SOME GEOMETRIC PROPERTIES OF THE DYNAMIC EXTENSION ALGORITHM*

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Abstract. In this paper, the Dynamic Extension Algorithm (DEA) is studied in the context of an infinite dimensional geometric differential approach recently introduced in control theory. Some known properties of the DEA are revisited in this setting. These properties are also generalized for nonaffine and time-varying nonlinear systems. To illustrate the usefulness of these results, we develop some characterizations of flatness of nonlinear systems and we establish the uniqueness of the notion of differential dimension for connected smooth systems that admits state representations around every point.

Key words. Nonlinear systems, dynamic extension algorithm, flatness.

1. Introduction. The input-output decoupling problem consists in finding feedback such that for the closed loop system each output component is affected by one and only one input component. Its solution by regular dynamic state feedback has been widely studied for nonlinear systems by ([60, 13, 16, 14, 44, 17, 26, 39, 42]. A weaker version of this problem is the block input-output decoupling problem ([43, 15, 17]). A related problem to input-output decoupling is input-output linearization of nonlinear systems. This problem consists in finding a feedback in such a way that the input-output behavior of the closed loop system is linear ([36, 56, 35, 10]). For the disturbance decoupling problem one is interested in finding feedback which render the output insensible to the disturbance variables. As the input-output decoupling problem, it has been widely considered in the context of regular dynamic state feedback ([16, 30, 54, 31, 7, 47, 52, 48, 4]).

The "Structure Algorithm" was introduced for linear systems in [59] for studying the inversion of linear systems. Motivated by the study of the left-inversion and the problem input-output decoupling , this algorithm was extended for nonlinear systems in [29, 61, 40]. The "Dynamic Extension Algorithm" [13, 14, 44] was motivated for studying the right-inversion of nonlinear systems and was shown to be useful for solving the nonlinear input-output decoupling problem. An interpretation of both algorithms in an algebraic framework was introduced in [15], showing that they are essentially a tool for computing a basis of some spaces. In [11, 12], based on the differential algebraic approach of [17], it is show that these algorithms are strongly related to the notion of quasi-static state feedback, a class of feedback that is rich enough in order to solve most of the nonlinear synthesis problems (input-output decoupling, input-output linearization and disturbance decoupling).

The notion of differential flatness, introduced by Fliess et al [18, 20], corresponds to a complete and finite parametrization of all solutions of a control system by a differentially independent family of functions. Feedback linearization, strongly related to the notion of differential flatness, is an important structural problem in control systems theory. This problem was completely solved in static-state feedback case [37, 32] but necessary and sufficient conditions for feedback linearizability by dynamic state feedback are not yet known (see [8, 57, 9, 25, 62, 64, 58, 3, 55, 63, 28, 66]).

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The aim of this paper is to establish the interpretations of [15] and [11] in the context of the infinite dimensional geometric setting of [19, 53, 21, 22, 23]. We do not study any control synthesis problem here. Nevertheless, this algorithm is instrumental for studying several important problems in control theory. From this point of view this paper establishes a tool for future works based on the geometric approach of [19, 53, 23] (see for instance [51, 49, 50]). These interpretations are generalized for nonlinear time-varying systems. To illustrate the usefulness of these results, we develop some characterizations of flatness of nonlinear systems. These characterizations are instrumental for developing necessary and sufficient conditions of flatness of nonlinear systems (see [49, 50]).

The paper is organized as follows. In section 2 the notation and some mathematical background is presented. The infinite dimensional differential geometric approach of [19, 53] is briefly summarized in section 3. The Dynamic Extension Algorithm is presented in section 4. The geometric interpretation of this algorithm is discussed in section 5. The extensions for time-varying systems are discussed in section 6. Some applications of these results for nonlinear system theory is presented in section 7. The analytic case of the geometric interpretation of DEA is developed in section 8. Finally, some conclusions are stated in section 9.

2. Preliminaries and notation. The field of real numbers will be denoted by $I\!R$. The set of natural numbers $\{0, 1, 2, \ldots\}$ will be denoted by $I\!N$. The subset of natural numbers $\{1, \ldots, k\}$ will be denoted by $\lfloor k \rceil$. If H is a finite set then card H stands for the number of elements of H. We will use the standard notations of differential geometry in the finite and infinite dimensional case [68, 69]. Let us briefly recall the main definitions of the infinite dimension setting introduced in control systems theory [19, 53, 23]. This approach is mainly based on the differential geometry of jets and prolongations (see for instance [1, 2, 27, 33, 38, 46, 67, 69, 65]) whereas the approach of [34] and [45] is based on finite dimensional differential geometry [68].

Let A be a countable set. Denote by $I\!R^A$ the set of functions from A to $I\!R$. One may define the coordinate function $x_i : I\!R^A \to I\!R$ by $x_i(\xi) = \xi(i), i \in A$. This set can be endowed with the Fréchet topology (*i.e.*, an *inverse limit* topology, [5, 6, 69]). A basis of this topology is given by the subsets of the form $\mathcal{B} = \{x \in I\!R^A \mid |x_i - \delta_i| < \epsilon_i, i \in F\}$, where F is a finite subset of A and $\delta_i \in I\!R$ and ϵ_i is a positive real number for $i \in A$. A function $\phi : I\!R^A \to I\!R$ is smooth if $\phi = \psi(x_{i_1}, \ldots, x_{i_s})$, where $\psi : I\!R^s \to I\!R$ is a smooth function. Only the dependence on a finite number of coordinates is allowed.

From this notion of smoothness, one can easily state the notions of vector fields and differential forms[1] on $I\!R^A$ and smooth mappings from $I\!R^A$ to $I\!R^B$. The notion of $I\!R^A$-manifold can be also established easily as in the finitely dimensional case [69].

Given an $I\!R^A$-manifold $\mathcal{P}$, $C^\infty(\mathcal{P})$ denotes the set of smooth maps from $\mathcal{P}$ to $I\!R$. Let $\mathcal{Q}$ be an $I\!R^B$-manifold and let $\phi : \mathcal{P} \to \mathcal{Q}$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_p\mathcal{P} \to T_{\phi(p)}\mathcal{Q}$ and $\phi^* : T^*_{\phi(p)}\mathcal{Q} \to T^*_p\mathcal{P}$.

The map ϕ is called an *immersion* if there exists local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates $\phi(x) = (x, 0)$. The map ϕ is called a *submersion* if there exists local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates, $\phi(x, y) = x$.

In the finite dimensional case, immersion and submersions are locally character-

[1] We stress that the forms are finite combinations of the form $\sum_i a_{I_i} dx_{I_i}$, where I_i is the multi-index $(j_{i,1}, \ldots, j_{i,r_i})$, the a_{I_i} are smooth functions, $dx_{I_i} = dx_{j_{i,1}} \wedge \ldots \wedge dx_{j_{i,r_i}}$. On the other hand, the fields are (possibly) infinite sums of the form $\sum_{i \in A} a_i \frac{\partial}{\partial x_i}$.

ized respectively by the injectivity and surjectivity of the tangent mappings. However, in the infinite dimensional case this is no longer true. Moreover, the inverse function Theorem and the classical Frobenius Theorem (for distributions) do not hold and a field does not admit a flow in general [69].

Given a field f and a 1-form ω on $\mathcal{P}$, we denote $\omega(f)$ by $\langle f, \omega\rangle$. The set of smooth k-forms on $\mathcal{P}$ will be denoted by $\Lambda_k(\mathcal{P})$ and $\Lambda(\mathcal{P}) = \cup_{k\in I\!N}\Lambda_k(\mathcal{P})$.

A smooth codistribution J is a $C^\infty(\mathcal{P})$-submodule $J \subset T^*\mathcal{P}$. If J is a codistribution, then $J(p)$ denotes the subspace of $T_p^*\mathcal{P}$ given by $\mathrm{span}_{I\!R}\,\{\omega(p)|\ \omega \in J\}$.

Remark 2.1. One may accept an alternative definition of a codistribution J as a map that associates to $p \in \mathcal{P}$ a subspace of $T_p^*\mathcal{P}$. (*i.e.*, a section of the cotangent bundle $T^*\mathcal{P}$). ♠

Assume that a codistribution I is locally generated by $\eta_1, \ldots, \eta_k$ and that $\Psi = \{x_i|\ i \in A\}$ is a local coordinate system around some open set $U \subset \mathcal{P}$. As $\eta_i = \sum_{\text{finite}} \alpha_{ij} dx_i$ for convenient smooth functions α_{ij}, then there must exist some finite subset $A_0 \subset A$ such that all the functions α_{ij} depend only on $\{x_i|\ i \in A_0\}$ and $\eta_i \in \mathrm{span}\,\{dx_j|\ j \in A_0\}$. Consider the finite dimensional vector space $I\!R^{A_0}$ and the canonical submersion $\pi : U \to I\!R^{A_0}$ such that $\pi \circ \Psi^{-1}(x_i|\ i \in A) = (x_i|\ i \in A_0)$. It is clear that the one-forms $\widetilde{\eta}_i = \sum_{\text{finite}} \alpha_{ij} dx_i$ on the open neighborhood $\pi(U) \subset I\!R^{A_0}$ are such that $\widetilde{\eta}_i = \pi^*\eta_i, i \in \lfloor k\rceil$. Furthermore, if $\widetilde{I} = \mathrm{span}\,\{\widetilde{\eta}_i|\ i \in \lfloor k\rceil\}$, then $I = \pi^*\widetilde{I}$. In other words one may apply to (locally) finitely generated codistributions the standard techniques of differential geometry, for instance the Frobenius Theorem, by "pulling-back" the results that hold on the finite dimensional case [53].

Proposition 2.1. *Let Ω be a codistribution defined on $\mathcal{P}$ generated by a finite set of one forms $\{\omega_1, \ldots, \omega_s\}$. Then the set of regular points of Ω is open and dense in $\mathcal{P}$.*

Proof. For each local coordinate system $\{x_i|i \in A\}$ around $\xi \in \mathcal{P}$, one may apply the construction above for each form ω_i. For convenient finite $A_0 \subset A$, there exists a surjective submersion $\pi : V \subset \mathcal{P} \to \tilde{V} \subset I\!R^{A_0}$, where $V, \tilde{V}$ are open neighborhoods respectively of ξ and $\pi(\xi)$, and there exist forms $\tilde{\omega}_i$ defined on $\tilde{V}$ such that $\omega_i = \pi^*\tilde{\omega}_i$. Let $\tilde{\Omega} = \mathrm{span}\,\{\tilde{\omega}_1, \ldots, \tilde{\omega}_s\}$ By construction, $\Omega = \pi^*\tilde{\Omega}$. Note also that π^* is injective and so it preserves dimensions of codistributions. In particular, $\dim \Omega(\nu) = \dim \tilde{\Omega}(\pi(\nu))$ for all $\nu \in V$. From a known result of finite dimensional differential geometry, we have that the set $\tilde{R}$ of regular points of $\tilde{\Omega}$ of is open and dense in $\tilde{V}$. Since π is surjective, it follows that the set of regular points of Ω, which is given by $\pi^{-1}(R)$, is open and dense on V. Hence the result follows from the fact that, for all $\xi \in \mathcal{P}$, the set of regular points of Ω is dense in some open neighborhood of ξ. □

The following Lemma will be useful for showing the uniqueness of the differential dimension.

Lemma 2.2. *Let S be an $I\!R^A$-manifold and let V be an open connected subset of S. Then V is pathwise connected, i.e., for every $\xi_0, \xi_1 \in S$ there exists a continuous map $\sigma : [0, 1] \to V$ such that $\sigma(0) = \xi_0$ and $\sigma(1) = \xi_1$.*

Proof. Let B be a basic open set (of the basis $\mathcal{B}$ of the Fréchet topology) of the linear vector space $I\!R^A$. Then B is convex, *i.e.*, the path $(t-1)\zeta + t\theta$ is contained in B for every $\zeta, \theta \in B$ and every $t \in [0, 1]$.

Let $\xi_0 \in V$ be fixed. Let $\tilde{V}$ be the subset of V of the points ξ_1 such that there exists a continuous map $\sigma : [0, 1] \to V$ such that $\sigma(0) = \xi_0$ and $\sigma(1) = \xi_1$. We will show that $\tilde{V}$ is open and closed in V, and since V is connected, we must have $\tilde{V} = V$. In fact, let $\xi_1 \in \tilde{V}$ and let ϕ be a local chart around ξ_1. Then ξ_1 is in the inverse image $\phi^{-1}(B)$ of a basic open set B and so $\tilde{V}$ contains $\phi^{-1}(B)$. In particular it follows that $\tilde{V}$ is open. We show now that $\tilde{V}$ is closed. Let $\xi_1 \notin \tilde{V}$. As an absurd, assume that every open neighborhood of $C \subset V$ of ξ_1 contains points that are in $\tilde{V}$. Take C as the inverse image $\phi^{-1}(B)$ of a basic open set B. In particular any point of C are pathwise connected to ξ_1. Hence $C \cap \tilde{V} = \emptyset$. This shows that $\tilde{V}$ is also open in V. □

3. Diffieties and Systems. In this section we recall the main concepts of the infinite dimensional geometric setting of [19, 53, 22, 21, 23]. We have chosen to present a simplified exposition. For a more complete and intrinsic presentation the reader may refer to the cited literature.

3.1. Diffieties. A *diffiety* M is a $\mathbb{R}^A$-manifold equipped with a distribution Δ of finite dimension r, called *Cartan distribution.* A section of the Cartan distribution is called a *Cartan field.* An *ordinary diffiety* is a diffiety for which $\dim \Delta = 1$ and a Cartan field ∂_M is distinguished and called *the Cartan field.* In this paper we will only consider ordinary diffieties, that will be called simply by *diffieties.*

A Lie-Bäcklund mapping $\phi : M \to N$ between diffieties is a smooth mapping that is compatible with the Cartan fields, *i.e.*, $\phi_* \partial_M = \partial_N \circ \phi$. A *Lie-Bäcklund immersion* (respectively, *submersion*) is a Lie-Bäcklund mapping that is an immersion (resp., submersion). A Lie-Bäcklund isomorphism between two diffieties is a diffeomorphism that is a Lie-Bäcklund mapping.

Context permitting, we will denote the Cartan field of an ordinary diffiety M simply by $\frac{d}{dt}$. Given a smooth object ϕ defined on M (a smooth function, field or form), then $L_{\frac{d}{dt}}(\phi)$ will be denoted by $\dot{\phi}$ and $L^n_{\frac{d}{dt}}(\phi)$ by $\phi^{(n)}, n \in \mathbb{N}$. In particular, if ω is a 1-form $\omega = \sum \alpha_i dx_i$, then $\dot{\omega} = \sum(\dot{\alpha}_i dx_i + \alpha_i d\dot{x}_i)$.

3.2. Systems. The set of real numbers $\mathbb{R}$ has a trivial diffiety structure with the Cartan field defined by the operation of differentiation of smooth functions. A *system* is a triple $(S, \mathbb{R}, \tau)$ where S is a diffiety equipped with Cartan field $\frac{d}{dt}$, $\tau : S \to \mathbb{R}$ is a Lie-Bäcklund submersion and $\frac{d}{dt}(\tau) = 1$. The function τ represents *time*, that is chosen once and for all. Context permitting, the system $(S, \mathbb{R}, \tau)$ is denoted simply by S. A *Lie-Bäcklund mapping between two systems* $(S, \mathbb{R}, \tau)$ and $(S', \mathbb{R}, \tau')$ is a *time-respecting* Lie-Bäcklund mapping $\phi : S \to S'$, *i.e.*, $\tau' = \tau \circ \phi$. The previous condition means that the notion of time of both systems coincide. This notion of system is *time-varying* as it will be explained bellow.

3.3. State Representation. We present a simplified definition of state representation that introduces the state and the input and its derivatives as a local coordinate system (see [19, 22, 21] for a more intrinsic presentation).

A local state representation of a system $(S, \mathbb{R}, \tau)$ is a local coordinate system $\psi = \{t, x, U\}$ where $x = \{x_i, i \in \lfloor n \rceil\}$, $U = \{u_j^{(k)} | \ j \in \lfloor m \rceil, k \in \mathbb{N}\}$, where $u_j^{(k)} = L^k_{\frac{d}{dt}} u_j$, and $\tau \circ \psi^{-1}(t, x, U) = t$. The set of functions $x = (x_1, \ldots, x_n)$ is called *state* and $u = (u_1, \ldots, u_m)$ is called *input.* In these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k \in \mathbb{N}} \sum_{j \in \lfloor m \rceil} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{3.1}$$

Note that f_i may depend on t, x and a finite number of elements of U. In this sense, the state representation defined here is said to be generalized, since one accepts that f_i may depend on the derivatives of the input. If the functions f_i depend only on $\{t, x, u\}$ for $i \in \lfloor n \rceil$, then the state representation is said to be *classical.* A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by $S : (x, u)$ or simply by (x, u), when clear from the context. A state representation is said to be *analytic* if the f_i are all analytic with respect to its arguments x and $\{u^{(j)} \mid j \in I\!N\}$[2].

For a connected system S that admits local state reprensentation around every $\xi \in S$, then the dimension of the input is a global invariant called the *differential dimension* of S (see Corollary 7.2).

3.4. Output. An output y of a system S is a set $y = (y_1, \ldots, y_p)$ of smooth functions defined on S. If (x, u) is a state representation of S, then it is clear that

$$y_j = y_j(t, x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p \rceil \tag{3.2}$$

If the y_j depend only on $\{t, x, u\}$ for $j \in \lfloor p \rceil$, then the output is said to be *classical* with respect to the state representation (x, u). A state representation (x, u) with output y is said to be *analytic* if the functions f_i and the y_j are all analytic with respect to its arguments x and $\{u^{(j)} \mid j \in I\!N\}$.

3.5. Time-invariant systems. Consider a system $(S, \tau, I\!R)$ where $S = I\!R \times M$ and $\tau : S \mapsto I\!R$ is the canonical projection in the first factor. By definition $I\!R$ is the diffiety with Cartan field $\partial_{I\!R}$ defined by the standard derivation of smooth functions. Assume that M is a diffiety with Cartan field ∂_M. Let $\pi : S \to M$ be the canonical projection and assume that the Cartan field $\frac{d}{dt}$ of S has the following properties :

$$\partial_{I\!R} \circ \tau = \tau_* \frac{d}{dt} \tag{3.3a}$$

$$\partial_M \circ \pi = \pi_* \frac{d}{dt} \tag{3.3b}$$

Then the system is said to be a time-invariant system.

A (local) time-invariant state representation (x, u) for a time-invariant system is a (local) coordinate system $\{x_i, u_k^{(j)} : i \in \lfloor n \rceil, k \in \lfloor m \rceil, j \in I\!N\}$ of M such that $u_k^{(j+1)} = L_{\partial_M} u_k^{(j)}$ for $k \in \lfloor m \rceil$ and $j \in I\!N$. In these coordinates we have :

$$\partial_M = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k \in I\!N} \sum_{j \in \lfloor m \rceil} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{3.4}$$

where f_i are functions of $x, u, \dot{u}, \ldots$.

Abusing notation, we denote $x_i \circ \pi$ and $u_k^{(j)} \circ \pi$ respectively by x_i and $u_k^{(j)}$. Note that $\{t, x_i, u_k^{(j)} : i \in \lfloor n \rceil, k \in \lfloor m \rceil, j \in \lfloor m \rceil\}$ is a local coordinate system for S. From (3.3) it is easy to verify that in these coordinates the Cartan field of S is given by (3.1). We stress however that in this case the f_i are time-invariant functions for $i \in \lfloor n \rceil$.

An output is of S is said to be time-invariant if $y = \tilde{y} \circ \pi$ where $\tilde{y}$ is a smooth function defined on M. In particular, in these coordinates we have

$$y_j = y_j(x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p \rceil \tag{3.5}$$

Now assume that a control system is defined by a set of equations

$$\begin{array}{rcl} \dot{t} &=& 1 \\ \dot{x}_i &=& f_i(x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n \rceil \\ y_j &=& y_j(x, u, \ldots, u^{(\beta_j)}),\ j \in \lfloor p \rceil \end{array} \tag{3.6}$$

[2]This definition is coordinate dependent since only smooth atlases are considered on diffities [69].

One can always associate to these equations a diffiety $S = I\!R \times M$ of global coordinates $\psi = \{t, x, U\}$ and Cartan fields given by (3.1), (3.4) defined respectively on S and M such that the properties (3.3) holds for the canonical projections $\tau : S \to I\!R$ and $\pi : S \to M$. The system S is the *time-invariant system associated to the equations* (3.6).

A time-invariant state representation (x, u) is said to be classical if span $\{d\dot{x}\} \subset$ span $\{dx, du\}$. An output y is said to be classical if span $\{dy\} \subset$ span $\{dx, du\}$.

3.6. Endogenous feedback and coordinate changes. Since a state representation is by definition a coordinate system, a new state representation (z, v) induces a coordinate change from $\{t, x, (u^{(i)} : i \in I\!N)\}$ to $\{t, z, (v^{(j)} : i \in I\!N)\}$. The coordinate changes of this kind are called *endogenous feedbacks*[3] :

Definition 3.1. *The following definitions classifies endogenous feedback as time-invariant, time-varying, quasi-static and static :*

(i) *Two local state representations (x, u) and (z, v) of system S around $\xi \in S$ are said to be linked by endogenous feedback. If S is time-invariant and the two state representations are time-invariant, then they are said to be linked by time-invariant endogenous feedback.*

(ii) *We say that two state representations (x, u) and (z, v) are linked by time-invariant static-state feedback if we locally have* span $\{dx\}$ = span $\{dz\}$ *and* span $\{dx, du\}$ = span $\{dz, dv\}$.

(iii) *We say that two state representations (x, u) and (z, v) are linked by time-invariant quasi-static state feedback if we locally have* span $\{dx\}$ = span $\{dz\}$[4].

(iv) *We say that two state representations (x, u) and (z, v) are linked by time-varying static-state feedback if we locally have* span $\{dt, dx\}$ = span $\{dt, dz\}$ *and* span $\{dt, dx, du\}$ = span $\{dt, dz, dv\}$.

(v) *We say that two state representations (x, u) and (z, v) are linked by time-varying quasi-static state feedback if we locally have* span $\{dt, dx\}$ = span $\{dt, dz\}$.

Remark 3.1. An example of endogenous feedback is putting integrators in series with the first k inputs of the system (**??**). This procedure induces a state representation (z, v) of the system S, where $z = (x_1, \ldots, x_n, u_1, \ldots, u_k)$ and $v = (\dot{u}_1, \ldots, \dot{u}_k, u_{k+1}, \ldots, u_m)$, called *dynamic extension* of the state. ♠

The next proposition shows a characterization of static-state feedback for classical systems.

Proposition 3.2. *Let (x, u) be a classical state representation for system S. Let z and v be sets of functions defined on S such that* card z = card x *and* card v = card u.

- *Let S and (x, u) be time-invariant. Assume that we locally have* span $\{dx\}$ = span $\{dz\}$ *and* span $\{dx, du\}$ = span $\{dz, dv\}$. *Then (z, v) is a local state representation that is linked to (x, u) by static-state time-invariant feedback.*
- *Assume that we locally have* span $\{dt, dx\}$ = span $\{dt, dz\}$ *and* span $\{dt, dx, du\}$ = span $\{dt, dz, dv\}$. *Then (z, v) is a local state representation that is linked to (x, u) by time-varying static-state feedback.*

Proof. By definition it suffices to show that (z, v) is also a state representation, *i.e.*, $\{t, z, (v^{(k)} : k \in I\!N)\}$ is a local coordinate system. We will do this only in the time-invariant case, since the time-varying case is analogous (but is local in time).

For this it suffices to note from the finite dimensional Inverse Function Theorem

[3]Here we adopt a simplified definition. See [19, 22, 21] for a more intrinsic setting.
[4]See [11] for further details about quasi-static state feedback.

that the map $\phi : V \to W$ defined by $(t,x,u) \mapsto (t,z,v)$ is a local diffeomorphism such that $x \mapsto z$ is also a local diffeomorphism. Note that $\frac{\partial v}{\partial u}$ must be nonsingular and depends only on x and u. It follows that

$$\dot{v} = \frac{\partial v}{\partial x}\dot{x} + \frac{\partial v}{\partial u}\dot{u}$$

Since (x,u) is classical, then span $\{d\dot{x}\} \subset$ span $\{dx, du\}$. By derivation one has

$$v^{(k)} = \phi_k(x,u,\ldots,u^{(k-1)}) + \frac{\partial v}{\partial u}\dot{u}^{(k)}.$$

Let $U \subset S$ be the open set where the local state representation (x,u) is defined and $\frac{\partial v}{\partial u}$ is nonsingular.

We can define a local map $\Phi : U \to W \times (\mathbb{R}^m)^\infty$ by the rule $(t,x,u,\dot{u},\ldots) \mapsto (t,z,v,\dot{v},\ldots)$.

By the same arguments we may write

$$\dot{u} = \frac{\partial u}{\partial z}\dot{z} + \frac{\partial u}{\partial v}\dot{v}$$

Note that span $\{d\dot{z}\} \subset$ span $\{dx, d\dot{x}\} \subset$ span $\{dx, du\} =$ span $\{dz, dv\}$. By derivation one has

$$u^{(k)} = \psi_k(z,v,\ldots,v^{(k-1)}) + \frac{\partial u}{\partial v}\dot{v}^{(k)}$$

This defines a local map $\Psi : W \times (\mathbb{R}^m)^\infty \to V \times (\mathbb{R}^m)^\infty$ by the rule $(t,z,v,\dot{v},\ldots) \mapsto (t,x,u,\dot{u},\ldots)$. Let $\tilde{U} = \Psi^{-1}(U)$. By construction it is easy to show that we have that $\Phi : U \to \Phi(U)$ is the inverse of $\Psi|\tilde{U} : \tilde{U} \to \Psi(\tilde{U})$. □

3.7. Flatness. We present now a simple definition of flatness in terms of coordinates[5]. A system S equipped with Cartan field $\frac{d}{dt}$ and time function $t = \tau$ is locally flat around $\xi \in S$ if there exists a set of smooth functions $y = (y_1,\ldots,y_m)$, called *flat output*, such that the set $\{t, y_i^{(j)} \mid i \in \lfloor m \rceil, j \in \mathbb{N}\}$ is a (local) coordinate system of S around $\xi \in S$, where $y_i^{(j)} = L^j_{\frac{d}{dt}} y_i$. Note that the Cartan field is locally given by :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{j\in\mathbb{N}}\sum_{i\in\lfloor m\rceil} y_i^{(j+1)}\frac{\partial}{\partial y_i^{(j)}} \tag{3.7}$$

Let $\Psi : S \to T$ be a Lie-Bäcklund isomorphism between two systems. Then S is flat if and only if T is flat, also. If $y = (y_1,\ldots,y_m)$ is a flat output of T then $\{y_1 \circ \Psi, \ldots, y_m \circ \Psi\}$ is a flat output of S.

A (time-varying) system S with state-representation (x,u) is said to be 0-flat if there exists a flat output $y = h(t,x)$ and y is said to be a 0-flat output.

A system S is said to be k-flat if there exists a flat output $y = h(t,x,u,\ldots,u^{(k-1)})$ and y is said to be a k-flat output.

[5]For more intrinsic definitions and some variations, see [19, 22, 21, 23].

3.8. Flatness and time-invariant systems. Let $S = I\!R \times M$ be a time-invariant system equipped with Cartan field $\frac{d}{dt}$ and time function $t = \tau$. Let $\pi : S \to M$ be the canonical projection. Let ∂_M be the Cartan field on M that obeys condition (3.3).

Let $\xi \in S$ and let $\zeta = \pi(\zeta)$. Then S is locally flat around $\xi \in S$ if there exists a set of time-invariant smooth functions $y = (y_1, \ldots, y_m)$, called *flat output*, such that the set $\{y_i^{(j)} \mid i \in \lfloor m \rceil, j \in I\!N\}$ is a (local) coordinate system of M around $\zeta \in M$, where $y_i^{(j)} = L^j_{\partial_M} y_i$. Note that the Cartan field ∂_M is locally given by :

$$\sum_{j \in I\!N} \sum_{i \in \lfloor m \rceil} y_i^{(j+1)} \frac{\partial}{\partial y_i^{(j)}} \tag{3.8}$$

Abusing notation, we denote the $y_i^{(j)} \circ \pi$ simply by $y_i^{(j)}$. Notice that $\{t, y_i^{(j)} \mid i \in \lfloor m \rceil, j \in I\!N\}$ is a local coordinate system around $x \in S$ and in these coordinates the Cartan field is given by (3.7). The only difference from the time-varying definition is the fact that the flat output is a time invariant-function. In theory, a time-invariant system could be flat when regarded as a time-varying system (*i.e.*, accepting time-varying flat-outputs) but not flat when regarded as a time-invariant system (*i.e.*, accepting only time-invariant flat-outputs). However, we state as a conjecture that *if it exists a time-varying flat output for a time-invariant system then there exists a time-invariant flat output for the same system.*

A time-invariant system S with state-representation (x, u) is said to be 0-flat if there exists a flat output $y = h(x)$ and y is said to be a 0-flat output.

A system S is said to be k-flat if there exists a flat output $y = h(x, u, \ldots, u^{(k-1)})$ and y is said to be a k-flat output.

Remark 3.2. It is important to stress that the time-varying notions of state-representation, endogenous feedback and flatness are local in time, whereas the corresponding notion for the case of time-varying systems are global in time. ♠

4. Dynamic Extension Algorithm. Consider an affine nonlinear system of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{4.1a}$$

$$y(t) = a(x(t)) + b(x(t))u(t) \tag{4.1b}$$

where $x(t) \in X \subset I\!R^n$ is the state vector, $y(t) \in I\!R^p$ is the output, and $u(t) \in I\!R^m$ is the input. Assume that all the components of $f(x)$, $g(x)$, $a(x)$ and $b(x)$ are *analytical* functions of x.

Let us recall the main aspects of the dynamic extension algorithm (in the version of [15]). Given an analytic system (4.1a)–(4.1b), the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators. Denote a system (4.1a)–(4.1b) with state x, input u and output y by (f, g).

Step 1. Let σ_0 be the generical rank of $b(x)$. There exists a partition[6] of the outputs $y = (\bar{y}_0^T, \widehat{y}_0^T)^T$, such that $\bar{y}_0$ has dimension σ_0, and we may write

$$\begin{aligned} \bar{y}_1 &= \bar{a}(x) + \bar{b}(x)u \\ \widehat{y}_1 &= \widehat{a}(x) + \widehat{b}(x)u \end{aligned}$$

[6] Including a possible reordering of the outputs.

where $\bar{b}$ has generical rank equal to σ_0. Up to a reordering of the input components, assume that $\bar{b} = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \end{pmatrix}$ where $\bar{b}_{11}(x)$ is generically nonsingular. Construct a static-state feedback (defined generically on X)

$$u = \alpha_0(x) + \beta_0(x)v_0$$

where

$$\beta_0(x) = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} \bar{b}_{11}^{-1} & -\bar{b}_{11}^{-1}\bar{b}_{12} \\ 0 & I \end{pmatrix}$$

$$\alpha_0(x) = \beta_0(x) \begin{pmatrix} -\bar{a}_0(x) \\ 0 \end{pmatrix}$$

Let $v_0 = \begin{pmatrix} \bar{v}_0^T & \widehat{v}_0^T \end{pmatrix}^T$ where $\bar{v}_0$ has σ_0 components. By construction we have

$$\begin{aligned} \bar{y}_0 &= \bar{v}_0 \\ \widehat{y}_0 &= \widehat{y}_0(x, \bar{v}_0) \end{aligned}$$

Add a dynamic extension :

$$\begin{aligned} \bar{u}_0 &= \dot{\bar{v}}_0 \\ \widehat{u}_0 &= \widehat{v}_0 \end{aligned}$$

and let $u_0 = \left(\bar{u}_0^T, \widehat{u}_0^T\right)^T$.

Step k. After step $k-1$ we have constructed a system (f_{k-1}, g_{k-1}) with state $x_{k-1} = (x^T, \bar{v}_1^T, \ldots, \bar{v}_{k-1}^T)$ input u_{k-1} and output $y^{(k-1)} = h_{k-1}(x_{k-1})$. Compute

$$\begin{aligned} y^{(k)} &= \langle dy^{(k-1)}, f_{k-1} + g_{k-1}u_{k-1} \rangle \\ &= a_k(x_{k-1}) + b_k(x_{k-1})u_{k-1} \end{aligned} \tag{4.2}$$

Let σ_k be the generic rank of b_k. There exists a partition[7] of the outputs $y = (\bar{y}_k^T, \widehat{y}_k^T)^T$, such that $\bar{y}_k$ has dimension σ_k and an analytic regular static state feedback[8]

$$u_{k-1} = \alpha_k(x_{k-1}) + \beta_k(x_{k-1})v_k$$

where $v_k = \begin{pmatrix} \bar{v}_k^T & \widehat{v}_k^T \end{pmatrix}^T$ is such that :

$$\begin{aligned} \bar{y}_k^{(k)} &= \bar{v}_k \\ \widehat{y}_k^{(k)} &= \widehat{y}_k(x_{k-1}, \bar{v}_k) \end{aligned}$$

Add a dynamic extension :

$$\begin{aligned} \bar{u}_k &= \dot{\bar{v}}_k \\ \widehat{u}_k &= \widehat{v}_k \end{aligned}$$

and let $u_k = \left(\bar{u}_k^T, \widehat{u}_k^T\right)^T$ $\square$.

[7]Including a possible reordering of the outputs as we have seen in the step zero.

[8]Defined on an open and dense subset of $X_{k-1} = X \times \mathbb{R}^{\sigma_1} \times \ldots \times \mathbb{R}^{\sigma_{k-1}}$ and constructed in a similar way as the one of the step zero. As in the first step, we may consider a reordering of the input u_{k-1}.

5. Geometric Interpretation of the Dynamic Extension Algorithm. It is well known (see [15, 11]) that the dynamic extension algorithm has an intrinsic interpretation. However, since the approaches of [15, 11] are algebraic it may be useful to to adapt these results to the differential geometric setting of [23].

Remark 5.1. It is important to stress that in this section we consider *smooth* state representations (that are not necessary analytic). ♠

Consider that system S is a time-invariant system with classical time-invariant state representation (x, u) and classical time-invariant output $y = h(x, u)$. Recall that the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators. By § 3.6, one sees that this algorithm can be regarded as the choice of a new local state representation of system S. Let $(x_{-1}, u_{-1}) = (x, u)$ be the original state representation of S with output y. In step $k-1$ of this algorithm ($k = 0, 1, 2, \ldots$) one has constructed a classical local state representation (x_{k-1}, u_{k-1}) with output $y^{(k-1)} = h_{k-1}(x_{k-1})$ defined on an open neighborhood U_{k-1} of $\xi \in S$. Assume that span $\left\{dx_{k-1}, dy^{(k)}\right\}$ is nonsingular in ξ. Note that

- (S1) $x_k = (x_{k-1}, \bar{y}_k^{(k)})$, where $\bar{y}_k$ is chosen among the components of y by completing $\{dx_{k-1}\}$ into a basis $\{dx_{k-1}, d\bar{y}_k^{(k)}\}$ for span $\left\{dx_{k-1}, dy^{(k)}\right\}$;
- (S2) $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$, where $\hat{u}_k$ is chosen among the components of u_{k-1} by completing $\{dx_{k-1}, d\bar{y}_k^{(k)}\}$ into a basis $\{dx_{k-1}, d\bar{y}_k^{(k)}, d\hat{u}_k\}$ for span $\{dx_k, du_{k-1}\}$.

By Prop. 3.2 and Rem. 3.1, it follows that (S1) and (S2) produces a new local state representation (x_k, u_k) of system S defined in an open neighborhood $U_k \subset U_{k-1}$ of ξ.

Remark 5.2. Note that the steps $(S1)$ and $(S2)$ describes the procedure of the Dynamic Extension Algorithm that could be performed, at least theoretically, for nonaffine systems[9] of the form :

$$\begin{aligned} \dot{x} &= F(x, u) \\ y &= G(x, u) \end{aligned}$$

In particular our geometric interpretation of Lemma 5.2 holds for nonaffine systems. ♠

Definition 5.1. *In the sequel we shall consider the following filtrations of T^*S :*

$$\mathcal{Y}_{-1} = \text{span}\,\{dx\} \tag{5.1a}$$

$$\mathcal{Y}_k = \text{span}\left\{dx, dy, \ldots, dy^{(k)}\right\} \text{ for all } k \in I\!N \tag{5.1b}$$

$$Y_{-1} = \{0\} \tag{5.2a}$$

$$Y_k = \text{span}\left\{dy, \ldots, dy^{(k)}\right\} \text{ for all } k \in I\!N \tag{5.2b}$$

The following result summarizes the main geometric properties of the DEA for time-invariant nonlinear systems. We stress that the list of integers $\{\sigma_0, \ldots, \sigma_n\}$,

[9] In this case the computations are much more difficult since one may apply the inverse function theorem to compute the feedback $u_{k-1} = \gamma(x_{k-1}, v_k)$ in each step of the algorithm. A description of a version of the DEA for nonaffine systems can be found in [40].

where $n = \dim x$, is strongly related to the *algebraic structure at infinity* (see [15]) and the integer $\rho = \sigma_n$ is called *output rank* at ξ.

LEMMA 5.2. *Let $S = I\!R \times M$ be a time-invariant system with classical state representation (x, u) and classical time-invariant output $y = h(x, u)$. Let V_k be the open and dense set of regular points of the codistributions Y_i and $\mathcal{Y}_i$ for $i = 0, \ldots, k$ defined in* (5.1) *and* (5.2) *(see Prop. 2.1). In the* kth *step of the dynamic extension algorithm, one may construct a new local classical state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$, input $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$ and output $y^{(k)} = h_k(x_k)$ defined in an open neighborhood U_k of ξ, such that*

1. $\operatorname{span}\{dx_k\} = \operatorname{span}\left\{dx, dy, \ldots, dy^{(k)}\right\}$.
2. $\operatorname{span}\{dx_k, du_k\} = \operatorname{span}\left\{dx, dy, \ldots, dy^{(k+1)}, du\right\}$.
3. *It is always possible to choose $\bar{y}_{k+1}^{(k+1)}$ in a way that $\dot{\bar{y}}_k^{(k)} \subset \bar{y}_{k+1}^{(k+1)}$.*
4. *It is always possible to choose $\hat{u}_{k+1} \subset \hat{u}_k$.*
5. *Let $\xi \in V_n$. Let S_k be the open neighborhood of ξ such that the dimensions of $Y_j, \mathcal{Y}_j$ $j \in \{0, \ldots, k\}$ are constant inside S_k. The sequence $\sigma_k = \dim(\mathcal{Y}_k|_\xi)) - \dim(\mathcal{Y}_{k-1}|_\xi)$ is nondecreasing, the sequence $\rho_k = \dim(Y_k|_\xi) - \dim(Y_{k-1}|_\xi)$ is nonincreasing, and both sequences converge to the same integer ρ, called the* output rank *at ξ, for some $k^* \le n = \dim x$.*
6. *$S_k = S_{k^*}$ for $k \ge k^*$.*
7. *$Y_k \cap \operatorname{span}\{dx\}|_\nu = Y_{k^*-1} \cap \operatorname{span}\{dx\}|_\nu$ for every $\nu \in S_{k^*}$ and $k \ge k^*$.*
8. *For $k \ge k^*$, one may choose $\bar{y}_k = \bar{y}_{k^*}$ in U_{k^*}. Furthermore, $Y_{k+1} = Y_k + \operatorname{span}\left\{\bar{y}_k^{(k+1)}\right\}$ for $k \ge k^*$.*

Proof. **(1 and 2).** We show first that the state representation (x_k, u_k) is classical *i.e.*, $\operatorname{span}\{d\dot{x}_k\} \subset \operatorname{span}\{dx_k, du_k\}$. This property holds for (x, u). By induction, assume that it holds for (x_k, u_k). Then from (S1) and (S2) we have $\operatorname{span}\{d\dot{x}_{k+1}\} \subset \operatorname{span}\left\{dx_k, d\dot{x}_k, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\right\} \subset \operatorname{span}\{dx_{k+1}, du_{k+1}\}$.

In step $k = 0$, we choose a partition $y^{(0)} = (\bar{y}_0^{(0)}, \hat{y}_0^{(0)})$ in a way that (S1) is satisfied for $k = 0$ and construct $\hat{u}_0$ satisfying (S2). Then $d\hat{y}_0^{(0)} \in \operatorname{span}\{dx, d\bar{y}_0^{(0)}\}$. Thus, $d\dot{\hat{y}}_0^{(0)} \in \operatorname{span}\{dx, d\dot{x}, d\bar{y}_0^{(0)}, d\dot{\bar{y}}_0^{(0)}\} \subset \operatorname{span}\{dx, du, d\bar{y}_0^{(0)}, d\dot{\bar{y}}_0^{(0)}\}$. So, $d\dot{y} \in \operatorname{span}\{dx_0, du_0\}$. Then it is easy to see that 1 and 2 are satisfied for $k = 0$. Now assume that, in the step $k-1$ we have a local state representation (x_{k-1}, u_{k-1}) satisfying 1 and 2. Choose a partition $y^{(k)} = (\bar{y}_k^{(k)}, \hat{y}_k^{(k)})$ in a way that (S1) is satisfied and construct $\hat{u}_k$ satisfying (S2). By 1 for $k - 1$ and (S1) it follows that, $\operatorname{span}\{dx_k\} = \operatorname{span}\{dx, dy, \ldots, dy^{(k)}\}$. By construction, notice that $d\dot{\hat{y}}_k^{(k+1)} \in \operatorname{span}\{dx_{k-1}, d\dot{x}_{k-1}, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\} \subset \operatorname{span}\{dx_{k-1}$ $du_{k-1}, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\}$. So, $dy^{(k+1)} \in \operatorname{span}\{dx_k,\, du_k\}$. We show now that if 2 holds for $k-1$, then $\operatorname{span}\{dx_k,\, du_k\} = \operatorname{span}\{dx,\, dy, \ldots,\, dy^{(k+1)},\, du\}$, completing the induction. In fact, note that $\operatorname{span}\{dx_k,\, du_k\} = \operatorname{span}\{dx_{k-1},\, d\bar{y}_k^{(k)},\, d\hat{u}_k\} + \operatorname{span}\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. By (S2) and the induction hypothesis it follows that $\operatorname{span}\{dx_k,\, du_k\} = \operatorname{span}\{dx, du, dy, \ldots,$ $dy^{(k)}\} + \operatorname{span}\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. Since $dy^{(k+1)} \in \operatorname{span}\{dx_k,\, du_k\}$, then 2 holds for k. This shows 1 and 2.

(3, 5, 6). We show first that

$$\dim Y_k(\nu) - \dim Y_{k-1}(\nu) \ge \dim Y_{k+1}(\nu) - \dim Y_k(\nu) \text{ for every } \nu \in S_k \tag{5.3}$$

For this note that, if the 1-forms $\{\eta_1, \ldots, \eta_s\} \subset Y_k$ are linearly dependent mod

Y_{k-1}, *i. e.*, if $\sum_{i=1}^{s}\alpha_i\eta_i + \sum_{i=1}^{p}\sum_{j=0}^{k-1}\beta_{ij}dy_i^{(j)} = 0$ then, differentiation in time gives $\sum_{i=1}^{s}(\dot\alpha_i\eta_i+\alpha_i\dot\eta_i)+\sum_{i=1}^{p}\sum_{j=0}^{k-1}(\dot\beta_{ij}dy_i^{(j)}+\beta_{ij}dy_i^{(j+1)}) = 0$. In other words, $\dot\eta_1,\ldots,\dot\eta_s$ are linearly dependent mod Y_{k+1}. Let $\xi \in S_k$. From the nonsingularity of $Y_j, \mathcal{Y}_j, j = 0,\ldots,k$ in S_k, if $\dim Y_k - \dim Y_{k-1} = r$ in $\xi \in S_k$, then we may choose a partition $y = (\bar y^T, \hat y^T)$ such that $\bar y$ has r components and we locally have $Y_k = \operatorname{span}\left\{d\bar y^{(k)}\right\} + Y_{k-1}$. Let $\hat y_j$ be any component of $\hat y$ for $j \in \lfloor p-r\rceil$. By construction we have that $\{d\hat y_j^{(k)}, d\bar y^{(k)}\}$ is linearly dependent $\bmod$ Y_{k-1} for every $j \in \lfloor p-r\rceil$. From the remark above it follows that the set $\{d\hat y_j^{(k+1)}, d\bar y^{(k+1)}\}$ is (locally) dependent $\bmod$ Y_k for every $j \in \lfloor p-r\rceil$, showing (5.3). In particular the sequence ρ_k is nonincreasing.

We show now that

$$\dim\mathcal{Y}_k(\nu) - \dim\mathcal{Y}_{k-1}(\nu) \le \dim\mathcal{Y}_{k+1}(\nu) - \dim\mathcal{Y}_k(\nu) \text{ for every } \nu\in S_k \tag{5.4}$$

Assume that (x_k, u_k) is a state representation constructed around a neighborhood U_k of a point $\xi \in S_k$ and satisfying (S1), (S2), 1 and 2. Since $\operatorname{span}\{dx_k\} = \mathcal{Y}_k$ and $d\dot{\bar y}_k^{(k)} \subset u_k$, it follows that the components of $d\dot{\bar y}_k^{(k)}$ are independent mod $\mathcal{Y}_k$ since they are also components of the input and u_k and furthermore $\operatorname{span}\{dx_k\} = \mathcal{Y}_k$. Hence $\bar y_{k+1}^{(k+1)}$ may be chosen satisfying 3, showing (5.4). In particular, $\sigma_{k+1} \ge \sigma_k$.

To show the convergence of sequences ρ_k and σ_k for some $k^* \le n$, assume that $\nu \in S_k$. Denote $\operatorname{span}\{dx\}$ by X. Then $\mathcal{Y}_k = X + Y_k$ and thus

$$\dim\mathcal{Y}_k(\nu) = \dim X(\nu) + \dim Y_k(\nu) - \dim(Y_k(\nu)\cap X(\nu)).$$

Denote for $k \in I\!N$:

$$\begin{array}{rcl} s_k(\nu) & = & \dim\mathcal{Y}_k(\nu) - \dim\mathcal{Y}_{k-1}(\nu) \\ p_k(\nu) & = & \dim Y_k(\nu) - \dim Y_{k-1}(\nu) \end{array}$$

Note that $\rho_k = p_k(\nu)$ and $\sigma_k = s_k(\nu)$ are constant for every $\nu \in S_k$. We also have

$$s_k(\nu) = p_k(\nu) - \dim(Y_k(\nu)\cap X(\nu)) + \dim(Y_{k-1}(\nu)\cap X(\nu)) \tag{5.5}$$

We show now that

$$\begin{array}{c} \text{if there exists } k^* \text{ and some } \nu\in S_k \text{ such that } s_{k^*}(\nu) = p_{k^*}(\nu) = \rho, \\ \text{then } s_{k^*+1}(\xi) = p_{k^*+1}(\xi) = \rho \text{ for every } \xi \in S_{k^*}. \end{array} \tag{5.6}$$

Note that, from (5.6), a simple induction shows that $s_k(\xi) = p_k(\xi) = \rho$ for every $k \ge k^*$ and $\xi \in S_{k^*}$. Furthermore, this last affirmation implies that $S_k = S_{k^*}$ for $k \ge k^*$.

To show (5.6), assume that $p_{k^*}(\nu) = s_{k^*}(\nu) = \rho$ for some $\nu \in S_{k^*}$ From (5.5), it follows that

$$-\dim(Y_{k^*}(\nu)\cap X(\nu)) + \dim(Y_{k^*-1}(\nu)\cap X(\nu)) = 0.$$

Since the dimensions of $Y_{k^*}\cap X$ and of $Y_{k^*-1}\cap X$ are constant in S_{k^*}, it follows that, for every $\xi \in S_{k^*}$, we have

$$p_{k^*}(\xi) = s_{k^*}(\xi) = \rho$$

and

$$-\dim(Y_{k^*}(\xi)\cap X(\xi)) + \dim(Y_{k^*-1}(\xi)\cap X(\xi)) = 0.$$

Note from (5.5) that

$$s_{k^*+1}(\xi) - p_{k^*+1}(\xi) = -\dim(Y_{k^*+1}(\xi) \cap X(\xi)) + \dim(Y_{k^*}(\xi) \cap X(\nu)) \tag{5.7}$$

for every $\xi \in S_{k^*}$. By (5.3) and (5.4) it follows that

$$s_{k^*+1}(\xi) - p_{k^*+1}(\xi) \geq 0.$$

Since

$$-\dim(Y_{k^*+1}(\xi) \cap X(\xi)) + \dim(Y_{k^*}(\xi) \cap X(\xi)) \leq 0,$$

the only possibility is to have both sides of (5.7) equal to zero for every $\xi \in S_{k^*}$. Using (5.3) and (5.4) again, then (5.6) follows. Note that a simple induction shows that (5.6) implies 7.

To complete the proof of 5, 6 and 7 it suffices to show the existence of k^* such that (5.6) holds. For this note that $\dim(Y_k(\nu) \cap X(\nu))$ is nondecreasing for $k = 0, \ldots, n$ and it is least than or equal to $n = \dim X$. In particular, there exists some $k^* \leq n$ such that $\dim(Y_{k^*}(\nu) \cap X(\nu)) = \dim(Y_{k^*-1}(\nu) \cap X(\nu))$.

(4). Easy consequence of 1, 2 and (S2).

(8). The first part of 8 follows easily from 3 from the fact that card $\bar{y}_k = \sigma_k$ and from 5. The second part of 8 follows easily from the equality card $\bar{y}_k = \sigma_k$, from the fact that the components of $d\bar{y}_k^{(k+1)}$ are independent $\mod Y_k$ and from the fact that $\sigma_k = \rho_k = \rho$ for $k \geq k^*$. □

6. Geometric interpretation of DEA for time-varying systems. Consider that system S is a time-varying system with classical time-varying state representation (x, u) and classical time-varying output $y = h(t, x, u)$.

REMARK 6.1. It is important to stress that in this section we consider *smooth* state representations (that are not necessary analytic). ♠

Recall that the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators. By § 3.6, one sees that this algorithm can be regarded as the choice of a new local state representation of system S. Let $(x_{-1}, u_{-1}) = (x, u)$ be the original state representation of S with output y. In step $k-1$ of this algorithm ($k = 0, 1, 2, \ldots$) one has constructed a classical local state representation (x_{k-1}, u_{k-1}) with output $y^{(k-1)} = h_{k-1}(x_{k-1})$ defined on an open neighborhood U_{k-1} of $\xi \in S$. Assume that span $\left\{dt, dx_{k-1}, dy^{(k)}\right\}$ is nonsingular in ξ. Note that

- (S1) $x_k = (x_{k-1}, \bar{y}_k^{(k)})$, where $\bar{y}_k$ is chosen among the components of y by completing $\{dt, dx_{k-1}\}$ into a basis $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}\}$ for span $\left\{dt, dx_{k-1}, dy^{(k)}\right\}$;
- (S2) $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$, where $\hat{u}_k$ is chosen among the components of u_{k-1} by completing $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}\}$ into a basis $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}, d\hat{u}_k\}$ for span $\{dt, dx_k, du_{k-1}\}$.

By Prop. 3.2 and Rem. 3.1, it follows that (S1) and (S2) produces a new local state representation (x_k, u_k) of system S defined in an open neighborhood $U_k \subset U_{k-1}$ of ξ.

DEFINITION 6.1. *In the sequel we shall consider the following filtrations of T^*S :*

$$\mathcal{Y}_{-1} = \text{span}\,\{dt, dx\} \tag{6.1a}$$

$$\mathcal{Y}_k = \text{span}\left\{dt, dx, dy, \ldots, dy^{(k)}\right\} \text{ for all } k \in I\!N \tag{6.1b}$$

$$Y_{-1} = \text{span}\,\{dt\} \tag{6.2a}$$

$$Y_k = \text{span}\left\{dt, dy, \ldots, dy^{(k)}\right\} \text{ for all } k \in I\!N \tag{6.2b}$$

The following result summarizes the main geometric properties of the DEA for time-varying nonlinear systems. We stress that the list of integers $\{\sigma_0, \ldots, \sigma_n\}$, where $n = \dim x$, is strongly related to the *algebraic structure at infinity* (see [15]) and the integer $\rho = \sigma_n$ is called *output rank* at ξ. The main difference with respect to the time-invariant case is that the *results are local in time.*

LEMMA 6.2. *Let S be a time-varying system with classical state representation (x, u) and classical output $y = h(x, u)$. Let V_k be the open and dense set of regular points of the codistributions Y_i and $\mathcal{Y}_i$ for $i = 0, \ldots, k$ defined in* (6.1) *and* (6.2) *(see Prop. 2.1). In the* kth *step of the dynamic extension algorithm, one may construct a new local classical state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$, input $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$ and output $y^{(k)} = h_k(x_k)$ defined in an open neighborhood U_k of ξ, such that*

1. $\text{span}\,\{dt, dx_k\} = \text{span}\left\{dt, dx, dy, \ldots, dy^{(k)}\right\}$.
2. $\text{span}\,\{dt, dx_k, du_k\} = \text{span}\left\{dt, dx, dy, \ldots, dy^{(k+1)}, du\right\}$.
3. *It is always possible to choose $\bar{y}_{k+1}^{(k+1)}$ in a way that $\dot{\bar{y}}_k^{(k)} \subset \bar{y}_{k+1}^{(k+1)}$.*
4. *It is always possible to choose $\hat{u}_{k+1} \subset \hat{u}_k$.*
5. *Let $\xi \in V_n$. Let S_k be the open neighborhood of ξ such that the dimensions of $Y_j, \mathcal{Y}_j$ $j \in \{0, \ldots, k\}$ are constant inside S_k. The sequence $\sigma_k = \dim(\mathcal{Y}_k|_\xi)) - \dim(\mathcal{Y}_{k-1}|_\xi)$ is nondecreasing, the sequence $\rho_k = \dim(Y_k|_\xi) - \dim(Y_{k-1}|_\xi)$ is nonincreasing, and both sequences converge to the same integer ρ, called the* output rank at ξ, *for some $k^* \leq n = \dim x$.*
6. *$S_k = S_{k^*}$ for $k \geq k^*$.*
7. *$Y_k \cap \text{span}\,\{dx\}|_\nu = Y_{k^*-1} \cap \text{span}\,\{dx\}|_\nu$ for every $\nu \in S_{k^*}$ and $k \geq k^*$.*
8. *For $k \geq k^*$, one may choose $\bar{y}_k = \bar{y}_{k^*}$ in U_{k^*}. Furthermore, $Y_{k+1} = Y_k +$ $\text{span}\left\{\bar{y}_k^{(k+1)}\right\}$ for $k \geq k^*$.*

Proof. The proof is very similar to the proof of Lemma 8.2 and is left to the reader. □

7. Applications for Smooth Systems.

7.1. Uniqueness of differential dimension. We give two applications of Lemma 5.2 for smooth systems. The first result implies, using Prop. 7.3, that the flat output of a flat system has the same dimension than the input[10]. The second one says that the dimension of the input of a smooth connected system is an invariant called *differential dimension.*

PROPOSITION 7.1. *Consider a system S with classical time-invariant state representation (x, u) and classical time-invariant output $y = h(x)$. Suppose that this system is well formed, i.e.,* $\text{span}\,\{d\dot{x}\} \subset \text{span}\,\{dx, du\}$. *Assume that, for some k^* there exist an open neighborhood V of $\xi \in S$ such that* $\text{span}\,\{dx\}\,|_\nu \subset \text{span}\left\{dy, \ldots, dy^{(k^*-1)}\right\}|_\nu$ *for all ν in V. Then $\sigma_{k^*}|_\xi = \text{card}\ u = m$ for all ξ in the open and dense set V_{k^*} of regular points of the codistributions Y_k and $\mathcal{Y}_k$ for $k = 0, \ldots, k^*$ (see* (5.1)

[10]See [53] for an alternative proof of this fact.

and (5.2)*). Furthermore, if the set* $\{dy, \dots, dy^{(k^*)}\}$ *is linearly independent in* ξ *then* $\mathrm{card}\, y = \mathrm{card}\, u = m$.

Proof. Let $W \subset V$ be the open neighborhood of ξ such that the dimensions of $Y_j, \mathcal{Y}_j$ are constant in W for $j = 0, \dots, k^*$. By Lemma 5.2 part 7, in W we may consider without loss of generality that k^* is the same integer stated in that Lemma. We may execute the DEA and in the step $k^* - 1$ one will construct a state representation (x_{k^*-1}, u_{k^*-1}) such that, By Lemma 5.2 part 2, we locally have $\mathrm{span}\,\{dx_{k^*-1}, du_{k^*-1}\} = \mathrm{span}\,\{dy, \dots, dy^{(k^*)}, du\}$. Since the system is well formed and $Y_{k^*-1} \supset \mathrm{span}\,\{dx\}$, by derivation it we have $Y_{k^*} \supset \mathrm{span}\,\{dx, du\}$. In particular, it follows that $\mathrm{span}\,\{dx_{k^*-1}, du_{k^*-1}\} = \mathcal{Y}_{k^*} = Y_{k^*}$. Taking dimensions and applying Lemma 5.2 parts 1 and 2 it follows that

$$\begin{aligned}\dim(\mathrm{span}\,\{dx_{k^*-1}, du_{k^*-1}\}) &= \dim \mathcal{Y}_{k^*-1} + \dim(\mathrm{span}\,\{du_{k^*-1}\}) \\ &= \left(n + \textstyle\sum_{k=0}^{k^*-1} \sigma_k\right) + m.\end{aligned}$$

On the other hand, we have $\dim Y_{k^*} = \dim \mathcal{Y}_{k^*} = \left(n + \sum_{k=0}^{k^*} \sigma_k\right)$. It follows that $\sigma_{k^*} = m = \mathrm{card}\, u$.

If we the set $\{dy, \dots, dy^{(k^*)}\}$ is independent in ξ then, there exist an open neighborhood $U \subset V$ of ξ such that $\{dy, \dots, dy^{(k^*)}\}$ is independent in $\nu \in U$. Let $W_1 = W \cap U$. In W_1 we may apply Lemma 5.2 part 5 showing that $m = \sigma_{k^*} = \rho_{k^*} = \dim Y_{k^*}/Y_{k^*-1}$. Note that, from the independence of $\{dy, \dots, dy^{(k^*)}\}$ it follows that $\rho_j = \mathrm{card}\, y$, $k = 0, \dots, k^*$. We conclude that $\mathrm{card}\, y = \mathrm{card}\, u = m$. □

The following result shows that two inputs of system S has always the same number of components. This invariant is called the *differential dimension* of S.

COROLLARY 7.2. *Let* $S = I\!R \times M$ *be a time-invariant system that admits a local state representation around every point* $\xi \in S$. *Assume that* S *is a connected diffiety. Then the dimension of any input of the system is an invariant called* differential dimension *of the system.*

Proof. We show first that we cannot have two state representations (x, u) and (z, v) around ξ such that $\mathrm{card}\, v \neq \mathrm{card}\, u$.

Without loss of generality, assume that both state representations are defined on the same open neighborhood U of ξ.

As (x, u) induces a local coordinate system, for r big enough we have span $\{d\dot{x}, dz, dv\} \subset \mathrm{span}\,\{dx, du, \dots, du^{(r)}\}$. In particular we can write $z = z(x, u, \dots, u^{(r)})$ and $v = v(x, u, \dots, u^{(r)})$.

Consider state representation $(\tilde{x}, \tilde{u})$ where $\tilde{x} = (x, u, \dots, u^{(r)})$ and $\tilde{u} = u^{(r+1)}$. Then consider $\tilde{y} = (z, v)$ as an output of the system. Then $(\tilde{x}, \tilde{u})$ is a classical state representation and $\tilde{y}$ is a classical output. Let $n = \dim \tilde{x}$ and choose a point $\xi \in U$ that is a regular point of $\tilde{\mathcal{Y}}_j = \mathrm{span}\,\{d\tilde{x}, d\tilde{y}, \dots, d\tilde{y}^{(j)}\}$ and $\tilde{Y}_j = \mathrm{span}\,\{d\tilde{x}, d\tilde{y}, \dots, d\tilde{y}^{(j)}\}$ for $j = 0, \dots, n$.

By Lemma 5.2, from the independence of $\{dz, dv, \dots, dv^{(k)}\}$ for $k \in I\!N$ and from the fact that $\mathrm{span}\,\{d\dot{z}\} \subset \mathrm{span}\,\{dz, dv, \dots, dv^{(l)}\}$ for some l, it follows that $\rho_k = \mathrm{card}\, v$ for $k \geq q^*$. Since $\mathrm{span}\,\{d\tilde{x}\} \subset \mathrm{span}\,\{d\tilde{y}, \dots, d\tilde{y}^{(r^*)}\}$ for r^* big enough, then the application of the first part of Prop. 7.1 furnishes $\sigma_{k^*} = m = \mathrm{card}\, u$. By Lemma 5.2 part 5, it follows that $\mathrm{card}\, u = \mathrm{card}\, v = m$.

Now assume that there exists two points ξ_0 and ξ_1 of S such and two state representations defined respectively around ξ_0 and ξ_1 for which the dimensions of the inputs do not coincide.

Since S is connected and open, by Lemma 2.2, S is pathwise connected and so there exists a continuous map $\sigma : [0,1] \to S$ such that $\sigma(0) = \xi_0$ and $\sigma(1) = \xi_1$. Let $\theta \in [0,1]$ be the least number such that there a state representation (x_θ, u_θ), defined on an open neighborhood U_θ of $\sigma(\theta)$, for which the dimension of its input do not coincide with the one of the state representation (x,u) defined around ξ_0. Hence, for ϵ small enough, $\sigma(\theta - \epsilon) \in U_\theta$ and hence U_θ will contain points for which there exists local state representations with inputs of different dimensions. □

REMARK 7.1. The reader will have no difficult to establish the corresponding results of Prop. 7.1 and Cor. 7.2 for the time-varying state representations. ♠

7.2. A characterization of flatness : time-invariant case. In this subsection we will consider the following assumption

ASSUMPTION 7.1. The System S is *time-invariant* with classic time-invariant state representation (x,u) and time-invariant output $y = h(x)$. ♣

Consider the filtrations of T^*S defined in (5.1) and (5.2). The following proposition gives a well known characterization of 0-flat outputs[11]

PROPOSITION 7.3. *[41] Let S be a system that obeys Assumption 7.1. Assume that the state representation is (x,u) is well-formed i.e.,* $\mathrm{span}\,\{d\dot{x}\} \subset \mathrm{span}\,\{dx, du\}$. *Then S is locally 0-flat around $\xi \in S$ with flat output $y = h(x)$ if and only if there exist $k^* \in I\!N$ such that $\{dy, \ldots, dy^{(k^*)}\}$ is linearly independent in ξ and* $\mathrm{span}\,\{dx\}\,|_\nu \subset Y_{k^*}|_\nu$ *for ν in some open neighborhood of ξ.*

REMARK 7.2. Proposition 7.3 was originally stated in [41, 24] with the extra assumption that $\mathrm{span}\,\{du\}\,|_\nu \subset Y_{k^*}|_\nu$, $\nu \in U$, that is not necessary for well-formed systems. Note that, by Prop. 7.1 and Prop. 7.3, if y is a flat output, then $\mathrm{card}\; y = \mathrm{card}\; u = m$. An alternative proof of this fact can be found in [53]. ♠

Proof. Note first that time-invariant flatness implies that $\{dy^{(k)} : k \in I\!N\}$ is a basis for T^*M. It follows that there exists k^* such that $\mathrm{span}\,\{dx\} \subset Y_{k^*}$ and $\{dy, \ldots, dy^{(k^*)}\}$ is linearly independent in ξ.

We show now that this condition implies 0-flatness.

As the system is well-formed, it follows that $\mathrm{span}\,\{dx, du\} = \mathrm{span}\,\{dx, d\dot{x}\}$. Since $\mathrm{span}\,\{dx, du\}$ is nonsingular, we may locally write $u = \psi(x, \dot{x})$.

Let $y = h(x)$ and let $y^{(k)} = L_{\frac{d}{dt}} h, k \in I\!N$.

Let $\zeta = (y, \ldots, y^{(k^*)})|_\xi \in I\!R^q$. From the fact that Y_{k^*} is nonsingular and $\mathrm{span}\,\{dx\} \subset Y_{k^*}$ then, an application of finite the dimensional inverse function theorem shows that there exist an open neighborhood $\hat{V} \subset I\!R^q$ of ζ and a function $\phi : \hat{V} \subset I\!R^q \to I\!R^n$ such that $x = \phi(y, \ldots, y^{(k^*)})$. By derivation $\dot{x} = \sum_{j=0}^{k} \frac{\partial \phi}{\partial y^{(j)}} y^{(j+1)} = \gamma(y, \ldots, y^{(k+1)})$. Hence, $u = \theta(y, \ldots, y^{(k^*+1)}) = \psi(\phi, \gamma)$. Consider the projection $\pi : S \to I\!R^q$ such that $\pi(\nu) = (y, \ldots, y^{(k^*)})|_\nu$. Let $H \subset S$ be the open neighborhood of ξ such that (x,u) is defined on H. Let $U \subset S$ be the open neighborhood of ξ defined by $U = \pi^{-1}(\hat{V}) \cap H$. Let $\delta : U \subset S \to I\!R^q$ defined by $\delta(\nu) = (y, \ldots, y^{(k^*+1)})|_\nu$. By construction we have

$$x = \phi \circ \delta(t, x, u, \ldots) \tag{7.1a}$$

$$u = \theta \circ \delta(t, x, u, \ldots) \tag{7.1b}$$

at every point $(t, x, u, \ldots) \in U$. We have $y = h(x)$ and so

$$y = h \circ \phi(y, \ldots, y^{(k^*)}) \tag{7.2}$$

[11] See and [41, 24].

for all $(y, \ldots, y^{(k^*)}) \in \hat{V}$.

Now consider the flat system Y with global coordinates $\{\tilde{t}, \tilde{y}_j^{(k)} : k \in I\!N, j \in \lfloor m \rceil\}$. and Cartan-field

$$\partial_Y = \frac{\partial}{\partial \tilde{t}} + \sum_{j \in I\!N} \sum_{i \in \lfloor m \rceil} \tilde{y}_i^{(j+1)} \frac{\partial}{\partial \tilde{y}_i^{(j)}} \tag{7.3}$$

Let $\epsilon : Y \to I\!R^q$ defined by $\epsilon(a) = (\tilde{y}, \ldots, \tilde{y}^{(k^*)})|_a$. Let V be the open set of Y given by $V = \epsilon^{-1}(\hat{V})$. Define the Lie-Bäcklund mapping $\Gamma : V \subset Y \to S$ by

$$\begin{array}{rcl} t & = & \tilde{t} \\ x & = & \phi(\tilde{y}, \ldots, \tilde{y}^{(k^*)}) \\ u^{(r)} & = & L^r_{\partial_Y} \theta|_{(\tilde{y}, \ldots, \tilde{y}^{(r+k^*+1)})}, r \in I\!N. \end{array}$$

Without loss of generality, we may assume that $\Gamma(V) \subset U$. If this is not the case we can restrict Γ to the open set $V_1 = \Gamma^{-1}(U)$.

Let $\Lambda : U \subset S \to \Lambda(U) \subset Y$ be the Lie-Bäcklund mapping defined by

$$\begin{array}{rcl} \tilde{t} & = & t \\ \tilde{y}^{(k)} & = & L^k_{\frac{d}{dt}} h, k \in I\!N \end{array}$$

We show now now that Γ is a Lie-Bäcklund isomorphism with inverse Λ[12]. For this, we show first that $\imath = \Gamma \circ \Lambda$ is the identity map. Note that Γ is defined locally by the rule $(x, u_j^{(k)}, j \in \lfloor m \rceil, k \in I\!N) \mapsto (x \circ \imath, u_j^{(k)} \circ \imath, j \in \lfloor m \rceil, k \in I\!N)$

From the application of the inverse function theorem above, we have $t = t \circ \imath$, $x = x \circ \imath$ and $u \circ \imath$. In fact, the first identity is obvious from the definition of Γ and Λ. On the other hand, from (7.1a) and (7.1b) it follows that :

$$x \circ \imath = \phi \circ \delta = x$$

and

$$u \circ \imath = \theta \circ \delta = u.$$

Note that the composition of Lie-Bäcklund mappings is also Lie-Bäcklund. So $\imath$ is a Lie-Bäcklund from $U \subset S$ to S, *i.e.*, $\imath_* \frac{d}{dt} = \frac{d}{dt} \circ \imath$. By induction assume that $u^{(k)} \circ \imath = u^{(k)}$. It follows that

$$\begin{array}{rcl} u^{(k+1)} \circ \imath & = & \langle du^{(k)}, \frac{d}{dt} \circ \imath \rangle \\ & = & \langle du^{(k)}, \imath_* \frac{d}{dt} \rangle \\ & = & \langle \imath^* du^{(k)}, \frac{d}{dt} \rangle \\ & = & \langle d(u^{(k)} \circ \imath), \frac{d}{dt} \rangle \\ & = & \langle du^{(k)}, \frac{d}{dt} \rangle \\ & = & u^{(k+1)} \end{array}$$

To show that $\jmath = \Lambda \circ \Gamma$ is the identity map, note first that, by construction (using the inverse function theorem), we have $\tilde{t} \circ \jmath = \tilde{t}$, $\tilde{y} \circ \jmath = \tilde{y}$. In fact, the first identity is obvious from the definition of Γ and Λ. On the other hand, from (7.2) we have :

$$y \circ \jmath = h \circ \phi \circ \pi = y.$$

[12] See the proof of [53, Prop. 3, p. 3] for similar arguments.

By induction assume that $\tilde{y}^{(k)} \circ \jmath = \tilde{y}^{(k)}$. As $\jmath$ is a Lie Bäcklund mapping from $V \subset Y$ to Y, we have $\jmath_* \partial_Y = \partial_Y \circ \jmath$. It follows that

$$\begin{aligned} \tilde{y}^{(k+1)} \circ \jmath &= \langle d\tilde{y}^{(k)}, \partial_Y \circ \jmath \rangle \\ &= \langle d\tilde{y}^{(k)}, \jmath_* \partial_Y \rangle \\ &= \langle \jmath^* d\tilde{y}^{(k)}, \partial_Y \rangle \\ &= \langle d(\tilde{y}^{(k)} \circ \jmath), \partial_Y \rangle \\ &= \langle d\tilde{y}^{(k)}, \partial_Y \rangle \\ &= \tilde{y}^{(k+1)} \end{aligned}$$

□

With some regularity assumptions, the characterization of 0-flatness above may be related to the algebraic structure at infinity [15].

PROPOSITION 7.4. *Consider the nonlinear system S obeying the assumption 7.1. Suppose that* card y = card $u = m$ *and* card $x = n$. *Assume that $\xi \in S$ is a regular point of Y_k and $\mathcal{Y}_k, k \in \{0, \ldots, n\}$ (see* (5.1) *and* (5.2)*). Then S is (locally)* 0*-flat around ξ with (local) flat output y if and only if there exist $k^* \in \lfloor n \rceil$ such that one of the following equivalent conditions are satisfied :*

(i) *The structure at infinity $\{\sigma_1, \ldots, \sigma_n\}$ at $\xi \in S$ obeys the following condition*

$$\begin{aligned} n + \textstyle\sum_{i=1}^{k^*-1} \sigma_i &= mk^* \\ \sigma_{k^*} &= m \end{aligned} \tag{7.4}$$

(ii) span $\{dx\} \subset Y_{k^*-1}$.

Proof. It is important to note that $y = h(x)$ implies $\sigma_0 = \dim \mathcal{Y}_0 - \dim \operatorname{span} \{dx\} = 0$.

We show first that 0-flatness implies (ii). This is an easy consequence of the fact that $T^*M = \operatorname{span} \left\{dy^{(k)} : k \in I\!N\right\}$[13] and of Lemma 5.2 part 7.

We show now that (i) is equivalent to (ii). To show that (i) implies (ii), note from Lemma 5.2 part 5 that $m = \text{card } y \geq \rho_k \geq \sigma_k$. Hence, $\sigma_{k^*} = m$ implies that $\rho_k = m$ for $k \in I\!N$ and so $\dim Y_{k^*-1} = mk^*$. From (i) we have $\dim \mathcal{Y}_{k^*-1} = n + \sigma_1 + \ldots + \sigma_{k^*-1} = mk^*$. In particular $Y_{k^*-1} = \mathcal{Y}_{k^*-1}$ and so (ii) holds.

To show that (ii) implies (i), assume that (ii) holds. Since the system is well formed, by derivation it follows that $Y_{k^*} \supset \operatorname{span} \{dx, du\}$. By Lemma 5.2 part 2, note that $\operatorname{span} \{dx_{k^*-1}, du_{k^*-1}\} = \mathcal{Y}_{k^*}$. Note that $\dim (\operatorname{span} \{dx_{k^*-1}, du_{k^*-1}\}) = \dim (\operatorname{span} \{dx_{k^*-1}\}) + \dim (\operatorname{span} \{du_{k^*-1}\}) = \dim \mathcal{Y}_{k^*-1} + m = n + \sum_{i=1}^{k^*-1} \sigma_i + m$. We conclude that $n + \sum_{i=1}^{k^*-1} \sigma_i + m = n + \sum_{i=1}^{k^*} \sigma_i$. It follows that $\sigma_{k^*} = m$ and from Lemma 5.2 part 5 we have $\rho_0 = \ldots = \rho_{k^*} = \sigma_{k^*} = m$. This show that the set $\{dy, \ldots, dy^{(k^*)}\}$ is independent in ξ. Then, the equality $Y_{k^*-1} = \mathcal{Y}_{k^*-1}$ implies that (7.4) holds.

To show that (i) implies 0-flatness, assume that (i) is true. By the proof above (i) implies the equality of $Y_{k^*-1} = \mathcal{Y}_{k^*-1}$ and the independence of $\{dy, \ldots, dy^{(k^*)}\}$. Since $\sigma_{k^*} = m$, Lemma 5.2 implies that in step k^* one constructs a local state representation with state $x_{k^*} = (y, \ldots, y^{(k^*)})$ and input $y^{(k^*+1)}$. In particular the system is locally 0-flat around ξ with flat output $y = h(x)$. □

REMARK 7.3. If $\xi \in S$ is a regular point of Y_k and $\mathcal{Y}_k, k \in \{0, \ldots, k^* - 1\}$ and $\sigma_{k^*} = m$ then Lemma 5.2 parts 5 and 6 implies that ξ is a regular point of Y_k and $\mathcal{Y}_k, k \in I\!N$. ♠

[13]See § 3.5 for the definition of flatness for time-invariant systems.

7.3. Characterization of Flatness — time-varying case. In this subsection we will consider the following assumption :

ASSUMPTION 7.2. The System S is *time-varying* with classical time-varying state representation (x, u) and classical output $y = h(t, x)$. ♣

Consider the filtrations of T^*S defined in (6.1) and (6.2). The following proposition gives a a time-varying characterization of 0-flatness.

PROPOSITION 7.5. *Let S be a system obeying Assumption 7.2. Assume that the time-varying system S is well-formed i.e.,* $\operatorname{span}\{d\dot{x}\} \subset \operatorname{span}\{dt, dx, du\}$*. Then S is locally 0-flat around $\xi \in S$ with flat output $y = h(t, x)$ if and only if there exist $k^* \in I\!\!N$ such that such that $\{dt, dy, \ldots, dy^{(k^*)}\}$ is linearly independent in ξ and* $\operatorname{span}\{dt, dx\}|_\nu \subset Y_{k^*}|_\nu$ *for ν in some open neighborhood of ξ.*

Proof. Very similar to the time-invariant case and is left to the reader. □

We state now the time-varying version of Prop. 7.4.

PROPOSITION 7.6. *Let S be a system obeying Assumption 7.2. Suppose that* $\operatorname{card} y = \operatorname{card} u = m$ *and* $\operatorname{card} x = n$*. Assume that $\xi \in S$ is a regular point of Y_k and $\mathcal{Y}_k, k \in \{0, \ldots, n\}$.*

Then S is (locally) 0-flat around ξ with (local) flat output y if and only if there exist $k^* \in \lfloor n \rceil$ such that one of the following equivalent conditions are satisfied :

(i) *The algebraic structure at infinity $\{\sigma_1, \ldots, \sigma_n\}$ of [15] obeys the following condition*

$$\begin{array}{rcl} n + \sum_{i=1}^{k^*-1} \sigma_i & = & mk^* \\ \sigma_{k^*} & = & m \end{array} \tag{7.5}$$

(ii) $\operatorname{span}\{dt, dx\} \subset Y_{k^*-1}$.

Proof. Very similar to the time-invariant case and is left to the reader. □

8. Some remarks about analytical state representations. Most of the results of this work consider smooth state representations In this case, the output-rank may change from point to point to point. For instance consider the system :

$$\begin{array}{rcl} \dot{x} & = & u \\ y & = & h(x) \end{array}$$

where $h(x)$ is a smooth function such that $h(x) = 0$ for $x \leq 0$ and $h(x) = x$ for $x \geq 1$. It is clear that the output rank may be zero or one, depending on the point.

The main role of analycity in the proofs of many results is the existence of generical dimensions. Hence, assuming analycity of the state representation we will obtain a version of Lemma 5.2 that assures that the structure at infinity $(\sigma_1, \ldots, \sigma_n)$ is a global feature of the system that coincides with the one of [15]. Apart singular points, the output rank is then an invariant of the system in a global fashion.

We state first the time-invariant version :

LEMMA 8.1. *Assume that the System S is* time-invariant *with classic analytic time-invariant global state representation (x, u) and classic time-invariant analytic output $y = h(x, u)$ defined in the entire S. Let S_k be the open and dense set of regular points of the codistributions Y_i and $\mathcal{Y}_i$ for $i = 0, \ldots, k$ defined in* (5.1) *and* (5.2) *(see Prop. 2.1). In the* kth *step of the dynamic extension algorithm, one may construct, around $\xi \in S_k$, a new local classical state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$ and input $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$ such that*

1. $\operatorname{span}\{dx_k\} = \operatorname{span}\left\{dx, dy, \ldots, dy^{(k)}\right\}$.
2. $\operatorname{span}\{dx_k, du_k\} = \operatorname{span}\left\{dx, dy, \ldots, dy^{(k+1)}, du\right\}$.

3. *It is always possible to choose* $\bar{y}_{k+1}^{(k+1)}$ *in a way that* $\dot{\bar{y}}_k^{(k)} \subset \bar{y}_{k+1}^{(k+1)}$.
4. *It is always possible to choose* $\hat{u}_{k+1} \subset \hat{u}_k$.
5. *Let* $\mathcal{D}(\mathcal{C})$ *denote the generic dimension of a codistribution* $\mathcal{C}$ *generated by the differentials of a finite set of analytic functions. The sequence* $\sigma_k = \mathcal{D}(\mathcal{Y}_k) - \mathcal{D}(\mathcal{Y}_{k-1})$ *is nondecreasing, the sequence* $\rho_k = \mathcal{D}(Y_k) - \mathcal{D}(Y_{k-1})$ *is nonincreasing, and both sequences converge to the same integer* ρ*, called the* output rank*, for some* $k^* \leq n = \dim x$.
6. $S_k = S_{k^*}$ *for* $k \geq k^*$.
7. $Y_k \cap \operatorname{span}\{dx\}|_\nu = Y_{k^*-1} \cap \operatorname{span}\{dx\}|_\nu$ *for every* $\nu \in S_{k^*}$ *and* $k \geq k^*$.
8. *For* $k \geq k^*$*, one may choose in* U_k $\bar{y}_k = \bar{y}_{k^*}$*. Furthermore,* $Y_{k+1} = Y_k + \operatorname{span}\left\{\bar{y}_k^{(k+1)}\right\}$ *for* $k \geq k^*$.

Proof. Very similar to the proof of Lemma 8.2 and is left to the reader. □

We state now the analytical version of Lemma 5.2. Refer to § 6 for the description of steps (S1), (S2) of the time-varying dynamical extension algorithm. We stress again that the next result is local in time.

LEMMA 8.2. *Assume that the System S is* time-varying *with classic analytic time-varying global state representation* (x, u) *and classic time-varying analytic output* $y = h(t, x, u)$ *defined in the entire S.. Let* S_k *be the open and dense set of regular points of the codistributions* Y_i *and* $\mathcal{Y}_i$ *for* $i = 0, \ldots, k$ *defined in* (6.1) *and* (6.2) *(see Prop. 2.1). In the* kth *step of the dynamic extension algorithm, one may construct, around* $\xi \in S_k$*, a new local classical state representation* (x_k, u_k) *of the system S with state* $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$ *and input* $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$ *such that*

1. $\operatorname{span}\{dt, dx_k\} = \operatorname{span}\left\{dt, dx, dy, \ldots, dy^{(k)}\right\}$.
2. $\operatorname{span}\{dt, dx_k, du_k\} = \operatorname{span}\left\{dt, dx, dy, \ldots, dy^{(k+1)}, du\right\}$.
3. *It is always possible to choose* $\bar{y}_{k+1}^{(k+1)}$ *in a way that* $\dot{\bar{y}}_k^{(k)} \subset \bar{y}_{k+1}^{(k+1)}$.
4. *It is always possible to choose* $\hat{u}_{k+1} \subset \hat{u}_k$.
5. *Let* $\mathcal{D}(\mathcal{C})$ *denote the generic dimension of a codistribution* $\mathcal{C}$ *generated by the differentials of a finite set of analytic functions. The sequence* $\sigma_k = \mathcal{D}(\mathcal{Y}_k) - \mathcal{D}(\mathcal{Y}_{k-1})$ *is nondecreasing, the sequence* $\rho_k = \mathcal{D}(Y_k) - \mathcal{D}(Y_{k-1})$ *is nonincreasing, and both sequences converge to the same integer* ρ*, called the* output rank*, for some* $k^* \leq n = \dim x$.
6. $S_k = S_{k^*}$ *for* $k \geq k^*$.
7. $Y_k \cap \operatorname{span}\{dx\}|_\nu = Y_{k^*-1} \cap \operatorname{span}\{dx\}|_\nu$ *for every* $\nu \in S_{k^*}$ *and* $k \geq k^*$.
8. *For* $k \geq k^*$*, one may choose in* U_k $\bar{y}_k = \bar{y}_{k^*}$*. Furthermore,* $Y_{k+1} = Y_k + \operatorname{span}\left\{\bar{y}_k^{(k+1)}\right\}$ *for* $k \geq k^*$.

REMARK 8.1. The proof of Lemma 8.2 is very similar to the proof of Lemma 5.2 with a few adaptations for considering generical dimensions and the time-varying case. The steps (S1) and (S2) of the DEA referred in this proof are described in section 6. ♠

Proof. **(1 and 2).** We show first that the state representation (x_k, u_k) is classical *i.e.*, $\operatorname{span}\{d\dot{x}_k\} \subset \operatorname{span}\{dt, dx_k, du_k\}$. This property holds for (x, u). By induction, assume that it holds for (x_k, u_k). Then from (S1) and (S2) we have $\operatorname{span}\{dt, d\dot{x}_{k+1}\} \subset \operatorname{span}\left\{dt, dx_k, d\dot{x}_k, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\right\} \subset \operatorname{span}\{dt, dx_{k+1}, du_{k+1}\}$.

In step $k = 0$, we choose a partition $y^{(0)} = (\bar{y}_0^{(0)}, \hat{y}_0^{(0)})$ in a way that (S1) is satisfied for $k = 0$ and construct $\hat{u}_0$ satisfying (S2). Then $d\hat{y}_0^{(0)} \in \operatorname{span}\{dt, dx, d\bar{y}_0^{(0)}\}$. Thus, $d\dot{\hat{y}}_0^{(0)} \in \operatorname{span}\{dt, dx, d\dot{x}, d\bar{y}_0^{(0)}, d\dot{\bar{y}}_0^{(0)}\} \subset \operatorname{span}\{dt, dx, du, d\bar{y}_0^{(0)}, d\dot{\bar{y}}_0^{(0)}\}$. So, $d\dot{y} \in \operatorname{span}\{dt, dx_0, du_0\}$. Then it is easy to see that 1 and 2 are satisfied for $k = 0$. Now assume that, in the step $k-1$ we have a local state representation (x_{k-1}, u_{k-1}) satisfying 1 and 2. Choose a partition $y^{(k)} = (\bar{y}_k^{(k)}, \hat{y}_k^{(k)})$ in a way that (S1) is satisfied and construct $\hat{u}_k$ satisfying (S2). By 1 for $k-1$ and (S1) it follows that, $\text{span}\{dt, dx_k\} = \text{span}\{dt, dx, dy, \ldots, dy^{(k)}\}$. By construction, notice that $d\dot{\hat{y}}_k^{(k+1)} \in \text{span}\{dt, dx_{k-1}, d\dot{x}_{k-1}, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\} \subset \text{span}\{dt, dx_{k-1}\, du_{k-1}, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\}$. So, $dy^{(k+1)} \in \text{span}\{dt, dx_k,\ du_k\}$. We show now that if 2 holds for $k-1$, then $\text{span}\{dt, dx_k, du_k\} = \text{span}\{dt, dx, dy, \ldots, dy^{(k+1)}, du\}$, completing the induction. In fact, note that $\text{span}\{dt, dx_k, du_k\} = \text{span}\{dt, dx_{k-1}, d\bar{y}_k^{(k)}, d\hat{u}_k\} + \text{span}\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. By (S2) and the induction hypothesis it follows that $\text{span}\{dt, dx_k, du_k\} = \text{span}\{dt, dx, du, dy, \ldots, dy^{(k)}\} + \text{span}\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. Since $dy^{(k+1)} \in \text{span}\{dt, dx_k,\ du_k\}$, then 2 holds for k. This shows 1 and 2.

(3, 5, 6). Recall that the generical dimension of a codistribution is the maximal dimension that occurs in any nonsingular points. Hence for every $\xi \in S_k$ we have that $\mathcal{D}(Y_j) = \dim(Y_j|_\xi)$ and $\mathcal{D}(\mathcal{Y}_j) = \dim(\mathcal{Y}_j|_\xi)$ for $j \in \{0, 1, \ldots, k\}$.

We show first that

$$\dim Y_k(\nu) - \dim Y_{k-1}(\nu) \geq \dim Y_{k+1}(\nu) - \dim Y_k(\nu) \text{ for every } \nu \in S_k \tag{8.1}$$

For this note that, if the 1-forms $\{\eta_1, \ldots, \eta_s\} \subset Y_k$ are linearly dependent mod Y_{k-1}, *i. e.*, if $\alpha_0 dt + \sum_{i=1}^{s} \alpha_i \eta_i + \sum_{i=1}^{p} \sum_{j=0}^{k-1} \beta_{ij} dy_i^{(j)} = 0$ then, differentiation in time gives $\dot{\alpha}_0 dt + \sum_{i=1}^{s} (\dot{\alpha}_i \eta_i + \alpha_i \dot{\eta}_i) + \sum_{i=1}^{p} \sum_{j=0}^{k-1} (\dot{\beta}_{ij} dy_i^{(j)} + \beta_{ij} dy_i^{(j+1)}) = 0$. In other words, $\dot{\eta}_1, \ldots, \dot{\eta}_s$ are linearly dependent mod Y_{k+1}. Let $\xi \in S_k$. From the nonsingularity of $Y_j, \mathcal{Y}_j, j = 0, \ldots, k$ in S_k, if $\dim Y_k - \dim Y_{k-1} = r$ in $\xi \in S_k$, then we may choose a partition $y = (\bar{y}^T, \hat{y}^T)$ such that $\bar{y}$ has r components and we locally have $Y_k = \text{span}\left\{d\bar{y}^{(k)}\right\} + Y_{k-1}$. Let $\hat{y}_j$ be any component of $\hat{y}$ for $j \in \lfloor p-r \rceil$. By construction we have that $\{d\hat{y}_j^{(k)}, d\bar{y}^{(k)}\}$ is linearly dependent mod Y_{k-1} for every $j \in \lfloor p-r \rceil$. From the remark above it follows that the set $\{d\hat{y}_j^{(k+1)}, d\bar{y}^{(k+1)}\}$ is (locally) dependent mod Y_k for every $j \in \lfloor p-r \rceil$, showing (8.1). In particular the sequence ρ_k is nonincreasing.

We show now that

$$\dim \mathcal{Y}_k(\nu) - \dim \mathcal{Y}_{k-1}(\nu) \leq \dim \mathcal{Y}_{k+1}(\nu) - \dim \mathcal{Y}_k(\nu) \text{ for every } \nu \in S_k \tag{8.2}$$

Assume that (x_k, u_k) is a state representation constructed around a neighborhood U_k of a point $\xi \in S_k$ and satisfying (S1), (S2), 1 and 2. Since $\text{span}\{dx_k\} = \mathcal{Y}_k$ and $d\dot{\bar{y}}_k^{(k)} \subset u_k$, it follows that the components of $d\dot{\bar{y}}_k^{(k)}$ are independent mod $\mathcal{Y}_k$ since they are also components of the input and u_k and furthermore span $\{dx_k\} = \mathcal{Y}_k$. Hence $\bar{y}_{k+1}^{(k+1)}$ may be chosen satisfying 3, showing (8.2). In particular, $\sigma_{k+1} \geq \sigma_k$.

To show the convergence of sequences ρ_k and σ_k for some $k^* \leq n$, assume that $\nu \in S_k$. Denote $\text{span}\{dx\}$ by X. Then $\mathcal{Y}_k = X + Y_k$ and thus

$$\dim \mathcal{Y}_k(\nu) = \dim X(\nu) + \dim Y_k(\nu) - \dim(Y_k(\nu) \cap X(\nu)).$$

Denote for $k \in I\!\!N$:

$$\begin{array}{rcl} s_k(\nu) & = & \dim \mathcal{Y}_k(\nu) - \dim \mathcal{Y}_{k-1}(\nu) \\ p_k(\nu) & = & \dim Y_k(\nu) - \dim Y_{k-1}(\nu) \end{array}$$

Note that $\rho_k = p_k(\nu)$ and $\sigma_k = s_k(\nu)$ are constant for every $\nu \in S_k$. We also have

$$s_k(\nu) = p_k(\nu) - \dim(Y_k(\nu) \cap X(\nu)) + \dim(Y_{k-1}(\nu) \cap X(\nu)) \tag{8.3}$$

We show now that

$$\begin{array}{c}\text{if there exists } k^* \text{ and some } \nu \in S_k \text{ such that } s_{k^*}(\nu) = p_{k^*}(\nu) = \rho, \\ \text{then } s_{k^*+1}(\xi) = p_{k^*+1}(\xi) = \rho \text{ for every } \xi \in S_{k^*}.\end{array} \tag{8.4}$$

Note that, from (8.4), a simple induction shows that $s_k(\xi) = p_k(\xi) = \rho$ for every $k \geq k^*$ and $\xi \in S_{k^*}$. Furthermore, this last affirmation implies that $S_k = S_{k^*}$ for $k \geq k^*$.

To show (8.4), assume that $p_{k^*}(\nu) = s_{k^*}(\nu) = \rho$ for some $\nu \in S_{k^*}$ From (8.3), it follows that

$$-\dim(Y_{k^*}(\nu) \cap X(\nu)) + \dim(Y_{k^*-1}(\nu) \cap X(\nu)) = 0.$$

Since the dimensions of $Y_{k^*} \cap X$ and of $Y_{k^*-1} \cap X$ are constant in S_{k^*}, it follows that, for every $\xi \in S_{k^*}$, we have

$$p_{k^*}(\xi) = s_{k^*}(\xi) = \rho$$

and

$$-\dim(Y_{k^*}(\xi) \cap X(\xi)) + \dim(Y_{k^*-1}(\xi) \cap X(\xi)) = 0.$$

Note from (8.3) that

$$s_{k^*+1}(\xi) - p_{k^*+1}(\xi) = -\dim(Y_{k^*+1}(\xi) \cap X(\xi)) + \dim(Y_{k^*}(\xi) \cap X(\nu)) \tag{8.5}$$

for every $\xi \in S_{k^*}$. By (8.1) and (8.2) it follows that

$$s_{k^*+1}(\xi) - p_{k^*+1}(\xi) \geq 0.$$

Since

$$-\dim(Y_{k^*+1}(\xi) \cap X(\xi)) + \dim(Y_{k^*}(\xi) \cap X(\xi)) \leq 0,$$

the only possibility is to have both sides of (8.5) equal to zero for every $\xi \in S_{k^*}$. Using (8.1) and (8.2) again, then (8.4) follows. Note that a simple induction shows that (8.4) implies 7.

To complete the proof of 5, 6 and 7 it suffices to show the existence of k^* such that (8.4) holds. For this note that $\dim(Y_k(\nu) \cap X(\nu))$ is nondecreasing for $k = 0, \ldots, n$ and it is least than or equal to $n = \dim X$. In particular, there exists some $k^* \leq n$ such that $\dim(Y_{k^*}(\nu) \cap X(\nu)) = \dim(Y_{k^*-1}(\nu) \cap X(\nu))$.

(4). Easy consequence of 1, 2 and (S2).

(8). The first part of 8 follows easily from 3 from the fact that card $\bar{y}_k = \sigma_k$ and from 5. The second part of 8 follows easily from the equality card $\bar{y}_k = \sigma_k$, from the fact that the components of $d\bar{y}_k^{(k+1)}$ are independent $\mod Y_k$ and from the fact that $\sigma_k = \rho_k = \rho$ for $k \geq k^*$. □

9. Conclusions. We have showed that the intrinsic interpretations of the dynamic extension algorithm of [15, 11] can be translated to the approach of [23] in a quite natural manner. These interpretations was generalized for time-varying an nonaffine nonlinear systems.

To illustrate the usefulness of Lemmas 5.2 and 6.2, we have studied a characterization of flatness for the time-invariant and the time-varying cases.

The uniqueness of the notion of differential dimension (cardinal of the input) is established for connected smooth systems that admits state representations around every point.

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