ON PROPER REALIZATIONS OF EXPLICIT AND IMPLICIT NONLINEAR SYSTEMS

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Abstract— Necessary and sufficient conditions for the existence of a proper state space realization for nonlinear systems is developed. This condition is an integrability test that is based on a derived flag. The result is generalized for implicit systems.

Keywords— Nonlinear systems; implicit systems; DAE's; differential geometric approach.

Resumo— Uma condição necessária e suficiente para existência de realizações próprias no espaço de estados para uma classe de sistemas não lineares é desenvolvida. Esta condição é um teste de integrabilidade que é baseada em um "flag derivado". O resultado é generalizada para sistemas implícitos

Palavras-chave— Sistemas não lineares; sistemas implícitos; DAE's; abordagem diferencial geométrica.

1 Introduction

In previous works, some sufficient conditions for the existence of proper realizations of nonlinear implicit system have been presented (Pereira da Silva and Batista, 2004; Batista, 2006). In this paper, the results of these previous works are generalized in order to get, under mild assumptions, necessary and sufficient conditions of the existence of a proper realization on nonlinear explicit and implicit system.

Our approach will follows the infinite dimensional geometric setting introduced in control theory by (Fliess et al., 1993; Pomet, 1995; Fliess et al., 1999) in combination with the ideas presented in (Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva and Batista, 2004).

We will use the standard notations of differential geometry in the finite and infinite dimensional case. A brief overview of the infinite dimensional approach of (Fliess et al., 1999) is presented in Appendix A. Some notations and definitions of Appendix A are used along the paper (*e. g.* the definition of system as a *diffiety*, and the definition of *state representation* as a local coordinate system).

The field of real numbers will be denoted by $I\!R$. The set of real matrices of n rows and m columns is denoted by $I\!R^{n\times m}$. The matrix M^T stands for the transpose of M. The set of natural numbers $\{1,\ldots,k\}$ will be denoted by $\lfloor k \rceil$. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x = (x_1,\ldots,x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1,\ldots,dx_n\}$. The time derivative of a function $\frac{d}{dt}u$ will be denoted by $\dot{u}$ (or $u^{(0)}$) and the k-fold time derivative of u will be denoted by $u^{(k)}$. If $u = (u_1,\ldots,u_m)$ is a set of functions then $u^{(k)} = (u_1^{(k)},\ldots,u_m^{(k)})$ and $\bar{u}$ stands for the set of functions $\{u^{(k)} : k \in I\!N\}$.

The problem of realization of input/output nonlinear differential equations was extensively studied in the literature (van der Schaft, 1987; Glad, 1988; Crouch and Lamnabhi-Lagarrigue, 1988; van der Schaft, 1989; Liu and Moog, 1994; Delaleau and Respondek, 1995; Moog et al., 2002). A comparison between all these works can been found in (Kotta and Mullari, 2003). The works (van der Schaft, 1987; Glad, 1988; Crouch and Lamnabhi-Lagarrigue, 1988; van der Schaft, 1989; Liu and Moog, 1994; Moog et al., 2002) consider the problem of giving proper realizations of input/output equations of the form

$$y^{(n)} = \phi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}) \tag{1}$$

where the highest derivative of y appears linearly.

Let S be a nonlinear system of the form

$$\dot{x}(t) = f(t, x(t), u(t), \ldots, u^{(\alpha)}) \tag{2}$$

A proper realization of this system is an equivalent system of the form

$$\dot{z}(t) = f(t, z(t), u(t)) \tag{3}$$

where z is a new state for the system with $\dim z = \dim x$ obtained by a generalized state transformation $z = z(t, x, u, \ldots, u^{(\gamma)})$. The necessary and sufficient conditions of the existence of proper realizations of (2) are given in (Delaleau and Respondek, 1995).

Recall that in the behavioral approach of (Willems, 1992) the input and the output are not chosen *a priori*. The same point of view is shared by the approach of (Fliess et al., 1999), and this fact is in accordance of what is found in physical systems. For instance, take an electrical transformer and consider the voltage of the primary circuit as the input and the voltage of the secondary circuit as the output. In the same way, one may consider the voltage of the secondary circuit as the input and the voltage of the primary circuit as the output. Another interesting example is the a DC motor, represented by the following model

$$
\begin{aligned}
Li^{(1)} + Ri + K\theta^{(1)} &= E &\quad (4a)\\
J\theta^{(2)} + B\theta^{(1)} &= Ki - \tau &\quad (4b)
\end{aligned}
$$

where θ is the shaft angle, i is the current in the motor (or in the load), L is is the inductance, E is the external voltage, R is the resistance of the motor, τ is the external torque (or the load torque), J is the inertia of the shaft, and K is the motor constant. When the input is the voltage E, the disturbance input is load torque τ, and the output is the shaft angle θ, the device is working as a motor and one may give the following state equations

$$
\begin{aligned}
\frac{d}{dt}i &= -(R/L)i - (K/L)\theta^{(1)} + E\\
\frac{d}{dt}\theta^{(0)} &= \theta^{(1)}\\
\frac{d}{dt}\theta^{(1)} &= -(K/J)i - (B/J)\theta^{(1)} - (1/J)\tau\\
y &= \theta^{(0)}
\end{aligned}
$$

with state $x = (i, \theta^{(0)}, \theta^{(1)})$ input $u = (E, \tau)$ and output $y = \theta^{(0)}$. If one considers that the device is working as a generator, one may take the external torque τ as an input, the current i on the load as a disturbance input and the voltage E as an output, which produces the following state equations

$$
\begin{aligned}
\frac{d}{dt}\theta^{(0)} &= \theta^{(1)}\\
\frac{d}{dt}\theta^{(1)} &= -(K/J)i - (B/J)\theta^{(1)} - (1/J)\tau\\
y &= K\theta^{(1)} + Ri + Li^{(1)}
\end{aligned}
$$

with state $x = (\theta^{(0)}, \theta^{(1)})$ input $u = (\tau, i)$ and output $y = E$. Note that the dimension of the states of the two state representations are different, and the second state representation is not proper (with respect to the output).

The equations (4) are the same for the motor and the generator, since they represent the model of the same physical system, but one may choose a different set of inputs and outputs, giving rise to different state equations. Motivated by this example, one may state the following problem. Given a system (2), when there exist a proper realization

$$\dot{z}(t) = f(t, z(t), v(t)) \qquad (5)$$

with a given input v? Now, v is a new input for the system, and so, the new state z has not necessarily the same dimension of the state x, as in the example of the DC motor.

Now consider the following implicit system[1]:

$$F(t, \dot{w}(t), w(t)) = 0 \qquad (6)$$

Given a system (6), when there exists an equivalent system (5) with a given input v? Note that the transformation of a system of the form (6) into the form (1) or (2) may be very difficult, since it may be necessary to apply the implicit function theorem several times. Hence, it is interesting to answer this question without rendering the system explicit.

The aim of this paper is to give necessary and sufficient conditions for the solution of these problems. It must be pointed out that the statement of the problems are not precise. They are given here in order to motivate the paper and will be stated in a precise manner later.

2 Explicit systems

Without loss of generality[2], consider a system S defined by

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad (7)$$

where f is smooth with respect to its arguments. In this paper, a system means a diffiety S defined in the sense of the appendix (see also (Fliess et al., 1999)). Let $v = (v_1, \ldots, v_m)$ be a set of functions defined on the diffiety S. Note that each v_i may be a function of $x, u^{(0)}, u^{(1)}, \ldots$. As stated in the appendix, a state representation (x, u) is a local coordinate system $\{t, x, (u^{(k)} : k \in I\!N)\}$. Hence, to say that there exists a local classical realization with input v is to assure that there exists a set of functions $z = (z_1, \ldots, z_p)$ such that $\{t, z, (v^{(k)} : k \in I\!N)\}$ is a local coordinate system and the corresponding state equations are of the form (5). Now we may state the following result

Theorem 1 *Consider the system S defined by (7). Then system (7) admits a local proper realization with input v if and only if there exists non-negative integers δ and γ, such that, for the codistribution Γ_0 defined on the diffiety S by*

$$\Gamma_0 = span\left\{dt, dx, du^{(0)}, \ldots, du^{(\gamma)}, dv^{(0)}, \ldots, dv^{(\delta)}\right\}$$

and for the codistributions

$$\Gamma_k = span\left\{\omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1}\right\} \qquad (8)$$

we have
(i) Γ_k is nonsingular for $k = 0, \ldots, \delta + 1$, and

[1] It is easy to show that this class of systems include input-output equations. Furthermore, it will be shown that is not important whether an input u of the implicit system is chosen *a priori* or not. The input v may be part of the system variables w or not.

[2] If (7) were not proper, *i. e.* , if the system is of the form (2), one may extend the state by taking $\tilde{x} = (x, u^{(0)}, \ldots, u^{(\alpha-1)})$ and $\tilde{u} = u^{(\alpha)}$.

$\dim \Gamma_{k-1} - \dim \Gamma_k = \dim v$ *for* $k = 1, \ldots, \delta+1$.
(ii) Γ_k *is involutive for* $k = 0, \ldots, \delta+1$.
(iii) $\Gamma_0 = \Gamma_1 \oplus \mathit{span}\left\{dv^{(\delta)}\right\}$.
(iv) The set $\{dv^{(k)}\}$ *is locally linearly independent for* $k = 0, \ldots, \delta$.

The proof of necessity of Theorem 1 relies on the next Lemma, whose proof can is an easy adaptation of the results of (Liu and Moog, 1994; Batista, 2006):

Lemma 2 *Let* S *be a system defined by* (5) *and let* $\delta \in I\!N$. *Let* $\Gamma_0 = \mathit{span}\left\{dt, dz, dv, \ldots, dv^{(\delta)}\right\}$ *and let* $\Gamma_k = \mathit{span}\,\{\omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1}\}$. *Then* $\Gamma_k = \mathit{span}\left\{dt, dz, dv, \ldots, dv^{(\delta-k)}\right\}$ *and* $\Gamma_{\delta+1} = \mathit{span}\,\{dt, dz\}$.

The proof of sufficiency is based on the following result whose proof is ommited[3]:

Lemma 3 *Let* (x, u) *be a local state representation of a system* S, *and let* $z = (z_1, \ldots, z_p)$ *and* $v = (v_1, \ldots, v_m)$ *be sets of functions defined on the diffiety* S *such that the set* $\mathbb{S} = \{dt, dz, dv, \ldots, dv^{(\alpha)}\}$ *is (locally) linearly independent and* $\mathit{span}\,\{dx, du\} \subset \mathbb{S}$. *Then* (z, v) *is also a local state representation of* S.

Proof: (of Theorem 1)
Necessity. Let (z, v) be a local classic state representation of system S defined by (7) with state equations 5. Since $\{t, z, (v^{(k)} : k \in I\!N)\}$ is a local coordinate system of the diffiety S, then $x = x(t, z, v^{(0)}, \ldots, v^{(\alpha)})$ and $u = u(t, z, v^{(0)}, \ldots, v^{(\beta)})$ for α, β big enough. Hence, $du^{(k)} \in \operatorname{span}\left\{dt, dz, dv^{(0)}, \ldots, dv^{(\beta+k)}\right\}$ for all $k \in I\!N$. In the same way, one may write $z = z(t, x, u^{(0)}, \ldots, u^{(r)})$ for some $r \in I\!N$. Now take $\delta = \max\{\alpha, \beta + r\}$ and $\gamma = r$. By construction, it follows that $\operatorname{span}\{dz\} \subset \operatorname{span}\left\{dt, dx, \ldots, du^{(\gamma)}\right\}$. Hence, $\operatorname{span}\left\{dt, dx, du, \ldots, du^{(\gamma)}\right\} \subset \operatorname{span}\left\{dt, dz, dv, \ldots, dv^{(\delta)}\right\}$. In particular, one shows that, for γ and δ constructed above, $\Gamma_0 = \operatorname{span}\left\{dt, dz, dv, \ldots, dv^{(\delta)}\right\}$. The proof of necessity then follows from Lemma 2.
Sufficiency. It will be shown first that

$$\operatorname{span}\left\{dv, \ldots, dv^{(\delta-k)}\right\} \subset \Gamma_k, k = 0, \ldots, \delta \quad (9a)$$

and that

$$\Gamma_k = \Gamma_{k+1} \oplus \operatorname{span}\left\{dv^{(\delta-k)}\right\}, k = 0, \ldots, \delta \quad (9b)$$

The condition (9a) is a straightforward consequence of the definition of (8). The equation (9b) will be shown by induction. By the assumptions of theorem 1, (9b) holds for $k = 0$. Assume that it holds for some k, with $0 < k < \delta$. By absurd, assume that $\Gamma_{k+2} \cap \operatorname{span}\left\{dv^{(\delta-k-1)}\right\} \neq \{0\}$. Since $\dim \Gamma_{k+1}$ and $\operatorname{span}\left\{dv^{(\delta-k-1)}\right\}$ are nonsingular, then $\Gamma_{k+2} \cap \operatorname{span}\left\{dv^{(\delta-k-1)}\right\}$ is a nonsigular smooth codistribution. Let $\omega = \sum_{i=1}^{\dim v} \alpha_i dv_i^{(\delta-k-1)}$ be a smooth one form in $\Gamma_{k+2} \cap \operatorname{span}\left\{dv^{(\delta-k-1)}\right\}$, where some α_i is not the null function. It follows that, $\dot{\omega} = \sum_{i=1}^{\dim v}[\dot{\alpha}_i dv_i^{(\delta-k-1)} + \alpha_i dv_i^{(\delta-k)}] \in \Gamma_{k+1}$. By (9a), one concludes that $\sum_{i=1}^{\dim v} \alpha_i dv_i^{(\delta-k)} \in \Gamma_{k+1} \cap \operatorname{span}\left\{dv^{(\delta-k)}\right\}$. This last condition contradicts the induction hypothesis. Now, by (i), one shows (9b). Now, let $\omega = dt$. Since $\dot{\omega} = 0$, it follows that $dt \in \Gamma_k, k = 0, \ldots \delta+1$. In particular, $\operatorname{span}\{dt\} \in \Gamma_{\delta+1}$. From the involutivity and nonsingularity of $\Gamma_{\delta+1}$, and from Frobenius theorem, there exists (locally) a set of functions $\{z_1, \ldots, z_p\}$ such that $\Gamma_{\delta+1} = \operatorname{span}\{dt, dz\}$. By (9b), it is clear that the set $\mathbb{S} = \{dt, dz, dv, \ldots, dv^{(\delta)}\}$ is (locally) linearly independent and $\Gamma_0 = \operatorname{span}\{\mathbb{S}\}$. So, by Lemma 3, it follows that (z, v) is a local state representation for the system. By (9b) for $k = \delta$, and from (8), one concludes that $d\dot{z}_j \in \operatorname{span}\{dt, dz, dv\}$, and so this local state representation is classic.

□

3 Implicit systems

3.1 Regular implicit systems

Now let Δ be an implicit system of the form (6). Let $x = (x_1, x_2)$, where $x_1 = w$ and $x_2 = \dot{w}$, and let $u = \ddot{w}$. It is clear that the system (6) is equivalent to the semi-implicit system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= F(x_1, x_2) = 0 \end{aligned}$$

which is in the form

$$\dot{x}(t) = f(x(t), u(t)) \quad (10a)$$
$$y(t) = h(x(t), u(t)) = 0 \quad (10b)$$

where $f(x, u) = (x_2, u)^T$ and $h(x, u) = F(x_1, x_2)$. Hence, without loss of generality, it will be assumed that the given implicit system Δ is in the form (10). Consider also that all the functions defining (10) are analytic. It must be pointed out that u is not the input of the implicit system, since the relation $y^{(k)} \equiv 0$ may induce differential relations among the components of u. For instance, in the example of the DC motor working as a generator, if one adds the constraint $y = R_L i - E = 0$ corresponding to an ohmic load R_L, it is clear that (τ, i) is no longer an input for the system. In fact, $y = 0$ induce the differential relation $K\theta^{(1)} + Li^{(1)} - R_L i = 0$ between the components[4]

[3]A similar result can be found in (Pomet, 1995).

[4]The definition of state representation given in the appendix considers that $\{t, x, u, \dot{u} \ldots\}$ is a local coordinate system, and so the variables $t, x, u, \dot{u} \ldots$ may not be linked by any relation.

of $x = (\theta^{(0)}, \theta^{(1)})$ and $u = (\tau, i)$. This explains why x and u are called respectively pseudo-input and pseudo state of (10).

The explicit system given by (10a) will be denoted by S. Consider the system S with Cartan field $\frac{d}{dt}$ and output[5] $y = h(x, u)$, in the framework of (Fliess et al., 1999) as defined in the last paragraph of the appendix. Then $y^{(k)}$ stands for the function $\frac{d^k}{dt^k} y$ defined on S, which may depend on $x, u^{(0)}, u^{(1)}, \ldots$.

We may construct the following codistributions defined on the explicit system S defined by (10a)

$$\left\{ \begin{array}{lcl} \mathcal{Y}_k & = & \text{span}\left\{dx, dy, \ldots, dy^{(k)}\right\} \\ Y_k & = & \text{span}\left\{dy, \ldots, dy^{(k)}\right\} \end{array} \right. \quad \text{for } k \in I\!N. \tag{11}$$

Let

$$\tilde{\Delta} = \{\xi \in S \mid y^{(k)}|_\xi = 0, k \in I\!N\} \tag{12}$$

(Prop. 1 shows that Δ is a immersed submanifold of S).

Definition 1 *The implicit system Δ is said to be regular the codistributions Y_k and $\mathcal{Y}_k$ are nonsingular for $k = 0, \ldots, n$ for every point $\xi \in \tilde{\Delta}$, where $n = \dim x$.*

The following theorem is a consequence of the properties of the dynamic extension algorithm.

Theorem 4 *(Pereira da Silva and Corrêa Filho, 2001, Theo. 4.3) Let S be the explicit system given by* (10a) *with output y. Around a point ξ such that the codistributions* (11) *are nonsingular for $k = 0, 1, \ldots, n$, where $n = \dim x$. Choose sets of functions $x_b \subset x$ and $u_b \subset u$ such that $span\{dx_b\} \oplus Y_n = span\{dx\} + Y_n$, $span\{dx_b, du_b\} \oplus Y_n = span\{dx, du\} + Y_n$. Then there exists a classic state representation $\widetilde{x} = (x_a, x_b)$, $\widetilde{u} = (u_a, u_b)$ of S of the form*

$$\dot{x}_a = f_a(t, x_a, u_a) \tag{13a}$$
$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \tag{13b}$$

in a way that $\mathcal{Y} = span\left\{dt, (dy^{(k)} : k \in I\!N)\right\} = span\left\{dt, dx_a, (du_a^{(j)} : j \in I\!N)\right\}$. This state representation is adapted to the output subsystem Y, i. e., (13a) *are local (classical) state equations for Y. Furthermore, x_b and u_b are such that $span\{dx_b\} \oplus \mathcal{Y} = span\{dx\} + \mathcal{Y}$ and $span\{dx_b, du_b\} \oplus \mathcal{Y} = span\{dx, du\} + \mathcal{Y}$ and the set of functions $\{x_a, u_a\}$ can be locally chosen as a subset of $\{y^{(k)} : k \in I\!N\}$.*

Remark 1 *A state representation $((x_a, x_b), (v_a, v_b))$ such that$\mathcal{Y} = span\left\{dt, (dy^{(k)} : k \in I\!N)\right\} = span\left\{dt, dx_a, (du_a^{(j)} : j \in I\!N)\right\}$ with state equations* (13) *is said to be adapted to the output subsystem Y.*

[5] We regard $y = h(x, u)$ as an output instead of being a constraint.

Definition 2 *Consider a implicit system Δ defined by* (10) *and let S be the explicit system defined by* (10a). *An equivalent system is a diffiety $\tilde{\Delta}$ such that there exists a Lie-Bäcklund immersion $\iota : \tilde{\Gamma} \to S$ with the property that, for every solution $\xi(t)$ of S respecting the restriction* (10b), *there exists a solution $\nu(t)$ of $\tilde{\Delta}$ such that $\xi(t) = \iota \circ \nu(t)$.*

It can be shown that a regular implicit system defined by (10a)–(10b) is equivalent to an immersed system in the explicit system S defined by (10a). This result is the Proposition 1 bellow, whose proof is based on the last theorem.

Proposition 1 *(Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva and Watanabe, 2002) Let Δ be the implicit system* (10). *Let S be the system associated to* (10a). *Let $\tilde{\Delta}$ be the subset of S defined by* (12). *Then the subset $\tilde{\Delta} \subset S$ has a canonical structure of immersed (embedded) submanifold of S such that the canonical insertion $\iota : \tilde{\Delta} \to S$ is a Lie-Bäcklund immersion. Furthermore $\tilde{\Delta}$ admits a local classical state representation around every point $\xi \in \Gamma$ and $\tilde{\Delta}$ is equivalent to Δ.*

The idea of the proof of Proposition 1 is to consider the local state representation of the last theorem. It is shown that, $\{t, x_a, x_b, U_a, U_b\}$ and $\{t, x_b, U_b\}$ are respectively local coordinates for S and Γ and, in these coordinates $\iota(t, x_b, U_b) = (t, 0, x_b, 0, U_b)$ where $U_a = \{u_a^{(j)} : j \in I\!N\}$ and $U_b = \{u_b^{(j)} : j \in I\!N\}$.

It must be pointed out that the local state equations of Γ are given by $\dot{x}_b = f_b(t, 0, x_b, 0, u_b)$.

3.2 Proper realizations of implicit systems

Definition 3 *A (proper) realization of an implicit system $\tilde{\Delta}$ given by* (10) *is a (proper) state representation (z, v) of an equivalent[6] system $\tilde{\Delta}$. It is said to be adapted to subsystem Y is the exists a local state representation of of S $((x_a, z), (u_a, v))$ adapted to the output subsystem Y.*

Consider a regular implicit system (10). Let $v = (v_1, \ldots, v_s)$ be a set of functions defined on the diffiety S associated[7] to (10a). The set v is the candidate for an input of the implicit system. The following theorem gives necessary and sufficient conditions solving the second problem stated in the introduction.

[6] Equivalent in the sense of definition 2.

[7] The components of v may depend on $x, u, u^{(1)}, \ldots$

Theorem 5 *Consider the explicit system* (10a). *Let* $\mathcal{Y}$ *be the codistribution, defined on* S*, given by* $\mathcal{Y} = span\left\{dt, (dy^{(k)} : k \in I\!N)\right\}$ *Then the implicit system* (10) *admits a local proper realization with input* v *adapted*[8] *to the output subsystem* Y*, if and only if there exist non-negative integers* δ *and* γ*, such that, for the codistribution* $\tilde{\Gamma}_0$*, defined on* S *by*

$$\tilde{\Gamma}_0 = span\left\{dx, du^{(0)}, \ldots, du^{(\gamma)}, dv^{(0)}, \ldots, dv^{(\delta)}\right\} + \mathcal{Y}$$

and for $\tilde{\Gamma}_k = span\left\{\omega \in \tilde{\Gamma}_{k-1} \mid \dot{\omega} \in \tilde{\Gamma}_{k-1}\right\}$*, we have*

(i) $\tilde{\Gamma}_k/\mathcal{Y}$ *is finite dimensional and nonsingular for* $k = 0, \ldots, \delta + 1$*, and* $\dim \frac{\tilde{\Gamma}_{k-1}}{\mathcal{Y}} - \dim \frac{\tilde{\Gamma}_k}{\mathcal{Y}} = \dim v$ *for* $k = 1, \ldots, \delta$.

(ii) $\tilde{\Gamma}_k$ *is involutive for* $k = 0, \ldots, \delta + 1$.

(iii) $\frac{\tilde{\Gamma}_0}{\mathcal{Y}} = \frac{\tilde{\Gamma}_1}{\mathcal{Y}} \oplus V^{(\delta)}$*, where* $V^{(\delta)} = \frac{span\{dv^{(\delta)}\}+\mathcal{Y}}{\mathcal{Y}}$.

(iv) The set $\{dv \mod \mathcal{Y}, \ldots, dv^{(\delta)} \mod \mathcal{Y}\}$ *is locally linearly independent.*

The proof of the last theorem mixes the results of section 2 with the ones of section 3.1, and is omitted because of the space limitations. The next version of theorem 5 is more suitable for computations (see Example 2).

Theorem 6 *Consider the explicit system* (10a). *Let* $\mathcal{Y}$ *be the codistribution, defined on* S*, given by* $\mathcal{Y} = span\left\{dt, dy^{(k)} : k \in I\!N\right\}$ *Let* $\delta, \gamma \in I\!N$ *and define the codistribution* $\tilde{\Gamma}_0$ *by*

$$\tilde{\Gamma}_0 = span\left\{dx, du^{(0)}, \ldots, du^{(\gamma)}, dv^{(0)}, \ldots, dv^{(\delta)}\right\} + \mathcal{Y}$$

Then the implicit system (10) *admits a local proper realization with input* v *adapted to the output subsystem* Y*, if and only if there exist non-negative integers* δ *and* γ*, and a smooth finite dimensional codistribution* Γ_0 *such that*

(1) $\tilde{\Gamma}_0 = \Gamma_0 \oplus \mathcal{Y}$.

(2) Let $\Gamma_k = span\{\omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1} + \mathcal{Y}\}$*. Then* Γ_k *is finite dimensional and nonsingular and* $\dim \Gamma_{k-1} - \dim \Gamma_k = \dim v$ *for* $k = 1, \ldots, \delta + 1$.

(3) $\Gamma_k \oplus \mathcal{Y}$ *is involutive for* $k = 0, \ldots, \delta + 1$.

(4) $\Gamma_0 \oplus \mathcal{Y} = \Gamma_1 \oplus span\left\{v^{(\delta)}\right\} \oplus \mathcal{Y}$.

(5) The set $\mathbb{V} = \{dv, \ldots, dv^{(\delta)}\}$ *is locally linearly independent and* $span\{\mathbb{V}\} \cap \mathcal{Y} = 0$.

4 Examples

Example 1. Consider the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_2x_3 - x_1u$ and $\dot{x}_3 = u$. Let $v = x_3$. Take $\Gamma_0 = \text{span}\{dt, dx_1, dx_2, dx_3, du, dv, d\dot{v}\} = \text{span}\{dx_1, dx_2, dv, d\dot{v}\}$. Simple calculations give $\Gamma_1 = \text{span}\{dt, dx_1, dx_2, dv\}$ and $\Gamma_2 = \text{span}\{dt, dx_1, dx_2 + x_1dv\}$. Hence, the conditions of theorem 1 holds. Taking $z_1 = x_1$ and $z_2 = x_2 + x_1v$, it follows that $\Gamma_2 = \text{span}\{dt, dz_1, dz_2\}$ and so (z, v) is a local state representation for the system, where $z = (z_1, z_2)$. The corresponding state equations are given by $\dot{z}_1 = z_2 - z_1v$ and $\dot{z}_2 = 0$ (note that the system is not controllable).

Example 2. One may rewrite the last example in an implicit form, obtaining $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$ and $\dot{x}_3 = u_2$, $\dot{x}_4 = u_1$ and the constraint $y = x_3 + x_2x_4 + x_1u_1 = 0$. Let $v = x_4$. Take $\tilde{\Gamma}_0 = \text{span}\{dt, dx_1, dx_2, dx_3, dx_4, du_1, du_2, dv, d\dot{v}, d\ddot{v}\} + \mathcal{Y} = \text{span}\{dx_1, dx_2, dx_4, du_1, d\dot{u}_1\} \oplus \mathcal{Y}$. Then, simple calculations give $\tilde{\Gamma}_1 = \text{span}\{dx_1, dx_2, dx_4, du_1\} \oplus \mathcal{Y}$, $\tilde{\Gamma}_2 = \text{span}\{dx_1, dx_2, dx_4\} \oplus \mathcal{Y}$, and $\tilde{\Gamma}_2 = \text{span}\{dx_1, dx_2 + x_1dx_4\} \oplus \mathcal{Y}$. The assumptions of theorem 5 holds. One may take $z = (z_1, z_2)$ where $z_1 = x_1$, $z_2 = x_2 + x_1x_4$. Noting that $x_4 = v$, one gets $\dot{z}_1 = x_2 = z_2 - z_1v$ and $\dot{z}_2 = \dot{x}_2 + \dot{x}_1v + x_1\dot{v} = x_3 + \dot{x}_1v + x_1\dot{v} = y - x_2v - x_1u_1 + \dot{x}_1v + x_1\dot{v}$. Since $\dot{v} = \dot{x}_4 = u_1$, it follows that $\dot{z}_2 = y$. Taking into account the constraint $y \equiv 0$, one gets the same state equations of the first example.

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[8] See Remark 1.

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A Diffieties and Systems

$I\!R^A$-Manifolds. Let A be a countable set. Denote by $I\!R^A$ the set of functions from A to $I\!R$. One may define the coordinate function $x_i : I\!R^A \to I\!R$ by $x_i(\xi) = \xi(i), i \in A$. This set can be endowed with the Fréchet topology (see (Fliess et al., 1999)). A function $\phi : I\!R^A \to I\!R$ is smooth if it depends on a finite number of coordinates and is smooth in the usual sense. From this notion of smoothness, one can easily state the notions of vector fields and differential forms on $I\!R^A$ and smooth mappings from $I\!R^A$ to $I\!R^B$. The notion of $I\!R^A$-manifold can be also established easily as in the finitely dimensional case.

Given an $I\!R^A$-manifold $\mathcal{P}$, $C^\infty(\mathcal{P})$ denotes the set of smooth maps from $\mathcal{P}$ to $I\!R$. Let $\mathcal{Q}$ be an $I\!R^B$-manifold and let $\phi : \mathcal{P} \to \mathcal{Q}$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_p\mathcal{P} \to T_{\phi(p)}\mathcal{Q}$ and $\phi^* : T^*_{\phi(p)}\mathcal{Q} \to T^*_p\mathcal{P}$. The map $\phi : \mathcal{P} \to \mathcal{Q}$ is called an *immersion* if, around every $\xi \in \mathcal{P}$ and $\phi(\xi) \in \mathcal{Q}$, there exist local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates $\phi(x) = (x, 0)$. The map ϕ is called a *submersion* if, around every $\xi \in \mathcal{P}$ and $\phi(\xi) \in \mathcal{Q}$, there exist local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates, $\phi(x, y) = x$.

Diffieties. A *diffiety* M is a $I\!R^A$ manifold equipped with a distribution Δ of finite dimension r, called *Cartan distribution.* A section of the Cartan distribution is called a *Cartan field.* An *ordinary diffiety* is a diffiety for which $\dim \Delta = 1$ and a Cartan field ∂_M is distinguished and called *the Cartan field.* In this paper we will only consider ordinary Diffieties that will be called simply by *Diffieties.*

A Lie-Bäcklund mapping $\phi : M \mapsto N$ between Diffieties is a smooth mapping that is compatible with the Cartan fields, *i. e.* , $\phi_*\partial_M = \partial_N \circ \phi$. A *Lie-Bäcklund immersion* (respectively, *submersion*) is a Lie-Bäcklund mapping that is an immersion (resp., submersion). A Lie-Bäcklund isomorphism between two diffieties is a diffeomorphism that is a Lie-Bäcklund mapping. Context permitting, we will denote the Cartan field of an ordinary diffiety M simply by $\frac{d}{dt}$. Given a smooth object ϕ defined on M (a smooth function, field or form), then $\dot\phi$ stands for $L_{\frac{d}{dt}}\phi$ and $L^n_{\frac{d}{dt}}\phi = \phi^{(n)}, n \in I\!N$.

Systems. The set of real numbers $I\!R$ have a trivial structure of diffiety with the Cartan field $\frac{d}{dt}$ given by the operation of derivation of smooth functions. A *system* is a triple $(S, I\!R, \tau)$ where S is a diffiety equipped with Cartan field ∂_S and $\tau : S \mapsto I\!R$ is a Lie-Bäcklund submersion. The global coordinate function t of $I\!R$ represents *time*, that is chosen for once and for all. A Lie-Bäcklund mapping between two systems $(S, I\!R, \tau)$ and $(S', I\!R, \tau')$ is a time-respecting Lie-Bäcklund mapping $\phi : S \mapsto S'$, *i. e.* , $\tau' = \tau \circ \phi$. Context permitting, the system $(S, I\!R, \tau)$ is denoted simply by S.

State Space Representation and Outputs. A local state representation of a system $(S, I\!R, \tau)$ is a local coordinate system, $\psi = \{t, x, U\}$ where $x = \{x_i, i \in \lfloor n\rceil\}$, $U = \{u_j^{(k)} | j \in \lfloor m\rceil, k \in I\!N\}$ where $\tau \circ \psi^{-1}(t, x, U) = t$. The set of functions $x = (x_1, \ldots, x_n)$ is called *state* and $u = (u_1, \ldots, u_m)$ is called *input.* In these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k\in I\!N,\\ j\in\lfloor m\rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{14}$$

A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by (x, u). An *output* y of a system S is a set of functions defined on S.

System associated to differential equations. Now assume that a control system is given by a set of equations

$$\begin{array}{rcl} \dot t &=& 1 \\ \dot x_i &=& f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n\rceil \\ y_j &=& \eta_j(x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p\rceil \end{array} \tag{15}$$

One can always associate to these equations a diffiety S of global coordinates $\psi = \{t, x, U\}$ and Cartan field given by (14).

Subsystems. A (local) subsystem S_a, of a system S is a pair (S_a, π), where S_a is a system with a time notion τ_a and Cartan field ∂_a, and π is a Lie-Bäcklund submersion $\pi : U \subset S \to S_a$ between the system $U \subset S$ and S_a. A local state representation $x = (x_a, x_b)$, $u = (u_a, u_b)$ is said to be adapted to a subsystem S_a if we locally have

$$\dot x_a = f_a(t, x_a, u_b) \tag{16a}$$

$$\dot x_b = f_b(t, x_a, x_b, u_a, u_b) \tag{16b}$$

and (x_a, u_a) is a local state representation of S_a with state equations (16a).