FLAT SYSTEMS, FLAT INFORMATION AND AN APPLICATION TO ATTITUDE CONTROL

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Abstract

In this work we consider a class of nonlinear flat systems evolving on the tangent bundle TG of a Lie-Group G. For this class of systems every set of local coordinate functions is a local flat output. As a consequence, the canonical projection of TG on G acts as a global flat output, called flat information. It is shown that a stabilization control law for such a system induces canonically a control law for tracking assintotically any desired trajectory on G. It is show that $\epsilon(t) = g(t)h^{-1}(t)$ is a "global" tracking error information, where $h(t)$ is the desired trajectory on G and $g(t)$ is the actual position. An application of these results to attitude control is presented. Based on a robust (nonlinearizing) global stabilization strategy, a control law for tracking assintotically a desired trajectory on $SO(3)$ is designed. Some computer simulations of the closed loop system are presented.

keywords Nonlinear systems, Lie-Groups, flatness, Lyapunov stability, tracking, attitude control.

1 Introduction and Motivation

The problem of feedback linearization is an important structural problem in control systems theory. This problem was completely solved by static-state feedback [13, 12] but the necessary and sufficient conditions for the solvability of this problem by dynamic state feedback is yet an open problem (see for instance [2, 23, 3, 28] for some results about this problem).

The notion of differential flatness, introduced by Fliess et al [8, 9, 10, 7], is strongly related to the problem of feedback linearization [23], [11], [27], [16], [20], [30], [24], [1], [32].

Roughly speaking, a system is *flat* if and only if there exists a differentially independent set of functions $y = (y_1, \dots, y_m)$, called *flat output*, such that every variable of the system is a function of the flat output and its derivatives.

In many applications (*e.g.* [10], [16], [22]) the flat output of a given model has a physical meaning. Hence, the problem of tracking a desired trajectory of the flat output is very important in pratice.

The idea is to compute the input as a function of the desired output $y(t)$ and its derivatives. Note that flatness of a system implies dynamic feedback linearizability and this could be a way to ensure the stabilization. However, the property of flatness can be used only for tracking a desired trajectory of the flat output, without having exact linearization.

In section 2 we will consider a class of nonlinear control systems evolving on the tangent bundle of a Lie-Group G. This class of systems will be shown to be flat. For these systems, the canonical projection of a point of TG on G plays the role of a "global" flat output, that will be called *flat information*. Furthermore, given a state feedback that stabilizes the system (not necessarily producing exact linearization), then this feedback induces canonically a control law that assures assintotic tracking of any desired trajectory $h(t) \in G$. We will se that a natural way to consider the tracking error is to compute $\epsilon(t) = g(t)h^{-1}(t)$ where $g(t)$ is the actual position on the Lie-Group and $h(t)$ the trajectory to be tracked.

In section 3 we present an application to attitude control. In fact the results of section 2 are inspired on the results of [17].

The idea of considering a tracking error of the form gh^{-1} for attitude control, at least in the case where h is a fixed

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point of G, was presented in [17] as well as a control law for stabilization based on Euler theorems. Our stabilization control law is a little bit different than the ones developed in [17, 18] and it is intrinsically robust since the closed loop stability does not depend on the knowledge of the system[1]. The Lyapunov based control laws of [19, 18, 34, 35, 31] are also closed related to ours. Other authors considered the control synthesis by a exact linearization strategy [5, 6, 25, 21, 26]. We stress that attitude control is a very old topic in control sciences and our list of citations of related results might be incomplete.

We will use standard notations of differential geometry and some elementary properties of Lie-Groups [33]. Given a matrix M, then M^T stands for its transpose. If G is a smooth manifold we denote a point of G by g and its expression in local coordinates by $\bar{g}$. Given a smooth map $\phi : G \mapsto H$ between smooth manifolds then $\phi_*(g) : T_gG \mapsto T_{\phi(g)}H$ denotes its tangent mapping.

2 A class of flat systems

In this section we will present a class of nonlinear systems evolving on the tangent bundle of a Lie-Group. For this class of systems it is shown that the solution of the stabilization problem induces a solution of a tracking problem.

Definition of the class of systems. Let G be a Lie-Group [33] with identity e. Denote by $g_1 g_2$ the product of two elements of G and let $L_g : G \mapsto G$ be the left translation given by $L_g(h) = gh$ and $R_g : G \mapsto G$ be the right translation given by $R_g(h) = hg$. Then L_g and R_g are both diffeomorphisms of G into G and $(L_g)^{-1} = L_{g^{-1}}$, $(R_g)^{-1} = R_{g^{-1}}$. A local coordinate system $\psi = \bar{g}$ induces a local coordinate system $\psi_* = (\bar{g}, \bar{g}^{(1)})$ of TG. Given a smooth curve $g(t)$ on G, then $\dot{g}(t) \in T_gG$ is defined by $g_*(\frac{d}{dt})(t)$ and expressed in local coordinates by $(\bar{g}(t), \bar{g}^{(1)}(t))$.

For each fixed $g \in G$, then $(L_g)_*(\xi) : T_\xi G \mapsto T_{L_g(\xi)}G$ is an isomorphism of finite dimensional vector spaces. In particular, taking $g = \xi^{-1}$ we see that the tangent bundle TG may be canonically identified with the trivial bundle $G \times T_eG$. In fact, we can define the diffeomorphism $\phi : TG \mapsto G \times T_eG$ by $\phi(\xi, v_\xi) = (\xi, (L_{\xi^{-1}})_* v_\xi)$. Note that $\phi^{-1}(\xi, v) = (\xi, (L_\xi)_* v)$.

Let $\pi_1 : G \times T_eG \mapsto G$ and $\pi_2 : G \times T_eG \mapsto T_eG$ be the canonical projections. Since $G \times T_eG$ is a trivial bundle then $T(G \times T_eG) \cong TG \times T(T_eG)$ This means that every field f on TG may be canonically written as (f^1, f^2) where f^1 and f^2 are given respectively by $f^1 = (\pi_1)_* f$ and $f^2 = (\pi_2)_* f$.

[1] The control laws developed in [17, 18] depend on the inverse of the inertia matrix J.

Denote a point of $G \times T_eG$ by (g, ω). Now consider the class of nonlinear control systems evolving on $TG \cong G \times T_eG$ of the form

$$\begin{array}{rcl} \dot{g}(t) &=& f^1(g(t), \omega(t)) \\ \dot{\omega}(t) &=& f^2(g(t), \omega(t), u(t)) \end{array} \tag{2.1}$$

where $f : G \times T_eG \times \mathcal{U} \mapsto TG \times T(T_eG)$ is a smooth field controlled by the input $u(t) \in \mathcal{U} \cong I\!R^m$. The field components $f^1 : G \times T_eG \times \mathcal{U} \mapsto TG$ and $f^2 : G \times T_eG \times \mathcal{U} \mapsto T(T_eG)$ are defined respectively by $f^1 = \pi_{1*} f$ and $f^2 = \pi_{2*} f$. We will assume that :

- (H1) The map f^1 does not depend on $u \in \mathcal{U}$. So it may be regarded as a map $f^1 : G \times T_eG \mapsto TG$. Furthermore it is assumed to be a diffeomorphism from $G \times T_eG$ to TG such that $\pi \circ f^1(g, \omega) = g$, where $\pi : TG \mapsto G$ is the canonical projection.
- (H2) There exists a smooth map $\mu : G \times T(T_eG) \mapsto \mathcal{U}$ such that $f^2(g, \omega, \mu(g, (\omega, v)) = (\omega, v)$ for all $g \in G, \omega \in T_eG$ and $v \in T_\omega(T_eG)$.

It is clear that (H1) means that the information of $\dot{g}(t)$ is the same of $(g(t), \omega(t))$. The assumption (H2) means that we can control $\dot{\omega}(t)$ freely.

Note that (H1) implies that there exists a map $\widehat{\omega} : TG \mapsto T_eG$ such that

$$f^1(g, \widehat{\omega}(g, v_g)) = v_g \tag{2.2}$$

for all $(g, v_g) \in T_G$. By (H2) it follows that for all fixed $(g, \omega) \in G \times T_eG$, the map $\widehat{f}^2 : \mathcal{U} \mapsto T_\omega(T_eG)$ defined by $\widehat{f}^2(u) = f^2(g, \omega, u)$ is a diffeomorphism (with inverse $\widehat{\mu}(v) = \mu(g, (\omega, v)))$. In particular we must have $\dim \mathcal{U} = \dim T_eG$.

Local flatness of the class of systems. Let $(\bar{g}, \bar{\omega})$ be a local coordinate chart of $G \times T_eG$ and consider that the local expression of (2.1) in these coordinates are given by

$$\begin{array}{rcl} \dot{\bar{g}}(t) &=& f^1(\bar{g}(t), \bar{\omega}(t)) \\ \dot{\bar{\omega}}(t) &=& f^2(\bar{g}(t), \bar{\omega}(t), u(t)) \end{array} \tag{2.3}$$

By (H1), the map ζ such that $\zeta(\bar{g}, \bar{\omega}) = (\bar{g}, f^1(\bar{g}, \bar{\omega}))$ is a dipheomorphism of open subsets of an Euclidean space. Let $\bar{g}^{(1)} = f^1(\bar{g}, \bar{\omega})$. Then $(\bar{g}, \bar{g}^{(1)})$ is also a new local coordinate system and $\bar{\omega} = \widehat{\omega}(\bar{g}, \bar{g}^{(1)})$. In these coordinates we have

$$\begin{array}{rcl} \dot{\bar{g}} &=& \bar{g}^{(1)} \\ \dot{\bar{g}}^{(1)} &=& \frac{\partial f^1}{\partial \bar{g}}|_{(\bar{g},\bar{\omega})} \bar{g}^{(1)} + \frac{\partial f^1}{\partial \bar{\omega}}|_{(\bar{g},\bar{\omega})} f^2(\bar{g}, \bar{\omega}) \end{array} \tag{2.4}$$

By (H1) it is clear that $\frac{\partial f^1}{\partial \bar{\omega}}|_{(\bar{g}, \widehat{\omega}(\bar{g}, \bar{g}^{(1)}))}$ is nonsingular. By (H2) it is not difficult to show that we may take the local state feedback $u = \mu(\bar{g}, \bar{\omega}, \theta(\bar{g}, \bar{\omega}, v))$, where $v(t) \in \mathcal{U}$ is the new input and $\theta = \{\frac{\partial f^1}{\partial \bar{\omega}}\}^{-1}[-\frac{\partial f^1}{\partial \bar{g}} \bar{g}^{(1)} + v]$. The closed loop system will locally read

$$\begin{array}{rcl} \dot{\bar{g}} &=& \bar{g}^{(1)} \\ \dot{\bar{g}}^{(1)} &=& v \end{array}$$

We conclude that any local set of coordinate functions $\bar{g}$ for G is in fact a local flat output of our class of nonlinear systems.

Flat information. The problem of taking local coordinate functions as a flat output for the system is that a control law based on this flat output might be defined only locally. We will show that the canonical projection of $G \times T_eG$ on G plays the role of a global flat output in the sense the canonical projection on G of a curve on $G \times T_eG$ determines completely the solution of system (2.1).

Proposition 1 *For every smooth curve $g(t)$ on G there exists a unique smooth curve $\Sigma(t)$ on $G \times T_eG$ and a unique $u(t) \in \mathcal{U}$ depending smoothly on $\dot{\Sigma}(t) \in T(G \times T_eG)$ such that $\Sigma(t)$ is a solution of system (2.1) with input $u(t)$ and $\pi_1(\Sigma(t)) = g(t)$.*

Proof. Let $\pi : TG \mapsto G$ be the canonical projection. Since $\dot{g}$ is a map from $I \subset I\!R$ to TG such that $\pi(\dot{g}(t)) = g(t)$, by (H1) and (2.2) it follows that there exists a unique smooth curve $\omega : I \mapsto T_eG$ given by $\omega(t) = \widehat{\omega}(\dot{g}(t))$ on T_eG such that $f^1(g(t), \omega(t)) = \dot{g}(t)$. By (H2) we may define $u(t) = \mu(g(t), \dot{\omega}(t))$. In this way we will have $f^2(g(t), \omega(t), u(t)) = \dot{\omega}(t)$. By construction, $\Sigma(t)$ has the claimed properties. □

Remark 1 *Given any smooth curve $h(t)$ on G we will denote $\Sigma(t) = (h(t), \omega_h(t))$.*

Assume that we have constructed a (global) assintotic stabilization feedback law such that e is the (unique) equilibrium point. Call the closed loop system by S_c. Then this control law induces canonically a solution of the problem of tracking a desired trajectory on G in a way that the error dynamics coincides with S_c.

Proposition 2 *Assume that $F : G \times T_eG \mapsto \mathcal{U}$ is a state feedback such that the closed loop system*

$$\begin{aligned} \dot{g}(t) &= f^1(g(t), \omega(t)) \\ \dot{\omega}(t) &= f^2(g(t), \omega(t), F(g(t), \omega(t))) \end{aligned} \tag{2.5}$$

has a (unique) equilibrium point $(e, 0) \in G \times T_eG$ which is (globally) assintotically stable. Let $h(t)$ be a smooth curve on G. Let $\epsilon(t) = g(t)h^{-1}(t)$. Denote by Σ_ϵ the curve on $G \times T_eG$ obtained from $\epsilon(t)$ by proposition 1. Then there exists a control law $u(t) = H(g(t), \omega(t), \epsilon(t), \omega_\epsilon(t), \dot{\omega}_\epsilon(t))$ such that

$$\begin{aligned} \dot{\epsilon}(t) &= f^1(\epsilon(t), \omega_\epsilon(t)) \\ \dot{\omega}_\epsilon(t) &= f^2(\epsilon(t), \omega_\epsilon(t), F(\epsilon(t), \omega_\epsilon(t))) \end{aligned} \tag{2.6}$$

In particular we have $lim_{t \mapsto \infty}(\epsilon(t), \omega_\epsilon(t)) \mapsto (e, 0)$ for all smooth curves $h(t)$ and all initial conditions $g(t_0)$ and $\omega(t_0)$.

Proof. Let $\phi = (\bar{g}, \bar{\omega})$ be a local coordinate chart of $G \times T_eG$ and let (2.3) be the local expression of (2.1). Let $\psi : G \times G \mapsto G$ defined by $\psi(\bar{h}, \bar{g}) = \bar{g}\bar{h}^{-1}$. Since $(L_h^{-1}(g) = L_{h^{-1}}(g) = \psi(h, g)$ is a diffeomorphism, it follows that, in local coordinates, the Jacobian $J\psi(\bar{h}, \bar{g})$ is of the form $J\Psi = \left(A(\bar{h}, \bar{g})\ B(\bar{h}, \bar{g})\right)$ where $B(\bar{h}, \bar{g})$ is a nonsingular matrix for every $(h, g) \in G \times G$. So, differentiating $\bar{\epsilon}(t)$ with respect to time

$$\begin{aligned} \dot{\bar{\epsilon}} &= \bar{\epsilon}^{(1)}(t) = J\psi \begin{pmatrix} \dot{\bar{h}}(t) \\ \dot{\bar{g}}(t) \end{pmatrix} \\ &= A(\bar{h}(t), \bar{g}(t))\dot{f}^1(\bar{h}, \bar{\omega}_h) + \\ &\quad B(\bar{h}(t), \bar{g}(t))f^1(\bar{g}(t), \bar{\omega}(t)). \end{aligned}$$

Differentiating once more, we obtain

$$\begin{aligned} \dot{\bar{\epsilon}}^{(1)}(t) &= C(\bar{h}(t), \bar{\omega}_h(t), \bar{g}(t), \bar{\omega}(t)) + \\ &\quad B\left[\tfrac{\partial f^1}{\partial \bar{g}} f^1 + \tfrac{\partial f^1}{\partial \bar{\omega}} f^2\right] \end{aligned} \tag{2.7}$$

Let $\omega_\epsilon(t) = \widehat{\omega}(\dot{\epsilon}(t))$ as defined by (2.2). In local the coordinates $(\bar{\epsilon}, \bar{\epsilon}^{(1)})$ for TG we may write $\dot{\bar{\epsilon}}(t) = (\bar{\epsilon}(t), \bar{\epsilon}^{(1)}(t))$ and $\widehat{\omega}(t) = \widehat{\omega}(\bar{\epsilon}(t), \bar{\epsilon}^{(1)}(t))$. Hence

$$\dot{\omega}_\epsilon(t) = \tfrac{\partial \widehat{\omega}}{\partial \bar{\epsilon}} \bar{\epsilon}^{(1)}(t) + \tfrac{\partial \widehat{\omega}}{\partial \bar{\epsilon}^{(1)}} \dot{\bar{\epsilon}}^{(1)} \tag{2.8}$$

Substituting (2.7) in (2.8) we obtain

$$\dot{\omega}_\epsilon(t) = S_1(\bar{h}, \bar{\omega}_h, \dot{\bar{\omega}}_h, \bar{g}, \bar{\omega}) + \frac{\partial \widehat{\omega}}{\partial \bar{\epsilon}^{(1)}} \frac{\partial f^1}{\partial \bar{\omega}} f^2(\bar{g}, \bar{\omega}) \tag{2.9}$$

By (H1) and (H2) it is easy to verify that the matrices $\frac{\partial \widehat{\omega}}{\partial \bar{\epsilon}^{(1)}}$ and $\frac{\partial f^1}{\partial \bar{\omega}}$ are both nonsingular. Hence the matrix $M(\bar{h}, \bar{\omega}_h, \dot{\bar{\omega}}_h, \bar{g}, \bar{\omega}) = \frac{\partial \widehat{\omega}}{\partial \bar{\epsilon}^{(1)}} \frac{\partial f^1}{\partial \bar{\omega}}$ is nonsingular. it follows that we may construct $u(t) = \mu(\bar{g}(t), \bar{\omega}(t), \bar{\omega}^{(1)}(t))$ where $\bar{\omega}^{(1)}(t) = M^{-1}[f^2(\bar{\epsilon}(t), \bar{\omega}_\epsilon(t),\ F(\bar{\epsilon}(t), \bar{\omega}_\epsilon(t)) - S_1]$ and μ is the map defined in the assumption (H2). □

Remark 2 *Taking $h(t) = (h_0)^{-1}, \forall t \in I\!R$, by the last result we conclude that for this class of systems, global assintotic stabilizability around $(e, 0)$ implies assintotic stabilizability around any point $(h_0, 0)$*

3 An application to attitude control

The equations of attitude control of a spacecraft modeled as a rigid body are

$$\begin{aligned} \dot{A}(t) &= S(\omega)A(t) \\ J\dot{\omega} &= S(\omega)J\omega + \tau \end{aligned} \tag{3.10}$$

where $A(t) \in SO(3)$ is the attitude matrix, J is the symmetric positive definite inertia matrix, τ is the torque produced by the actuators and $S(\omega)$ is the matrix such that

$S(\omega)v = v \wedge \omega$ for all $\omega, v \in I\!R^3$ and is given by :

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

The columns of $A(t)$ represents the three vectors of an inertial frame when written a frame of the spacecraft. The vector $\omega(t) \in I\!R^3$ is the angular velocity of the spacecraft. Note that this systems evolves on $TSO(3) \cong SO(3) \times I\!R^3$, the tangent space of the Lie Group of orthogonal matrices of positive determinant.

It is easy to verify that this system is of the class defined in section 2.

Consider the following problems :

Stabilization Problem. - Given a desired attitude $A_d \in SO(3)$ find a feedback law $\tau = \phi(A(t), A_d, \omega)$ in a way that $\lim_{t\to\infty} A(t) = A_d$ for any initial condition $(A(0), \omega(0)) \in SO(3) \times I\!R^3$.

Without loss of generality we shall consider $A_d = I$. (see the remark 2.

Tracking Problem. Given a desired smooth attitude trajectory $A_d(t) \in SO(3)$ find a control law

$$\tau = \phi(A(t), A_d(t), \dot{A}_d(t), \ddot{A}_d(t), \omega)$$

in a way that $\lim_{t\to\infty}[A(t) - A_d(t)] = 0$ for any initial condition $(A(0), \omega(0)) \in SO(3) \times I\!R^3$.

The well known Euler's theorem on rotations [17] can be used to solve this problem by a kind of *proportional-derivative* control law. Note that $c : SO(3) \mapsto S^2$ and $\varphi : SO(3) \mapsto [0, \pi]$ are respectively the Euler axis and the Euler angle and S^2 denotes the unity sphere in $I\!R^3$. Remember that the Euler angle $-\varphi$ is the angle that we have to rotate the body around the axis c in a way that the three axes the body frame become aligned with the inertial frame. Note that when these axes are aligned we have $A(t) = I$. It can be shown [29], [17] that c is the solution of $Ac = c$ and $\varphi \in [0, \pi]$ can be calculated by the relation $\cos\varphi = (trace(A) - 1)/2$. Then $\varphi(A) = 0$ if and only if $A = I$. The map φ is continuous. Because of singularity problems, we define $c(I) = (1, 0, 0)^T$, and

$$c(A) = \frac{1}{\sin\varphi} q(A)$$

when $\varphi \neq 0$ and $\varphi \neq \pi$, where $q : SO(3) \mapsto I\!R^3$ is defined by $q(A) = (a_{23} - a_{32}, a_{31} - a_{13}, a_{12} - a_{21})^T$ and a_{ij} are the elements of A. When $\varphi = \pi$, one we define c by choosing a solution of $cc^T = (I + A)/2$ (note that if $\bar{c}$ is a solution of this equation then $-\bar{c}$ is also a solution). With this definition, it can be shown that the map c is not continuous in the identity matrix and in the points of SO(3) such that $\varphi(A) = \pi$. In all other points this mapping is smooth. On the other hand, the map $c\varphi$ is continuous on the identity matrix but not on the matrices A that $\varphi(A) = \pi$ (this map changes the sign around these points).

We will denote the Euler's axis and angle of an orthogonal matrix A by $(c(A), \varphi(A))$ or simply (c, φ) when it is clear from the context. We stress that the map $d : SO(3) \times SO(3) \mapsto [0, \pi]$ such that $d(A, B) = \varphi(AB^{-1})$ is a metric on $SO(3)$.

Solution of the stabilization problem. Now we will show that a proportional derivative control assures global assintotic stabilization.

Theorem 1 *The following control law*

$$\tau = -(\alpha\varphi)c - \beta\omega \tag{3.11}$$

solves the global stabilization problem, where α and β are positive constants.

Sketch of Proof. One has to study the case when the trajectory may contain points where $\varphi = \pi$. Here we simplify the presentation, considering only the main ideas of the proof. Consider the Lyapunov candidate $V : SO(3) \times I\!R^3 \mapsto I\!R$ such that $V(A, \omega) = \varphi^2/2 + \omega^T J\omega$. Computing $\dot{V}$ along the trajectories of (3.10) and using proposition the fact that $\dot{\varphi} = \omega^T c$ [17], we obtain :

$$\dot{V}(\varphi, \omega) = -\omega^T(S(\omega)J\omega) - \beta\|\omega\|^2$$

Consider first the case $\varphi \neq \pi$. Since $S(\omega)J\omega = -\omega \wedge (J\omega)$ and the last term is orthogonal to ω, it follows that

$$\dot{V}(\varphi, \omega) = -\beta\|\omega\|^2$$

When $\varphi = \pi$, a similar analysis shows that if we assume that $\dot{V} = 0$ implies $\omega = 0$.

Let $\Omega = SO(3) \times I\!R^3$. It is easy to verify that the set $Q = \{x \in \Omega : \dot{V}(x) = 0\} \subset \{(A, \omega) \in \Omega : \omega = 0\}$ does not contain any nontrivial trajectory. Then by a convenient extension of LaSalle's theorem (see [15]) the result follows. $\square$

Remark 3 *It can be shown (see [14, 4]) that the Lyapunov function $V(\varphi, \omega) = \frac{1}{2}k\alpha\varphi^2 + \frac{1}{2}k\omega^T J\omega + \|c^T J\omega\varphi\|$ may be useful to conclude the exponential stability of the proposed control law by choosing a convenient $k > 0$.*

Solution of tracking problem. Define the tracking error $E(t) \in SO(3)$ by

$$E(t) = A(t)A_d^T(t) \tag{3.12}$$

differentiating we obtain

$$\dot{E}(t) = S(\omega)A(t)A_d^T(t) + A(t)\dot{A}_d^T(t)$$

Note that there exists some smooth curve $\omega_d(t)$ such that $\dot{A}_d(t) = S(\omega_d)A_d(t)$. Since $S(\omega_d)$ is skew symmetric, and as $E(t) = A(t)A_d^T(t)$, the last equation implies that

$$\dot{E}(t) = S(\omega)E(t) - E(t)S(\omega_d)$$

now using the fact that $S(y)x = -y \wedge x$ and that $E(y \wedge x) = Ex \wedge Ey$ for an orthogonal matrix E and for all $x, y \in I\!R^3$, it is easy to verify that $ES(y) = S(Ey)E$ for all $y \in I\!R^3$. So

$$\dot{E}(t) = [S(\omega) - S(E\omega_d)]E(t).$$

Since $S(x+y) = S(x) + S(y)$, we conclude that $\dot{E}(t) = S(\omega - E\omega_d)E(t)$. By the uniqueness of $\omega_E(t)$ in the expression $\dot{E}(t) = S(\omega_E)E(t)$, it follows that $\omega_E(t) = \omega - E\omega_d$.

Differentiating $J\omega_E$ we obtain

$$J\dot{\omega}_E = S(\omega)J\omega + \tau - JS(\omega_E)E\omega_d - JE\dot{\omega}_d.$$

Define

$$\tau = -S(\omega)J\omega + JS(\omega_E)E\omega_d - JE\dot{\omega}_d + \tau_E \tag{3.13}$$

where

$$\tau_E = -(\alpha\varphi_E)c_E - \beta\omega_E \tag{3.14}$$

and α, β are positive constants. By construction we have

$$\begin{array}{rcl} \dot{E}(t) & = & S(\omega_E)E(t) \\ J\dot{\omega}_E & = & S(\omega_E)J\omega_E + \tau_E \end{array} \tag{3.15}$$

where τ_E is given by (3.14). By theorem 1 applyed to the closed loop system (3.15)-(3.14) it follows that $\lim_{t\to\infty}(E(t), \omega_E(t)) = (I, 0)$.

We present now some computer simulations for the tracking problem. We have chosen "low" gains $\alpha = 20$ and $\beta = 10$. To give an idea of the what is going on with A we presented the evolution of the Euler angle $\varphi(A(t))$. A plot of the tracking error angle $\varphi(E(t))$ is also presented as well as a plot of $\|\tau(t)\|$

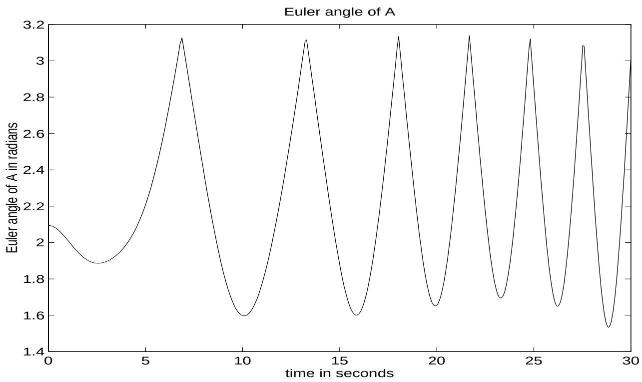

Figure 1: Plot of Euler angle $\varphi(t) = \varphi(A(t))$.

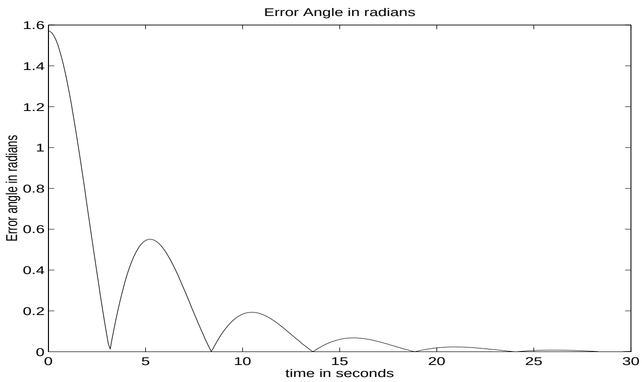

Figure 2: Plot of error angle $\varphi_E = \varphi(E(t))$.

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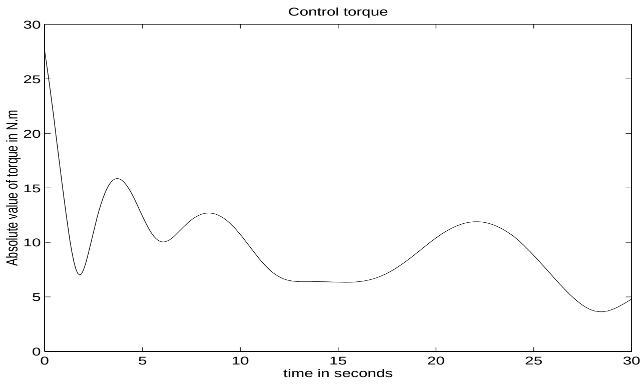

Figure 3: Plot of the control torque $\tau(t)$.

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