now that an output of a system S in the sense of Section 2.1 is any set of functions defined on S. In particular, the input of a system is also an output. Hence one may introduce the following notion of input-output subsystem.

Definition 4 *Given a system S with (local) state representation (x, u) and output y. Consider the output $w = (y, u)$. The* input-output subsystem *is the output subsystem*[10] *W corresponding to the output w.*

A notion of decomposition of systems by a subsystem was introduced in (Pereira da Silva and Corrêa Filho, 2001). The next definition generalizes this concept.

Definition 5 *(i-decomposition and decomposition of systems) Let S be a system and let $\mathcal{F} = \{S_i, i \in \lfloor h \rceil\}$ be a family of subsystems with local coordinates respectively $(t, x_i), i \in \lfloor h \rceil$. The system S is said to be (locally) incompletely decomposed, or simply* i-decomposed *by $\mathcal{F}$ if there exists a (local) coordinate system $(t, x_1, \ldots, x_h, x_{h+1})$ defined in $U \subset S$ and a family of Lie-Bäcklund submersions $\{\pi_i : U \to S_i, i \in \lfloor h \rceil\}$ such that the local expression of π_i in these coordinates is given by $\pi_i(t, x_1, \ldots, x_h, x_{h+1}) = (t, x_i), i = 1, \ldots, h$. The system S is (locally)* decomposed *by $\mathcal{F}$ if it is i-decomposed by $\mathcal{F}$ and $x_{h+1} = \emptyset$.*

[8] In particular the existence of Y is generically assured.

[9] The uniqueness is implied by the existence.

[10] By (Pereira da Silva and Corrêa Filho, 2001, Theo. 4.3), this subsystem exists (generically) and is unique.

4.2 Relative-decoupling

In this section we introduce and solve the *Relative-Decoupling Problem (RDP).*

Definition 6 *(RDP) Let S be a system with output (y, z), where $z = (z_1, \ldots, z_p)$. Let Y be the (local) output subsystem corresponding to the output y. A (local) state representation (x, u), where $u = (u_1, \ldots, u_m)$ of S is said to be (locally)* relatively decoupled *with respect to Y if system S is (locally) i-decomposed by the family $\mathcal{F} = \{Y, S_1, \ldots, S_p\}$ where S_i is the input-output subsystem corresponding to input u_i and output $z_i, i = 1, \ldots, p$ and the output rank $\rho(z_i)$ of subsystem S_i is one.*

Remark 4 *The definition above is invariant with respect to the choice of the state,* i. e.*, if (x, u) is relatively decoupled w.r.t. Y, then $(\bar{x}, u)$ is also relatively decoupled w.r.t. Y.*

Since our definition of endogenous feedback is stated in Section 2.1 as the relation between two state representations, we may state and solve the *relative-decoupling problem* in the following way:

Theorem 1 *Given a system S with (local) state representation (x, u) and output (y, z), the RDP is the problem of finding an endogenous feedback,* i. e.*, a new state representation $(\bar{x}, \bar{u})$, in a way that the output z is relatively decoupled with respect to output subsystem Y. Then the RDP is solvable around a regular point $\xi \in S$ (see Def. 2) if and only if the relative output rank $\widetilde{\rho}(z)$ is equal to the number of components of z.*

For the proof of this theorem we need the following lemma

Lemma 3 *Let ξ be a regular point of system S. If $\widetilde{\rho}(z) = \operatorname{card} z$ then:*

(i) *The RDEA constructs around ξ a (local) state representation $(\bar{x}, \bar{u})$, where $\bar{x} = (\widehat{x}, \eta, z, \ldots, z^{(\widetilde{k}^*)})$, $\bar{u} = (\omega_{\widetilde{k}^*+1}, z^{(\widetilde{k}^*+1)}, \widehat{\mu}_{\widetilde{k}^*})$ such that* $\operatorname{span}\{dt, d\eta, (d\omega^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)\} = \operatorname{span}\left\{dt, (dy^{(k)} : k \in I\!N\right\} = \mathcal{Y}$.

(ii) $\dim \mathcal{L}_{\widetilde{k}^*} = \dim Y_{k^*-1} + (\rho(y) + \widetilde{\rho}(z))(\widetilde{k}^* + 1) + \dim \frac{\mathcal{L}_{\widetilde{k}^*}}{L_{\widetilde{k}^*}}$

(iii) *System S is i-decomposed by $\mathcal{F} = \{Y, Z\}$, where Y is a local output subsystem related to the output y and Z is a flat subsystem, with flat output z. In particular, system S has the strucuture of Figure 1.*

(iv) *The RDEA constructs in the step $\widetilde{k}^*$ a (local) solution of the relative decoupling problem.*

Proof. See appendix C. □

Proof. (of Theorem 1) The sufficiency is a consequence of part (iv) of Lemma 3. Now assume that the relative-decoupling problem is solvable. By definition, if $T^*Y = \mathcal{Y} = \operatorname{span}\left\{dt, (dy^{(j)} : j \in I\!N)\right\}$, it is easy to see from the definition 6 that $\dim \frac{\mathbb{Z}_{k+1}+\mathcal{Y}}{\mathbb{Z}_k+\mathcal{Y}} = \mathrm{card}\, z = p$. From this, one may show easily that $\dim \frac{L_{k+1}}{L_k} = \dim \frac{\mathbb{Z}_{k+1}+Y_{k+1+k^*}}{\mathbb{Z}_k+Y_{k+k^*}} = \dim \frac{Y_{k+1+k^*}}{Y_{k+k^*}} + \dim \frac{\mathbb{Z}_{k+1}}{\mathbb{Z}_k} = \rho(y) + p$, $k \in \mathbb{N}$. By part 5 of Lemma 2, it follows that $\widetilde{\rho}(z) = \mathrm{card}\, z = p$. $\square$

Remark 5 *The state of the zero dynamics*[11] $S/(Y \cup Z)$ *of Figure 1 is* $\widehat{x}$.

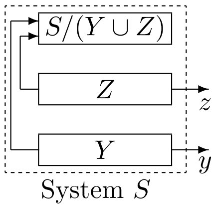

Figure 1: *Structure of a system S for which the output z is relatively decoupled with respect to subsystem Y. The subsystem Z is flat with flat output z.*

5 Relative-flatness and relatively flat outputs

In (Pereira da Silva and Corrêa Filho, 2001) the notion of relative-flatness with respect to a subsystem S_a is introduced. Using our previous definitions, we may translate this definition into the following form:

Definition 7 *A system S is said to be (locally) relatively flat with respect to a subsystem S_a if there exists a flat subsystem Z such that S is (locally) decomposed by the family $\mathcal{F} = \{S_a, Z\}$. A flat output z of Z is said to be a relatively flat output of S (with respect to S_a).*

The following result is a characterization of relatively flat outputs with respect to output subsystems:

Theorem 2 *Let S be a system* (14) *with state representation* (x, u) *and output* (y, z) *and assume that $\xi \in S$ is a regular point of S (see Definition 2). Assume that the system S is well formed,* i. e., $\mathrm{span}\,\{dt, dx, du\} = \mathrm{span}\,\{dt, dx, d\dot{x}\}$[12]. *Let Y be the local output subsystem corresponding to the output y*[13]. *The following affirmations are equivalent:*

[11] Note that $S/(Y \cup Z)$ is only a notation suggesting a quotient, but it does not have any precise meaning. Using Kähler differentials it is possible to translate part of the present results to the differential algebraic approach of (Fliess, 1989). In this setting, the zero dynamics may be interpreted as a quotient field.

[12] This is equivalent to say that $g(x)$ of (14) has independent columns (Rudolph, 1995). Note that, if the system is not well formed one my apply the Theorem 2 to system S with state (x, u) and input $\dot{u}$, which is well formed.

[13] The output subsystem Y exists and is unique according to (Pereira da Silva and Corrêa Filho, 2001, Theorem 4.3).

(i) *The system S is relatively flat around ξ with respect to subsystem Y and the output z is a (local) relatively flat output around ξ.*

(ii) *Around ξ we have $\widetilde{\rho}(z) = \mathrm{card}\, z$ and $\mathrm{span}\,\{dx\} \subset L_{\widetilde{k}^*-1}$.*

(iii) *Around ξ we have $\widetilde{\rho}(z) = \mathrm{card}\, z$ and $n - \dim Y_{k^*-1} + \sum_{i=0}^{k^*-1} \sigma_i - \widetilde{\rho}(z)(\widetilde{k}^*+1) + \sum_{j=0}^{\widetilde{k}^*} \widetilde{\sigma}_j = 0$.*

Remark 6 *Let z be a flat output of system Z. Remember that $\mathrm{card}\, z$ coincides with the differential dimension of Z (see Section 2.1). It follows easily from part (ii) of Lemma 2 and part (ii) of Theorem 2 that, for a relatively flat output z with respect to Y we must have $\mathrm{card}\, z = \widetilde{\rho}(z) = \mathrm{card}\, u - \rho(y) = m - \rho(y)$.*

Proof.
(i) $\Rightarrow$ (ii). Note first that, by similar arguments to the ones of the proof (of necessity) of Theorem 1, one shows easily that

$$\text{If } z \text{ is a relatively flat output, then } \widetilde{\rho}(z) = \mathrm{card}\, z. \tag{22}$$

By definition, (i) implies that $\mathrm{span}\left\{dt, (dy^{(k)} : k \in I\!N)\right\} + \mathrm{span}\,\{dt, (dz^{(k)} : k \in I\!N)\} = T^*S$ inside an open neighborhood U of ξ. In particular, for every $\nu \in U$, there must exist some k such that $L_k|_\nu \supset \mathrm{span}\,\{dx\}\,|_\nu$. By part 7 of Lemma 2 it follows that around an open neighborhood of ξ we must have $L_{\widetilde{k}^*} \supset \mathrm{span}\,\{dx\}$, showing (ii).
(ii) $\Rightarrow$ (i). Since $\widetilde{\rho}(z) = \mathrm{card}\, z$, by lema 3, S is i-decomposed by $\mathcal{F} = \{Y, Z\}$ where Y is the output subsystem and Z is a flat subsystem with flat output z. To show (i) it is enough to show that $T^*S = \mathrm{span}\left\{dt, (dy^{(k)} : k \in I\!N)\right\} + \mathrm{span}\left\{dt, (dz^{(k)} : k \in I\!N)\right\}$. Since $\mathrm{span}\,\{dx\} \subset L_{\widetilde{k}^*}$, then $\mathrm{span}\left\{dt, dx^{(k)} : k \in I\!N\right\} \subset \mathrm{span}\left\{dt, (dy^{(k)} : k \in I\!N)\right\} + \mathrm{span}\left\{dt, (dz^{(k)} : k \in I\!N)\right\}$. Note now that, the condition $\mathrm{span}\,\{dt, dx, d\dot{x}\} = \mathrm{span}\,\{dt, dx, du\}$ implies, by derivation, that we have $\mathrm{span}\left\{dt, dx^{(k)} : k \in I\!N\right\} = T^*S$, completing the proof of (ii)$\Rightarrow$(i).
(ii)$\Leftrightarrow$(iii). By Lemma 1 (see also remark 2) and Lemma 2 , it follows easily that

$$\dim \mathcal{L}_{k^*} = n + \sum_{i=0}^{k^*-1} \sigma_i + \sum_{j=0}^{\widetilde{k}^*} (\rho(y) + \widetilde{\sigma}_j) \tag{23}$$

By Lemma 2 we have that $\mathrm{span}\,\{dx\} \subset L_{\widetilde{k}^*-1}$ is equivalent to $\mathrm{span}\,\{dx\} \subset L_{\widetilde{k}^*}$, which is in turn equivalent to saying that

$$\left(\mathrm{span}\,\{dx\} \subset L_{\widetilde{k}^*-1}\right) \Leftrightarrow \left(\dim \frac{\mathcal{L}_{\widetilde{k}^*}}{L_{\widetilde{k}^*}} = 0\right) \tag{24}$$

By (23), (24) and part (ii) of Lemma 3, then the equivalence between (ii) and (iii) folows. $\square$

Remark 7 *Note that condition (iii) is equivalent to saying that the zero dynamics of Figure 1 is absent (see the remark after Lemma 3).*

6 DAEs

In this section we consider DAEs of the form (13). We show that the relative-decoupling and relative-flatness for system (14) implies respectively decoupling and flatness for system (13). We shall consider *regular* DAEs (see Definition 2). The following result is instrumental.

Theorem 3 *(Pereira da Silva and Corrêa Filho, 2001, Theo. 4.3) Let S be the system given by* (14a)–(14b) *with output y. Around a point ξ such that the codistributions* (15a)–(15b) *are nonsingular for $k = 0, 1, \ldots, n$, where $n = \dim x$, there exists a classic state representation $\widetilde{x} = (x_a, x_b)$, $\widetilde{u} = (u_a, u_b)$ of S of the form*

$$\dot{x}_a = f_a(t, x_a, u_a) \tag{25a}$$

$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \tag{25b}$$

in a way that $\mathcal{Y} = \text{span}\left\{dt, (dy^{(k)} : k \in I\!N)\right\} = \text{span}\left\{dt, dx_a, (du_a^{(j)} : j \in I\!N)\right\}$. *This state representation is adapted to the output subsystem Y, i. e.,* (25a) *are local (classical) state equations for Y. Furthermore,* $\text{span}\,\{x_b\} + \mathcal{Y} = \text{span}\,\{x\} + \mathcal{Y}$, $\text{span}\,\{x_b, u_b\} + \mathcal{Y} = \text{span}\,\{x, u\} + \mathcal{Y}$ *and the set of functions $\{x_a, u_a\}$ can be locally chosen as a subset of $\{y^{(k)} : k \in I\!N\}$.*

It can be shown that a regular DAE defined by (13a)–(13b) can be regarded as an immersed system in the explicit system S defined by (14). This result is the Proposition 1 bellow, whose proof is based on the last theorem.

Proposition 1 *(Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva and Watanabe, 2002) Let S be the system associated to* (14). *Let Γ be the subset of S defined by $\Gamma = \{\xi \in S \mid y^{(k)}(\xi) = 0, k \in I\!N\}$. Suppose that Γ is nonempty and every $\xi \in \Gamma$ is a regular point of the codistributions $\mathbb{Y}_k, Y_k, \mathcal{Y}_k, k = 0, \ldots, n$ (see* (15a)-(15b)*). Then the subset $\Gamma \subset S$ has a canonical structure of immersed (embedded) submanifold of S such that the canonical insertion is a Lie-Bäcklund immersion. Furthermore Γ admits a local classical state representation around every point $\xi \in \Gamma$.*

The idea of the proof of Proposition 1 is to consider the local state representation of the last theorem. It is shown that, $\{t, x_a, x_b, U_a, U_b\}$ and $\{t, x_b, U_b\}$ are respectively local coordinates for S and Γ and, in these coordinates[14]

$$\iota(t, x_b, U_b) = (t, 0, x_b, 0, U_b) \tag{26}$$

where $U_a = \{u_a^{(j)} : j \in I\!N\}$ and $U_b = \{u_b^{(j)} : j \in I\!N\}$.

By construction it is easy to verify that $\iota^*\omega \in \text{span}\,\{dt\}$ for a 1-form ω defined on S if and only if $\omega \in \mathcal{Y} = \text{span}\,\{dt, dx_a, dU_a\}$. The construction of the state x_b and the input u_b of Γ is canonical in the sense that

$$\iota^*\text{span}\,\{dt, dx\} = \text{span}\,\{dt, d(x_b \circ \iota)\} \tag{27}$$

$$\iota^*\text{span}\,\{dt, dx, du\} = \text{span}\,\{dt, d(x_b \circ \iota), d(u_b \circ \iota)\} \tag{28}$$

[14]Abusing notation, we let x_b and u_b stand respectively for $x_b \circ \iota$ and $u_b \circ \iota$.

Furthermore, the local state equations of Γ are given by $\dot{x}_b = f_b(t, 0, x_b, 0, u_b)$.

6.1 Relative-decoupling and DAEs

Let Γ be a regular implicit system defined by (13). Then the following result holds:

Theorem 4 *Let $\iota : \Gamma \to S$ and $(\tilde{x}, \tilde{u})$ be respectively the Lie-Bäcklund immersion and the state representation of the proof of Proposition 1, where $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$. Let $\widetilde{z} = z \circ \iota$, $\widetilde{x}_b = x_b \circ \iota$ and $\widetilde{u}_b = u_b \circ \iota$. Then $(\widetilde{x}_b, \widetilde{u}_b)$ is a classic local state representation for Γ, and $\widetilde{z}$ is a classic output. Furthermore, the relative structure at infinity of the output z of S with respect to output subsystem Y coincides with the structure at infinity of Γ with output z considering the state representation $(\widetilde{x}_b, \widetilde{u}_b)$. In particular the relative output rank $\widetilde{\rho}(z)$ of system* (14) *coincides with the output rank $\rho(z)$ of the DAE* (13).

Remark 8 *By the results of (Fliess, 1989; Di Benedetto* et al.*, 1989; Delaleau and Pereira da Silva, 1998*b*) it is clear that the dynamic input-ouput decoupling problem for a regular DAE defined by* (13) *is solvable if and only if $\widetilde{\rho}(z) = \mathrm{card}\, z$. It is now easy to verify that the Relative Dynamic Extension Algorithm, when specialized to $y \equiv 0$ furnishes a decoupling feedback law for the DAE. In fact, $y \equiv 0$ implies $\omega_k \equiv 0$, which simplifies a lot the RDEA. With this specialization, the RDEA becomes the DEA for system* (13).

Proof. (of Theorem 4) The fact that $(\widetilde{x}_b, \widetilde{u}_b)$ is a local state representation of Γ is a consequence of discussion above. To show that the sequence $\widetilde{\sigma}_k$ is the structure at infinity of the system Γ we must show that $\widetilde{\sigma}_k = \dim \frac{\widetilde{\mathcal{Z}}_k}{\widetilde{\mathcal{Z}}_{k-1}}$ where $\widetilde{\mathcal{Z}}_k = \mathrm{span}\left\{dt, d\widetilde{x}_b, d\widetilde{z}, \ldots, d\widetilde{z}^{(k)}\right\}$.

Now note that, since $\mathcal{Y} = \mathrm{span}\left\{dt, (dy^{(k)} : k \in I\!N)\right\} = \mathrm{span}\left\{dt, dx_a, dU_a\right\}$, from (26) we have that

$$\begin{aligned} \iota^* dt &= dt,\ \iota^*(\mathcal{Y}) = \iota^* \mathrm{span}\left\{dt, dx_a, dU_a\right\} = \mathrm{span}\left\{dt\right\} && (29)\\ \iota^* dx_b &= d\widetilde{x}_b,\ \iota^* dU_b = d\widetilde{U}_b && (30) \end{aligned}$$

From this and from (27), we have

$$\iota^* \mathcal{L}_k = \mathrm{span}\left\{\iota^* dt, \iota^* dx, \iota^* dy, \ldots, \iota^* dy^{(k^*+k)}, \iota^* dz, \ldots, \iota^* dz^{(k)}\right\} = \widetilde{\mathcal{Z}}_k \qquad (31)$$

Let $\gamma \in \Gamma$ such that $\iota(\gamma) = \xi$. Let $\mathcal{T} = \mathrm{span}\left\{dt\right\}|_\gamma \subset T^*_\gamma \Gamma$. Let $\pi_{\mathcal{T}} : T^*_\gamma \Gamma \to \frac{T^*_\gamma \Gamma}{\mathcal{T}}$ and $\pi_{\mathcal{Y}} : T^*_\xi S \to \frac{T^*_\xi S}{\mathcal{Y}|_\xi}$ be the canonical projections. It is not difficult to verify from (29) that the map $\Xi : \frac{T^*_\xi S}{\mathcal{Y}|_\xi} \to \frac{T^*_\gamma \Gamma}{\mathcal{T}}$ defined by $\Xi \circ \pi_{\mathcal{Y}} = \pi_{\mathcal{T}} \circ \iota^*$ is an isomorphism. In particular it follows from part 9 of Lemma 2 that $\widetilde{\sigma}_k = \dim \frac{\mathcal{L}_k + \mathcal{Y}}{\mathcal{L}_{k-1} + \mathcal{Y}} = \dim \frac{\mathcal{L}_k + \mathcal{Y}}{\mathcal{Y}} - \dim \frac{\mathcal{L}_{k-1} + \mathcal{Y}}{\mathcal{Y}} = \dim \Xi\left(\frac{\mathcal{L}_k + \mathcal{Y}}{\mathcal{Y}}\right) - \dim \Xi\left(\frac{\mathcal{L}_{k-1} + \mathcal{Y}}{\mathcal{Y}}\right) = \dim \frac{\iota^* \mathcal{L}_k}{\mathcal{T}} - \dim \frac{\iota^* \mathcal{L}_{k-1}}{\mathcal{T}} = \dim \frac{\widetilde{\mathcal{Z}}_k}{\mathcal{T}} - \dim \frac{\widetilde{\mathcal{Z}}_{k-1}}{\mathcal{T}} = \dim \frac{\widetilde{\mathcal{Z}}_k}{\widetilde{\mathcal{Z}}_{k-1}}$. $\square$

6.2 Relative-flatness and DAEs

In (Pereira da Silva and Corrêa Filho, 2001) it is shown that relative-flatness of (14) with respect to Y implies flatness of a regular DAE Γ defined by (13). Furthermore, it is shown there that the relatively flat output z of S with respect to Y is the flat output of Γ. In particular, the following result holds:

Corollary 1 *If one of the equivalent conditions of Theorem 2 holds for system* (14)*, then z is a local flat output of the regular DAE* (13)*.*

7 Examples

Reconsider now the system S of example 1, given in the introduction. It is easy to verify that the regularity conditions of Definition 2 hold for every point of the system with the exception of the singularities $\bar{v}_1 \neq \phi_1$. The computations Relative Dynamic Extension Algorithm have already been done in the introduction. For the preparation process, one gets $\sigma_0 = 0, \sigma_1 = 1$, $\sigma_2 = 2$, and so $k^* = 2$. For the algorithm itself, one obtains $\tilde{\sigma}_0 = 0$, $\tilde{\sigma}_1 = 0$, $\tilde{\sigma}_2 = 2$ and so $\tilde{k}^* = 2$. In particular, from Theorem 1, the problem of relative-decoupling with respect to subsystem Y is solvable. As the condition (iii) of Theorem 2 holds (it is easy to show that $\dim Y_1 = 4$), it follows that z is a relatively flat output with respect to subsystem Y. From Theorem 4 it follows that the implicit system Γ obtained by adding the constraint $y = 0$ can be decoupled by dynamic state feedback. From Corollary 1 it follows that Γ is flat with flat output z. It is also easy to show by direct computation, that the relative structure at infinity of S , with respect to subsystem Y, coincides with the structure at infinity of Γ (with ouput z for both systems).

Example 2 *Consider now a planar mechanical system that is formed by two punctual unitary masses of coordinates $x_1(t), x_2(t) \in I\!R^2$, that are connected by an ideal bar of lenght $L(t)$. Assume that, on the first mass, one can apply a control force of module $u_1(x_1 - x_2)$, with $u_1(t) \in I\!R$ (in other words, this force is in the direction defined by the bar). On the second mass, one can apply a control force $U = (u_2, u_3)^T \in I\!R^2$. One is interested in controlling the position $z_1 = (x_1 + x_2)/2$ of the center of mass of the two bodies and a variable defined by $z_2 = h^T(x_1 - x_2)$, which gives the information of the angle between the bar and a fixed direction $h \in I\!R^2$. Denoting the Lagrange multiplier associated to the constraint $\|x_1 - x_2\|^2 - L(t)^2 = 0$ by u_4, the following model can be easily obtained:*

$$
\begin{aligned}
\ddot{x}_1 &= (x_1 - x_2)u_1 + 2(x_1 - x_2)u_4 \\
\ddot{x}_2 &= U - 2(x_1 - x_2)u_4 \\
y(t) &= \|x_1 - x_2\|^2 - L^2(t) = 0 \\
z_1 &= (x_1 + x_2)/2 \\
z_2 &= h^T(x_1 - x_2)
\end{aligned}
$$

The state of the (nonconstrained) model is $(x_1, x_2, \dot{x}_1, \dot{x}_2)$*, but there is no need to write state equations, since the RDEA works only with derivatives. Computing the preparation process of RDEA, one gets* $\sigma_0 = 0, \sigma_1 = 0$*,* $\sigma_2 = 2$*,* $\bar{y}_2^{(2)} = \ddot{y} = -(2\dot{L}^2 + 2L\ddot{L}) + 2\|\dot{x}_1 - \dot{x}_2\|^2 + 2yu_1 + 8(y + L^2)u_4 - 2(x_1 - x_2)^T U$ *and the feedback* $u_4 = [2\|\dot{x}_1 - \dot{x}_2\|^2 + 2(x_1 - x_2)^T U + 2\dot{L}^2 + 2L\ddot{L} + \bar{y}_2^{(2)}]/(8y + 8L^2)$*. Computing the RDEA, one gets* $\tilde{\sigma}_0 = 0$*,* $\tilde{\sigma}_1 = 0$*,* $\tilde{\sigma}_2 = 3$ *and* $z^{(2)} = \bar{z}_2^{(2)} = a(L, \dot{L}, \ddot{L}, x_1, x_2, \dot{x}_1, \dot{x}_2, \bar{y}_2^{(2)}) + c(x_1, x_2, \dot{x}_1, \dot{x}_2)\bar{u}$*, where* $\bar{u} = (u_1 u_2 u_3)^T$*. The matrix* c*, when multiplied by a triangular matrix* R*, gives a triangular matrix* F*, and these matrices are given by*

$$c = \begin{pmatrix} (x_1 - x_2)/2 & I_2/2 \\ h^T(x_1 - x2)/2 & -h^T[I_2 + (x_1 - x_2)(x_1 - x_2)^T]/2 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 \\ -(x_1 - x2)/2 & I_2 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & I_2/2 \\ -h^T(x_1 - x_2)[1 + (y + L^2)/2] & -h^T[I_2 + (x_1 - x_2)(x_1 - x_2)^T]/2 \end{pmatrix}$$

where I_2 *is the identity matrix of dimension two. It follows that* c *is nonsingular iff* $y \neq -L^2$ *and* h *is not orthogonal to the bar. The feedback* $\bar{u} = -c^{-1}a + c^{-1}\bar{v}$ *produces* $z^{(2)} = \bar{v}$ *in closed loop, and so it is a solution of the relative-decoupling problem. Note that* $n = 8$*,* $k^* = \tilde{k}^* = 2$*,* $\dim z = \tilde{\rho}(z) = 3$ *and* $\dim Y_{k^*-1} = 2$*. Hence, from Theorem 2 part (iii),* z *is a relatively flat output. In particular, from Theorem 4 and Corollary 1, when the constraint* $y(t) = 0$ *is added, the given feedback is a decoupling and linearizing feedback law for the corresponding DAE.*

8 Conclusions

The results of this paper may be useful for studying flatness and the dynamic decoupling problem for implicit systems. It is important to point out that our results show effective ways for computing the output rank and control laws for dynamic feedback linearization and/or decoupling of an implicit system Γ, without the need to transform Γ into an explicit system. In fact, note that the relative dynamic extension algorithm for affine systems relies only on sums, multiplications and matrix inversions.

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A Computation of the static-feedback of the kth step of RDEA

Let

$$\begin{array}{rcl} \bar{z}_k^{(k)} & = & \bar{a}(t, x_{k-1}) + \bar{b}(t, \widetilde{x}_{k-1})\omega_k + \bar{c}(t, \widetilde{x}_{k-1})\mu_{k-1} \\ \widehat{z}_k^{(k)} & = & \widehat{a}(t, \widetilde{x}_{k-1}) + \widehat{b}(t, \widetilde{x}_{k-1})\omega_k + \widehat{c}(t, \widetilde{x}_{k-1})\mu_{k-1} \end{array}$$

Up to a reordering of the components of z, we may assume that rank c = rank $\bar{c} = \widetilde{\sigma}_k$ is locally constant. Up to a reordering of the components of μ_{k-1}, we may suppose that $\bar{c} = [\bar{c}_{11}\ \bar{c}_{12}]$,where $\bar{c}_{11}$ is locally nonsingular. Then define locally

$$\beta_k(t, \widetilde{x}_{k-1}) = \left(\begin{array}{cc} \bar{c}_{11} & \bar{c}_{12} \\ 0 & I \end{array} \right)^{-1} = \left(\begin{array}{cc} \bar{c}_{11}^{-1} & -\bar{c}_{11}^{-1}\bar{c}_{12} \\ 0 & I \end{array} \right)$$

$$\bar{\alpha}_k(t, \widetilde{x}_{k-1}) + \widehat{\alpha}_k(t, \widetilde{x}_{k-1})\omega_k = \beta_k \left(\begin{array}{c} -\bar{a} - \bar{b}\omega_k \\ 0 \end{array} \right)$$

and let

$$\mu_{k-1} = \bar{\alpha}_k(t, \widetilde{x}_{k-1}) + \widehat{\alpha}_k(t, \widetilde{x}_{k-1})\omega_k + \beta_k(t, \widetilde{x}_{k-1})v_k$$

Then it is easy to verify this choice of $(\bar{\alpha}_k, \widehat{\alpha}_k, \beta_k)$ is such that (19) holds.

B Proof of Lemma 2

Proof. Along this proof, we shall write $\omega = \omega_0$. By (20), it is clear that $\omega_k = \omega^{(k)}$ for $k = 0, 1, \ldots$. The following remark is instrumental for the proof:

Remark 9 *Assume that $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$ is a state representation around ξ. Then by definition, $\Psi = \{t, \widetilde{x}_{k-1}, (\omega_k^{(j)}, \mu_{k-1}^{(j)} : j \in I\!N)\}$ is a local coordinate chart around ξ. In particular, the differentials of the functions of Ψ are locally independent.*

We give first a geometric description of the RDEA. Let $(\widetilde{x}_{-1}, \widetilde{u}_{-1})$ be the state representation of system S with output $z^{(0)}$ defined by (17). In step $k-1$ of this algorithm ($k = 0, 1, 2, \ldots$) one has constructed a classical (local) state representation $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$, where $\widetilde{u}_{k-1} = (\omega_k, \mu_{k-1})$, with output $z^{(k)}$ defined on an open neighborhood U_{k-1} of $\xi \in S$. Assume that span $\{dt, d\widetilde{x}_{k-1}, d\omega_k, dz^{(k)}\}$

is nonsingular around ξ[15]. Note that we can give the following geometric description of the step k of RDEA:

- (S1) By remark 9, the set $\{dt, d\widetilde{x}_{k-1}, d\omega_k\}$ is locally independent. Choose $\bar{z}_k$ (possibly among the components of z) by completing $\{dt, d\widetilde{x}_{k-1}, d\omega_k\}$ into a basis $\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ for span $\{dt, d\widetilde{x}_{k-1}, d\omega_k, dz^{(k)}\}$.

- (S2) Now choose $\widehat{\mu}_k$ (possibly among the components of $\widetilde{\mu}_{k-1}$) by completing $\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ into a basis $\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}, d\widehat{\mu}_k\}$ of span $\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\}$. According to the Section 2.1, this defines a local state feedback with new input[16] (ω_k, v_k), where $v_k = (\bar{z}_k^{(k)}, \widehat{\mu}_k)$. By construction, this state feedback has the property (19).

- (S3) Define the new state representation $(\widetilde{x}_k, \widetilde{u}_k)$ by taking $\widetilde{x}_k = (\widetilde{x}_{k-1}, \omega_k, \bar{z}_k^{(k)})$, and $\widetilde{u}_k = (\dot{\omega}_k, \mu_k)$, where $\mu_k = (\bar{z}_k^{(k+1)}, \widehat{\mu}_k)$. This is an extension of the state of the form (20).

Note that, (see the end of Section 2.1), we have that (S1), (S2) and (S3) produce a new local state representation $(\widetilde{x}_k, \widetilde{u}_k)$ of system S defined in an open neighborhood $U_k \subset U_{k-1}$ of ξ. Note that the steps (S1)-(S2)-(S3) describe the procedure of the step k of RDEA, that could be performed, at least theoretically, for nonaffine systems[17]. In particular our geometric interpretation of Lemma 1 holds for nonaffine systems, if one considers that (S1)-(S2)-(S3) are the procedure of step k.

(1 and 2). We show first that the state representation $(\widetilde{x}_k, \widetilde{u}_k)$ is classical, *i. e.*, span $\left\{d\dot{\widetilde{x}}_k\right\} \subset$ span $\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$. This property holds for $(\widetilde{x}_{-1}, \widetilde{u}_{-1})$. By induction, assume that it holds for $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$. Then from (S1),(S2) and (S3) we have span $\left\{d\dot{\widetilde{x}}_k\right\} \subset$ span $\left\{dt, d\widetilde{x}_{k-1}, d\dot{\widetilde{x}}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\bar{z}_k^{(k+1)}\right\}$ $\subset$ span $\left\{dt, d\widetilde{x}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\bar{z}_k^{(k+1)}, d\mu_k\right\}$ = span $\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\bar{z}_k^{(k+1)}\} \subset$ span $\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$. By (S1), (S2), (S3) notice that $d\widehat{z}_k^{(k+1)} \in$ span$\{dt, d\widetilde{x}_{k-1}, d\dot{\widetilde{x}}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\dot{\bar{z}}_k^{(k)}\} \subset$ span$\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}, d\omega_k, d\dot{\omega}_k, d\bar{z}_k^{(k)}, d\dot{\bar{z}}_k^{(k)}\}$, and so, span$\{dz^{(k+1)}\} \subset$ span $\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$.

We show now 1 and 2 by induction. Since $\widetilde{x}_{-1} = x_{k^*-1}$, by part 1 of Lemma 1 it follows that span $\{dt, d\widetilde{x}_{-1}\} = \mathcal{Y}_{k^*} = \mathcal{L}_{-1}$. By remark 2 and from parts 1 and 8 of Lemma 1, it follows that span $\{dt, d\widetilde{x}_{-1}, d\widetilde{u}_{-1}\} = \mathcal{Y}_{k^*} +$ span $\{du\} = \mathcal{L}_0 +$ span $\{du\}$, where this last equality follows from the fact that $\mathbb{Z}_0 =$ span $\{dz\} \subset$ span $\{dt, dx, du\} \subset \mathcal{Y}_{k^*}$. Hence, one sees that 1 and 2 are satisfied for $k = 0$. Assume that, in the step $k-1$ we have a local state representation $(\widetilde{x}_{k-1}, \widetilde{u}_{k-1})$ satisfying 1 and 2. Choose a partition

[15]It is easy to show that this is equivalent to the fact that the matrix $c_k(t, \widetilde{x}_{k-1})$ of (6) has constant rank around ξ.

[16]In fact, by construction we have that $\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\}$ and $\{dt, d\widetilde{x}_{k-1}, d\omega_k, dv_k\}$ are both local basis of the same codistribution.

[17]In this case the computations are much more difficult since one may apply the inverse function theorem to compute the feedback.

$z^{(k)} = (\bar{z}_k^{(k)}, \widehat{z}_k^{(k)})$ in a way that (S1) is satisfied and construct $\widehat{\mu}_k$ satisfying (S2). By 1 for $k-1$ and (S1) and from the fact that $\omega_k = \omega^{(k)}$, it follows that $\text{span}\{dt, d\widetilde{x}_k\} = \text{span}\{dt, d\widetilde{x}_{k-1}, d\omega^{(k)}, d\bar{z}_k^{(k)}\} = \text{span}\{dt, d\widetilde{x}_{k-1}, d\omega^{(k)}, dz_k^{(k)}\} = \mathcal{L}_{k-1} + \text{span}\left\{d\omega^{(k)}, dz^{(k)}\right\}$. From the fact that $\omega_0 = \omega = \bar{y}_{k^*}^{(k*)}$, by (15) and part 8 of Lemma 1, it follows that $\mathcal{L}_{k-1} + \text{span}\left\{d\omega^{(k)}, dz^{(k)}\right\} = \mathcal{L}_k$, showing 1 for k.

We show now that if 2 holds for $k-1$, then $\text{span}\{dt, d\widetilde{x}_k,\ d\widetilde{u}_k\} = \mathcal{L}_{k+1} + \text{span}\{du\}$, completing the induction. By (S1),(S2) and (S3) and from the fact that $\text{span}\left\{d\bar{z}_k^{(k+1)}\right\} \subset \text{span}\left\{dz^{(k)}\right\} \subset \text{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\}$, it follows that

$$\begin{aligned}\text{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\} &= \text{span}\{dt, d\widetilde{x}_{k-1}, d\omega^{(k)}, d\bar{z}_k^{(k)}, d\mu_k\} + \\ &\quad\ \text{span}\{d\omega^{(k+1)}, d\bar{z}_k^{(k+1)}\} \\ &= \text{span}\{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\} + \text{span}\left\{d\omega^{(k+1)}, dz_k^{(k+1)}\right\}\end{aligned}$$

By the induction hypothesis, we have $\text{span}\{dt, d\widetilde{x}_k, d\widetilde{u}_k\} = \mathcal{L}_k + \text{span}\{du\} + \text{span}\left\{d\omega^{(k+1)}, dz_k^{(k+1)}\right\}$. By part 8 of Lemma 1 and the fact that $\omega = \bar{y}_{k^*}^{(k^*)}$, this shows 2 for k.

(3, 5, 6, 7). Note now that, since $\{dt, d\widetilde{x}_k\} = \{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ is a basis of $\mathcal{L}_k$ and $\{dt, d\widetilde{x}_{k-1}\}$ is a basis of $\mathcal{L}_{k-1}$, it follows that

$$\{d\omega_k\} \text{ is independent } \bmod\ \mathcal{L}_{k-1}. \tag{32a}$$

In particular, $\{d\omega_k\}$ is also independent $\bmod\ L_{k-1}$. Since $\omega = \bar{y}_{k^*}^{(k^*)}$ and $\text{card}\,\omega_k = \text{card}\,\omega = \rho(y)$, by remark 1, we see that

$$\dim \mathcal{L}_k - \dim \mathcal{L}_{k-1} \geq \rho(y) \tag{32b}$$

$$\dim L_k - \dim L_{k-1} \geq \rho(y) \tag{32c}$$

We show first that

$$\dim L_k(\nu) - \dim L_{k-1}(\nu) \geq \dim L_{k+1}(\nu) - \dim L_k(\nu) \text{ for every } \nu \in S_k \tag{33}$$

In fact, if the 1-forms $\{\eta_1, \ldots, \eta_s\} \subset L_k$ are linearly dependent mod L_{k-1}, *i. e.*, if $\alpha_0 dt + \sum_{i=1}^s \alpha_i \eta_i + \sum_{i=1}^r \sum_{j=0}^{k^*+k-1} \beta_{ij} dy_i^{(j)} + \sum_{i=1}^p \sum_{j=0}^{k-1} \gamma_{ij} dz_i^{(j)} = 0$, then differentiation in time gives $\dot{\alpha}_0 dt + \sum_{i=1}^s (\dot{\alpha}_i \eta_i + \alpha_i \dot{\eta}_i) + \sum_{i=1}^r \sum_{j=0}^{k^*+k-1} (\dot{\beta}_{ij} dy_i^{(j)} + \beta_{ij} dy_i^{(j+1)}) + \sum_{i=1}^p \sum_{j=0}^{k-1} (\dot{\gamma}_{ij} dz_i^{(j)} + \gamma_{ij} dz_i^{(j+1)}) = 0$. In other words, the 1-forms $\dot{\eta}_1, \ldots, \dot{\eta}_s$ are linearly dependent mod L_k. Let $\xi \in S_k$. From the nonsingularity of $L_j, \mathcal{L}_j, j = 0, \ldots, k$ in S_k, if $\dim L_k - \dim L_{k-1} = l + \rho(y)$ in $\xi \in S_k$, then by (32a) we may choose a partition $z = (\bar{z}^T, \widehat{z}^T)$ such that $\bar{z}$ has l components and we locally have $L_k = \text{span}\left\{d\omega^{(k)}, d\bar{z}^{(k)}\right\} + L_{k-1}$. Let $\widehat{z}_j$ be any component of $\widehat{z}$ for $j \in \lfloor p-l \rceil$. By construction we have that $\{d\widehat{z}_j^{(k)}, d\omega^{(k)}, d\bar{z}^{(k)}\}$ is linearly dependent $\bmod\ L_{k-1}$ for every $j \in \lfloor p-l \rceil$. From the remark above it follows that the set $\{d\widehat{z}_j^{(k+1)}, d\omega^{(k+1)}, d\bar{z}^{(k+1)}\}$ is (locally) dependent mod L_k for every $j \in \lfloor p-l \rceil$, showing (33). In particular the sequence $\widetilde{\rho}_k$ is nonincreasing.

We show now that

$$\dim \mathcal{L}_k(\nu) - \dim \mathcal{L}_{k-1}(\nu) \leq \dim \mathcal{L}_{k+1}(\nu) - \dim \mathcal{L}_k(\nu) \text{ for every } \nu \in S_k \tag{34}$$

Assume that $(\widetilde{x}_k, \widetilde{u}_k)$ is a state representation constructed around a neighborhood U_k of a point $\xi \in S_k$ and satisfying (S1), (S2), 1 and 2. Since $(d\dot{\omega}_k, d\dot{\bar{z}}_k^{(k)}) \subset \widetilde{u}_k$, by remark 9 it follows that the components of $\{d\dot{\omega}_k, d\dot{\bar{z}}_k^{(k)}\}$ are independent mod $\mathcal{L}_k$. Hence $\bar{z}_{k+1}^{(k+1)}$ may be chosen satisfying 3, showing 3 and (34). In particular, $\widetilde{\sigma}_{k+1} \geq \widetilde{\sigma}_k$.

To show the convergence of sequences $\widetilde{\rho}_k$ and $\widetilde{\sigma}_k$ for some $k^* \leq n$, assume that $\nu \in S_k$. Denote span$\{dx\}$ by X. Then $\mathcal{L}_k = X + L_k$ and thus

$$\dim \mathcal{L}_k(\nu) = \dim X(\nu) + \dim L_k(\nu) - \dim(L_k(\nu) \cap X(\nu)). \tag{35}$$

Denote for $k \in I\!N$:

$$\begin{array}{rcl} s_k(\nu) & = & \dim \mathcal{L}_k(\nu) - \dim \mathcal{L}_{k-1}(\nu) - \rho(y) \\ p_k(\nu) & = & \dim L_k(\nu) - \dim L_{k-1}(\nu) - \rho(y) \end{array}$$

Note that $\widetilde{\rho}_k = p_k(\nu)$ and $\widetilde{\sigma}_k = s_k(\nu)$ are constant for every $\nu \in S_k$. From (35) we also have

$$s_k(\nu) = p_k(\nu) - \dim(L_k(\nu) \cap X(\nu)) + \dim(L_{k-1}(\nu) \cap X(\nu)). \tag{36}$$

We show now that

$$\begin{array}{c} \text{if there exists } \widetilde{k}^* \text{ and some } \nu \in S_{\widetilde{k}^*} \text{ such that } s_{\widetilde{k}^*}(\nu) = p_{\widetilde{k}^*}(\nu) = \widetilde{\rho}, \\ \text{then } s_{\widetilde{k}^*+1}(\zeta) = p_{\widetilde{k}^*+1}(\zeta) = \widetilde{\rho} \text{ for every } \zeta \in S_{\widetilde{k}^*}. \end{array} \tag{37}$$

Note that, from (37), a simple induction shows that $s_k(\zeta) = p_k(\zeta) = \widetilde{\rho}$ for every $k \geq \widetilde{k}^*$ and $\zeta \in S_{\widetilde{k}^*}$. Furthermore, this last affirmation implies that $S_k = S_{\widetilde{k}^*}$ for $k \geq \widetilde{k}^*$.

To show (37), assume that $p_{\widetilde{k}^*}(\nu) = s_{\widetilde{k}^*}(\nu) = \widetilde{\rho}$ for some $\nu \in S_{\widetilde{k}^*}$. From (36), it follows that $-\dim(L_{\widetilde{k}^*}(\nu) \cap X(\nu)) + \dim(L_{\widetilde{k}^*-1}(\nu) \cap X(\nu)) = 0$. Since the dimensions of $L_{\widetilde{k}^*} \cap X$ and of $L_{\widetilde{k}^*-1} \cap X$ are constant in $S_{\widetilde{k}^*}$, it follows that

$$p_{\widetilde{k}^*}(\zeta) = s_{\widetilde{k}^*}(\zeta) = \widetilde{\rho}, \text{for every } \zeta \in S_{\widetilde{k}^*}. \tag{38}$$

So, for every $\zeta \in S_{\widetilde{k}^*}$, we have $-\dim(L_{\widetilde{k}^*}(\zeta) \cap X(\zeta)) + \dim(L_{\widetilde{k}^*-1}(\zeta) \cap X(\zeta)) = 0$. Note from (36) that

$$s_{\widetilde{k}^*+1}(\zeta) - p_{\widetilde{k}^*+1}(\zeta) = -\dim(L_{\widetilde{k}^*+1}(\zeta) \cap X(\zeta)) + \dim(L_{\widetilde{k}^*}(\zeta) \cap X(\zeta)) \tag{39}$$

for every $\zeta \in S_{\widetilde{k}^*}$. By (33), (34) and (38), it follows that $s_{\widetilde{k}^*+1}(\zeta) - p_{\widetilde{k}^*+1}(\zeta) \geq 0$. Since the sequence $l_k = \dim L_k \cap X|_\zeta$ is nondecreasing for a fixed $\zeta \in S_{\widetilde{k}^*}$, we have $-\dim(L_{\widetilde{k}^*+1}(\zeta) \cap X(\zeta)) + \dim(L_{\widetilde{k}^*}(\zeta) \cap X(\zeta)) \leq 0$, the only possibility is to have both sides of (39) equal to zero for every $\zeta \in S_{\widetilde{k}^*}$. Using (33) and (34) again, then (37) follows. Note that a simple induction shows that (37) implies 7.

To complete the proof of 5, 6 and 7 it suffices to show the existence of $\widetilde{k}^*$ such that the assumption of (37) holds. Now take $\nu = \xi$ and recall that $l_k = \dim(L_k(\nu) \cap X(\nu))$ is nondecreasing for $k = 0, \dots, n$ and it is least than or equal to $n = \dim X$. In particular, there exists some $\widetilde{k}^* \leq n$ such that $\dim(L_{\widetilde{k}^*}(\nu) \cap X(\nu)) = \dim(L_{\widetilde{k}^*-1}(\nu) \cap X(\nu))$. By the definition of $S_{\widetilde{k}^*}$ given in 6, we have $\nu = \xi \in S_{\widetilde{k}^*}$. From (36) it is clear that $s_{\widetilde{k}^*}(\nu) = p_{\widetilde{k}^*}(\nu)$.

(4). Note that, by 2,

$$\mathbb{B}_k = \{dt, d\widetilde{x}_{k-1}, d\widetilde{u}_{k-1}\} = \{\underbrace{dt, dx_{k-1}, d\omega_k, d\bar{z}_{k-1}^{(k)}}_{\mathbb{H}_k}, d\widehat{\mu}_{k-1}\}$$

is a basis of $\mathcal{L}_k + \operatorname{span}\{du\}$. By 1 and (S1), $\mathbb{R}_k = \{dt, d\widetilde{x}_{k-1}, d\omega_k, d\bar{z}_k^{(k)}\}$ is a basis of $\mathcal{L}_k$. By (S2), $\widehat{\mu}_k$ is chosen in a way to complete $\mathbb{R}_k$ into a basis of $\mathcal{L}_k + \operatorname{span}\{du\}$. By 3 we may choose $\bar{z}_k^{(k)} \supset \bar{z}_{k-1}^{(k)}$. With this choice we have $\mathbb{R}_k \supset \mathbb{H}_k$. Since one may complete the basis $\mathbb{R}_k$ by choosing elements of $\mathbb{B}_k$ and $\mathbb{B}_k = \mathbb{H}_k \cup \{d\widehat{\mu}_{k-1}\}$, it follows easily that $\widehat{\mu}_k$ may be chosen among the elements of $\widehat{\mu}_{k-1}$.

(8). The first part of 8 follows easily from 3, from the fact that $\operatorname{card} \bar{z}_k = \widetilde{\sigma}_k$, from 5 and from (32a). The second part of 8 follows easily from the equality $\operatorname{card} \bar{z}_k = \widetilde{\sigma}_k$, from the fact that the components of $(d\omega_k^{(k)}, d\bar{z}_k^{(k+1)})$ are independent mod L_k and from the fact that $\widetilde{\sigma}_k = \widetilde{\rho}_k = \widetilde{\rho}$ for $k \geq \widetilde{k}^*$.

(9) Recall that, by remark 9 (for k) and from the fact that $\omega_k = \omega^{(k)}$, we have that the set $\Psi = \{dt, d\widetilde{x}_k, (d\omega^{(k+1+j)}, d\bar{z}_k^{(k+1+j)}, \widehat{\mu}_k^{(j)} : j \in I\!N)\}$ is (locally) independent. Let $\Omega_k := \{dt, d\widetilde{x}_k, (d\omega^{(k+1+j)} : j \in I\!N)\} \subset \Psi$. By 1, from the fact that $\omega^{(\alpha)} = \bar{y}_{k^*}^{(k^*+\alpha)}$ and from part 8 of Lemma 1, it follows that Ω_k is a local basis of $\mathcal{L}_k + \mathcal{Y}$. In particular $\Omega_{k+1} = \{dt, d\widetilde{x}_{k+1}, (d\omega^{(k+2+j)} : j \in I\!N)\} = \{dt, (d\widetilde{x}_k, d\omega^{(k+1)}, d\bar{z}_{k+1}^{(k+1)}), (d\omega^{(k+2+j)} : j \in I\!N)\} = \Omega_k \cup \{d\bar{z}_{k+1}^{(k+1)}\}$ is a local basis of $\mathcal{L}_{k+1} + \mathcal{Y}$. Since $\operatorname{card} \bar{z}_{k+1}^{(k+1)} = \widetilde{\sigma}_{k+1}$, we have that 9 holds. □

C Proof of Lemma 3

Proof. (i), (ii) and (iii). If $\widetilde{\rho}(z) = \operatorname{card} z$, the RDEA will construct in the step $\widetilde{k}^*$ a state representation $(\widetilde{x}_{\widetilde{k}^*}, \widetilde{u}_{\widetilde{k}^*})$ of system S with the properties of Lemma 2. In particular $\operatorname{span}\left\{d\widetilde{x}_{\widetilde{k}^*}\right\} = \mathcal{L}_{\widetilde{k}^*} = \mathcal{Y}_{k^*+\widetilde{k}^*} + \mathbb{Z}_{\widetilde{k}^*}$.

By part 5 of Lemma 2, since $\widetilde{\rho}_k$ is nondecreasing, it follows that we (locally) have $\dim \frac{L_k}{L_{k-1}} = \dim \frac{Y_{k^*+k} + \mathbb{Z}_k}{Y_{k^*+k-1} + \mathbb{Z}_{k-1}} = \operatorname{card} z + \rho(y)$. By Lemma 1, we have $\dim \frac{Y_{k^*+k}}{Y_{k^*+k-1}} = \rho(y)$ for $k \in I\!N$. Then it is easy to show that $\dim (Y_{k^*+k} + \mathbb{Z}_k) = \dim Y_{k^*+k} + \dim \mathbb{Z}_k$ for $k \in I\!N$. In particular $Y_{k^*+k} \cap \mathbb{Z}_k = \{0\}$ for $k \in I\!N$.

Now choose a (local) state transformation $(t, \widetilde{x}_{\widetilde{k}^*}) \leftrightarrow (t, \xi)$ such that $\xi =$

$(\widehat{x}, \eta, z, \ldots, z^{(\widetilde{k}^*)})$, and

$$\begin{aligned} \operatorname{span}\{dt, d\eta\} &= Y_{k^*+\widetilde{k}^*} & (40a) \\ \operatorname{span}\left\{dt, d\eta, dz, \ldots, dz^{(\widetilde{k}^*)}, d\widehat{x}\right\} &= \mathcal{L}_{\widetilde{k}^*} = \mathcal{Y}_{k^*+\widetilde{k}^*} + \mathbb{Z}_{\widetilde{k}^*} & (40b) \end{aligned}$$

With this choice, note that

$$\operatorname{card} \widehat{x} = \dim \frac{\mathcal{L}_{\widetilde{k}^*}}{L_{\widetilde{k}^*}} \tag{41}$$

By (40) and (41) it follows that (iii) holds.

By construction we have, by part 8 of Lemma 1, that $\operatorname{span}\{dt, d\eta, (d\omega^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)\} = \operatorname{span}\left\{dt, (dy^{(k)} : k \in I\!N)\right\} = \mathcal{Y}$. Note first that $\operatorname{span}\{d\eta\} \subset \mathcal{Y}$ and so $\operatorname{span}\{d\dot{\eta}\} \subset \mathcal{Y}$. So it is easy to show that the system Y with local coordinates $(t, d\eta, (d\omega^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)$ and Cartan field $\partial_Y = \frac{\partial}{\partial t} + \frac{\partial}{\partial \eta}\dot{\eta} + \sum_{j \in I\!N} \frac{\partial}{\partial \omega^{(j)}_{\widetilde{k}^*+1}} \omega^{(j+1)}_{\widetilde{k}^*+1}$ is an output subsystem of S with respect to the output y. The map $\pi_Y\left(t, \widehat{x}, \eta, (z^{(j)}, \omega^{(j)}_{\widetilde{k}^*+1}, \widehat{\mu}^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)\right) = \left(t, \eta, (\omega^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)\right)$ is the corresponding Lie-Bäcklund submersion. Now let Z the subsystem with local coordinates $(t, (z^{(k)} : k \in I\!N))$ and Cartan field $\partial = \frac{\partial}{\partial t} + \sum_{j \in I\!N} \frac{\partial}{\partial z^{(j)}} z^{(j+1)}$ and corresponding Lie-Bäcklund submersion $\pi(t, \widehat{x}, \eta, (\omega^{(j)}_{\widetilde{k}^*+1}, z^{(j)}, \widehat{\mu}^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)) = (t, (z^{(j)} : j \in I\!N))$. By construction it is clear that S is i-decomposed by the family $\mathcal{F}_0 = \{Y, Z\}$ and the system has the structure of Figure 1. This shows (i), (ii) and (iii).

(iv). Now define the family of subsystems S_i with local coordinates $\{t, (z_i^{(j)} : j \in I\!N)\}$, Cartan field $\partial_i = \frac{\partial}{\partial t} + \sum_{j \in I\!N} \frac{\partial}{\partial z^{(j)}} z^{(j+1)}$ and corresponding Lie-Bäcklund submersions $\pi_i(t, \widehat{x}, \eta, (\omega^{(j)}_{\widetilde{k}^*+1}, z^{(j)}, \widehat{\mu}^{(j)}_{\widetilde{k}^*+1} : j \in I\!N)) = (t, (z_i^{(j)} : j \in I\!N))$. It is clear that the family of subsystems $\mathcal{F} = \{Y, (S_i : i \in \lfloor p \rceil)\}$ has the properties of Definition 6. □