ON GEOMETRIC CONTROL AND NUMERIC INTEGRATION OF DAE'S

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Abstract: Two methods of numerical integration of DAE's using mixed symbolic/numerical computations are studied in this paper. The first method, not new in the literature, is directly obtained by a decoupling and stabilizing control law. Numerical experiments shows that this first method may be numerically unstable for high index systems. The main problem of the first method is that the stabilization of the constraint manifold is the stabilization of a chain of integrators, and this may generate the numerical problems found in the experiments. The second method assures a direct convergence to the constraint manifold, and the numerical experiments show very good results.

Keywords: Nonlinear systems, implicit systems, DAE's, differential geometric approach, index, structure at infinity.

1. INTRODUCTION

Implicit systems have been deeply studied in the literature. Linear *singular* (or implicit, or descriptor) systems are an important class of control systems and many papers and books on this subject have been published (Campbell, 1982; Liyi, 1989; Campbell, 1990). Solvability of nonlinear implicit differential equations is addressed in (Brenan *et al.*, 1995; Rheinboldt, 1991). The index of general DAE's was studied for instance in (Campbell and Gear, 1995; Le Vey, 1998). Our paper is an attempt to study DAE's under a geometric approach, as done earlier for instance by (Rheinboldt, 1984; Reich, 1990; Rabier and Rheinboldt, 1991; Rabier and Rheinboldt, 1994; Fliess *et al.*, 1995). In previous works, the connections between implicit systems and the zero dynamics was pointed out (Byrnes and Isidori, 1988; Rouchon, 1990; Krishnan and McClamroch, 1994) Our work studies this relationship further and shows that the normal form associated to static decoupling theory (Isidori, 1995) may be used for the numerical integration of a class of DAE's, at least when one introduces a modification in the dynamics that assures a direct convergence to the constraint manifold, as it will be shown later. In this paper we consider a semi-implicit system Γ of of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1a}$$

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$$y(t) = h(x(t)) = 0 \tag{1b}$$

where $x(t) \in I\!R^n$ is the *pseudo-state* of the system, $u(t) \in I\!R^m$ is the *pseudo-input*[2], and $y_i \equiv 0, i = 1, \ldots, r$ are the constraints. In the language of some DAE's researchers, the components of u are called algebraic variables, since no derivatives of u are involved in (1). The explicit system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{2a}$$

$$y(t) = h(x(t)) \tag{2b}$$

will be denoted by S. Note that S is obtained from Γ by considering that y is an output, instead of being a constraint.

In this paper, two methods of numerical integration of DAE's will be studied. The first one (that is not a contribution of the paper, but it is presented only for comparison terms) is simply the decoupling and stabilization of constraint functions y. Some numerical experiments shows that it is not a reliable numerical scheme for high index DAEs. The second method is a symbolic/numeric scheme (see (Campbell and Marszalek, 1998)), and it is based on the following geometric result. Given a completely determined DAE (1) one can construct an explicit system $\mathcal{T}$ (using some symbolic differentiations of the constraints y, that are executed for once and for all), in a way that the solutions of $\mathcal{T}$ converge to the solutions of Γ. Such a method is in fact an explicit integrator of a DAE (see (Campbell and Zhong, 1997)) In fact, under some regularity assumptions we will show that there exists an explicit control system given by

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \tau(x(t), u(t)) \tag{3}$$

having the property, that the subvector $x(t)$ of the solutions of (3) converge (globally) to the solutions of the implicit system (1) with compatible initial conditions and input $u(t)$. It is shown that the field τ may be constructed using the symbolic derivatives $y^{(k)}$ and their differentials for $k = 0, \ldots k^*$, where k^* is the differential index. Based on this result, one may develop a numerical integration method for a class of high index DAEs. In this method, one has to compute the symbolic derivatives of the constraints but all the other computation may be performed numerically. Note that, if the equations of the system are sparse, then matrix inversions may destroy this property, whereas the symbolic derivatives of restrictions will preserve sparsity, which is a nice feature of this second method. The paper is organized as follows. In this section, we present some basic facts about decoupling theory. In section 2 we present the first method of numerical integration, in section 3 we establish the second method. In section 5 we state some conclusion remarks. Finally, in the appendix, we show the results of some numerical experiments.

[2] Note that u is not a differentially independent input for Γ, since the constraints $y \equiv 0$ induce differential relations linking the components of u. By the same reasons, x is not a state of Γ.

1.1 Notation

The field of real numbers will be denoted by $I\!R$. The matrix M^T stands for the transpose of M. Given a function $f(t)$ then $\dot{f}$ stands for $\frac{d}{dt}f$ and $\frac{d^k}{dt^k}f$ will be denoted by $f^{(k)}$. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x = (x_1, \ldots, x_n)$ be a vector of functions. Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$ and $x^{(k)}$ stands for $(x_1^{(k)}, \ldots, x_n^{(k)})$. Let $\rho = (\rho_1, \ldots, \rho_r)$ and $y = (y_1, \ldots, y_r)$. Then $y^{\langle\rho\rangle}$ stands for $(y_1^{(\rho_1)}, \ldots, y_r^{(\rho_r)})$, and $Y^{\langle\rho\rangle}$ stands for $(y_1^{(0)}, \ldots, y_1^{(\rho_1)}, \ldots, y_r^{(0)}, \ldots, y_r^{(\rho_r)})$. Furthermore $y^{\langle\rho-1\rangle}$ stands for $(y_1^{(\rho_1-1)}, \ldots, y_r^{(\rho_r-1)})$ and $Y^{\langle\rho-1\rangle}$ stands for $(y_1^{(0)}, \ldots, y_1^{(\rho_1-1)}, \ldots, y_r^{(0)}, \ldots, y_r^{(\rho_r-1)})$.

1.2 Decoupling, normal forms and implicit systems

For a system S with output y given by (2) one can define the relative degree at x_0 as follows (Isidori, 1995). For $i = 1, \ldots, r$. Compute the derivative $\dot{y}_i = \frac{\partial h_i}{\partial x}\dot{x} = \frac{\partial h_i}{\partial x}(f(x) + g(x)u) = h_i^{(1)}(x, u)$. If we have $\frac{\partial h_i}{\partial u} \equiv 0$, locally around x_0, this process may be continued[3]. So, for $k = 1, \ldots, \rho_i$ we may compute $h_i^{(k)}(x, u) = \frac{\partial h_i^{(k-1)}}{\partial x}(f(x) + g(x)u)$, where ρ_i is the least integer k such that $h_i^{(k)}(x, u)$ does depend on u around x_0. Hence one may write

$$y^{\langle\rho\rangle} = a(x) + b(x)u \tag{4}$$

If the $r \times m$ matrix $b(x)$, called decoupling matrix, has constant rank r around x_0, then the decoupling problem by static-state feedback is locally solvable around x_0. In this case the output y is said to have relative degree $\rho = (\rho_1, \ldots, \rho_r)$ and there exists a local coordinate change $x \mapsto (Y^{\langle\rho-1\rangle}, \widehat{x})$, for which, in these coordinates, we have the following canonical form

$$\dot{y}_1 = y_1^{(1)} \tag{5a}$$

$$\vdots \quad \vdots \quad \vdots \tag{5b}$$

$$\dot{y}_1^{(\rho_1-2)} = y_1^{(\rho_1-1)} \tag{5c}$$

[3] It is easy to see that $\frac{\partial h_i}{\partial u}$ does not depend on u

$$\dot{y}_1^{(\rho_1-1)} = a_1 + b_1 u \quad (5d)$$

$$\vdots\vdots\vdots$$

$$\dot{y}_r = y_r^{(1)} \quad (5e)$$

$$\vdots\vdots\vdots$$

$$\dot{y}_r^{(\rho_r-2)} = y_r^{(\rho_r-1)} \quad (5f)$$

$$\dot{y}_r^{(\rho_r-1)} = a_r + b_r u \quad (5g)$$

$$\dot{\widehat{x}} = \eta(\widehat{x}, Y^{\langle\rho-1\rangle}, u) \quad (5h)$$

where $a = (a_1, \ldots, a_r)^T$ and $b = (b_1^T, \ldots, b_r)^T$ are given by (4). In the literature, (5h) is called *zero dynamics* (Byrnes and Isidori, 1988) and is directly related to the subsystem obtained by *zeroing the output* and hence to the implicit system. Let $\widehat{u}$ be a set of $m - r$ components. Up to a reordering, we may write $u = (\bar{u}, \widehat{u})$ where $\bar{u}$ are the remaining r components of u. One obtains

$$y^{\langle\rho\rangle} = a(x) + \bar{b}(x)\bar{u} + \widehat{b}(x)\widehat{u} \quad (6)$$

Assume that $\bar{u}$ is such that $\bar{b}(x)$ is locally nonsingular around x_0 [4] . Now choose $v = (y^{\langle\rho\rangle}, \widehat{u})$. Then it is clear that

$$v = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{b} & \widehat{b} \\ 0 & I \end{pmatrix} u$$

and so the regular static-state feedback (see (Isidori, 1995))

$$u = \alpha(x) + \beta(x)v \quad (7a)$$

where α and β are defined by

$$\alpha = -\begin{pmatrix} \bar{b} & \widehat{b} \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (7b)$$

$$\beta = \begin{pmatrix} \bar{b} & \widehat{b} \\ 0 & I \end{pmatrix}^{-1} \quad (7c)$$

is a static-feedback that solves the decoupling problem. Note that the closed-loop system has state $z = (\widehat{x}, Y^{\langle\rho-1\rangle})$ and input $v = (v_1, \ldots, v_r, \widehat{u})$, where $v_i = y_i^{(\rho_i)}$, given by

$$\dot{y}_1 = y_1^{(1)} \quad (8a)$$

$$\vdots\vdots\vdots \quad (8b)$$

$$\dot{y}_1^{(\rho_1-2)} = y_1^{(\rho_1-1)} \quad (8c)$$

$$\dot{y}_1^{(\rho_1-1)} = y_1^{(\rho_1)} \quad (8d)$$

$$\vdots\vdots\vdots$$

$$\dot{y}_1 = y_1^{(1)} \quad (8e)$$

$$\vdots\vdots\vdots$$

$$\dot{y}_r^{(\rho_r-2)} = y_r^{(\rho_r-1)} \quad (8f)$$

$$\dot{y}_r^{(\rho_r-1)} = y_r^{(\rho_r)} \quad (8g)$$

$$\dot{\widehat{x}} = \theta(\widehat{x}, Y^{\langle\rho\rangle}, \widehat{u}) \quad (8h)$$

Where $\theta = \eta(\widehat{x}, Y^{\langle\rho-1\rangle}, \alpha + \beta v)$. Now consider a implicit system (1). It is clear that the implicit system is equivalent to the output nulling dynamics or *zero dynamics*

$$Y^{\langle\rho\rangle} = 0 \quad (9a)$$

$$\dot{\widehat{x}} = \theta(\widehat{x}, 0, \widehat{u}) \quad (9b)$$

Let $\Sigma = I\!R^n \times I\!R^m$ with global coordinates (x, u). Consider the submanifold $\Gamma \subset \Sigma$ defined by (9a). It is clear that the zero dynamics evolves on the submanifold [5] Γ. The state and the input of the implicit dynamics is respectively given by $\widehat{x}$ and $\widehat{u}$. Note that the pair $(\widehat{x}, \widehat{u})$ is in fact a local coordinate system of the submanifold Γ.

In this paper we assume that the input $\widehat{u}$ of the implicit system (1) one may perform the construction above for the system (2). Note that, if $r = m$, *i. e.*, if the number of constraints are equal to the number of inputs, then $\widehat{u}$ is absent and the implicit system will be completely determined.

Let $v_i = y_i^{(\rho_i)}, i = 1, \ldots, r$. A standard trick now may be applied in order to ensure the output stabilization. For this it suffices to apply the regular static-state feedback

$$v_i = -\sum_{j=1}^{\rho_i-1} \alpha_{ij} y_i^{(j)}, \; i = 1, \ldots, r \quad (10)$$

where all the roots of the polinomials $\pi_i(s) = s^{\rho_i} + \sum_{j=1}^{\rho_i-1} \alpha_{ij} s^j$ have negative real part.

2. FIRST METHOD

The first method of integration of DAE's we present here is a straightforward application of decoupling theory, that is summarized in section 1.2. In this paper, some numerical experiments are performed in order to show that this method is not numerically reliable in general, even for linear systems.

This first method is in fact the application of the decoupling feedback (7) along with the stabilization feedback (10). Furthermore, if the initial condition is already in Γ, then the stabilizing feedback will not act, ensuring that the closed loop dynamics will be in fact will converge to the implicit dynamics (9b). Then it suffices to

[4] Since $b(x)$ has rank r, one can always choose $\bar{u}$ in a way that $\bar{b}$ is nonsingular

[5] By the results of (Isidori, 1995), assuming that y possesses relative degree around every point of Γ, then it is easy to show that Γ is a embedded submanifold of Γ

integrate the (explicit) closed loop system with a standard numerical method. This seems very nice, but numerical experiments shows that this method is not reliable due to the following reasons. The convergence to the manifold Γ is rather indirect. In fact since one aims to stabilize a chain of integrators, for some initial conditions (for instance a positive y_i and a positive $\dot{y}_i$) then the function $y_i(t)$ will have a positive derivative at t_0. In one hand, this process, when regarded from the numerical point of view, generates the necessity of imposing fast poles in order to have fast convergence, otherwise one observes a non negligible numerical noise in the variables $Y^{\langle\rho\rangle}$. On the other hand, if one imposes fast poles, one will observe a very fast dynamics, related to the variables $Y^{\langle\rho\rangle}$ and a slow dynamics, related to zero dynamics (9b). From the numerical point of view, this generates stiffness. So, as the numerical experiments show, it is very difficult to tune the poles in order to have good results. To illustrate this, consider the following implicit linear system

$$\dot{z} = Az + Bu \tag{11a}$$

$$y = Cx = 0 \tag{11b}$$

where

$$A = R^{-1}\widetilde{A}R, \qquad B = R^{-1}\widetilde{B} \qquad C = \widetilde{C}R \tag{11c}$$

where

$$\widetilde{A} = \begin{pmatrix} 0 & -100 & -100 & -100 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11d}$$

$$\widetilde{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{11e}$$

$$\widetilde{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \tag{11f}$$

By construction, it is clear that, if $x = (x_1x_2x_3x_4)^T$ and $z = (z_1z_2z_3z_4)^T$ with $x = Rz$, then

$$\dot{x}_1 = -100x_2 - 100x_3 - 100x4 \tag{12}$$

$$\dot{x}_2 = x_3 \tag{13}$$

$$\dot{x}_3 = x_4 \tag{14}$$

$$\dot{x}_4 = u \tag{15}$$

$$y = x_2 = 0 \tag{16}$$

In particular $y = x_2, \dot{y} = x_3, y^{(2)} = x_4, y^{(3)} = u$. This system is an index 4 implicit system. Applying the first method, one has to integrate the explicit system $\dot{z} = Az + Bu$ in closed loop with the output stabilizing feedback $u = -a_1y - a_2\dot{y} - a_3\ddot{y}$. Choosing $a_1 = \gamma^3, a_2 = 3\gamma^2, a_3 = 3\gamma$, one imposes the poles $\{-\gamma, -\gamma, -\gamma\}$ to the output dynamics. Note that, for this example, Γ is the set $\{(z, u) \in I\!\!R^5 | z = T_1^{-1}(c, 0, 0, 0)^T, c \in I\!\!R \text{ and } u = 0\}$. The exact solution of the implicit system for some $(z_0, 0) \in \Gamma$ is $z(t) \equiv z_0$.

3. SECOND METHOD FOR COMPLETELY DETERMINED SYSTEMS

In this section we assume that the output y of system (2) possesses relative degree and $m = r$. In particular, the input $\widehat{u}$ of the implicit system (1) is absent and so it is completely determined. Note that, in this case $(\widehat{x}, Y^{\langle\rho\rangle})$ will be a local coordinate system of space $\Sigma = I\!\!R^n \times I\!\!R^m$ (which possesses canonical coordinates (x, u). In these coordinates the dynamics of the explicit system (2) reads like (8) where $\widehat{u} \equiv 0$ (no input). Recall from section 2 that a problem of the first method is that the stabilization of a chain of integrators is rather indirect. The idea of the second method is to forget that $y_i^{(j)}$ is the derivative of $y_i^{(j-1)}$ and to regard the vector $Y^{\langle\rho\rangle}$ as a vector of independent variables. So, in the coordinates $(\widehat{x}, Y^{\langle\rho\rangle})$ of the space $\Sigma = I\!\!R^n \times I\!\!R^m$, take $\gamma > 0$ and define the following auxiliary system obtained by performing what can be called a generalized output injection

$$\dot{y}_1 = -\gamma y_1 \tag{17a}$$

$$\vdots \tag{17b}$$

$$\dot{y}_1^{(\rho_1-1)} = -\gamma y_1^{(\rho_1-1)} \tag{17c}$$

$$\dot{y}_r^{(\rho_1)} = -\gamma \dot{y}_r^{(\rho_1)} \tag{17d}$$

$$\vdots$$

$$\dot{y}_r = -\gamma y_r \tag{17e}$$

$$\vdots \tag{17f}$$

$$\dot{y}_r^{(\rho_r-1)} = -\gamma y_r^{(\rho_r-1)} \tag{17g}$$

$$\dot{y}_r^{(\rho_r)} = -\gamma \dot{y}_r^{(\rho_r)} \tag{17h}$$

$$\dot{\widehat{x}} = \theta(\widehat{x}, Y^{\langle\rho\rangle}) \tag{17i}$$

By construction it is clear that the solution of the auxiliary system is such that $Y^{\langle\rho\rangle}(t) = Y^{\langle\rho\rangle}(t_0)e^{-\gamma(t-t_0)}$. In particular $\lim_{t\to\infty} Y^{\langle\rho\rangle} = 0$ (see Theorem 1).

Now, to establish our second method, it suffices to rewrite system (17) in the original coordinates (x, u). For this, denote by $\Psi(x, u) = (\widehat{x}, Y^{\langle\rho\rangle})$. Around (x_0, u_0) note that Ψ is a diffeomorphism. Hence, the matrix $T(x, u) = \frac{\partial\Psi}{\partial(x,u)}$ is inversible. Note that $\dot{\widehat{x}} = \frac{\partial\widehat{x}}{\partial x}(f(x) + g(u)) = \theta(\widehat{x}, Y^{\langle\rho\rangle})$. Now define the explicit completely determined system

$$\frac{d}{dt}\begin{pmatrix} x \\ u \end{pmatrix} = \tau(x, u) \tag{18}$$

where $\tau(x,u)$ is defined by the following rule

$$T(x,u)\tau(x,u) = \begin{pmatrix} \dfrac{\partial \widehat{x}}{\partial x}(f(x)+g(u)) \\ -\gamma Y^{\langle \rho \rangle} \end{pmatrix} \qquad (19)$$

Write

$$Y^{\langle \rho \rangle} = \begin{pmatrix} Y^{\langle \rho-1 \rangle} \\ y^{\langle \rho \rangle} \end{pmatrix}$$

Then note that $T(x,u)$ is of the following form

$$T(x,u) = \begin{pmatrix} \dfrac{\partial \widehat{x}}{\partial x} & 0 \\ \dfrac{\partial Y^{\langle \rho-1 \rangle}}{\partial x} & 0 \\ \dfrac{\partial y^{\langle \rho \rangle}}{\partial x} & \dfrac{\partial y^{\langle \rho \rangle}}{\partial u} \end{pmatrix}$$

Note that $T(x,u)$ may be determined by symbolic computations and the solution of the linear equation (19) may be computed pointwise by numerical methods. If the system is sparse, the the matrix T of equations (19) is also sparse.

Remark 1. Note that, from decoupling theory, to choose the state-variables $\widehat{x}$ of the zero dynamics one may assure that the matrix $T(x,u)$ of (19) is pointwise nonsingular. Two facts may arise from this remark. Sometimes it is easy to choose $\widehat{x}(x)$ using symbolic manipulations (for instance, in our example the choice of some components of x does the job). When it is not possible (or not easy) to choose $\widehat{x}(x)$ using symbolic manipulations, it can be shown that one may choose $\widehat{x}(x)$ pointwise using numerical methods in the following way. At $(\bar{x},\bar{u})$, choose a constant matrix R in way that the matrix $T(\bar{x},\bar{u})$ defined by

$$T(\bar{x},\bar{u}) = \begin{pmatrix} R & 0 \\ \dfrac{\partial Y^{\langle \rho-1 \rangle}}{\partial x} & 0 \\ \dfrac{\partial y^{\langle \rho \rangle}}{\partial x} & \dfrac{\partial y^{\langle \rho \rangle}}{\partial u} \end{pmatrix}$$

Then find $\tau(\bar{x},\bar{u})$ numerically by solving the following linear equation

$$T(\bar{x},\bar{u})\tau(\bar{x},\bar{u}) = \begin{pmatrix} R(f(\bar{x})+g(\bar{x})\bar{u}) \\ -\gamma Y^{\langle \rho \rangle} \end{pmatrix} \qquad (20)$$

It can be shown that this technique ensures the convergence of $(x(t),u(t))$ to the manifold Γ and for a point $(\bar{x},\bar{u})$ over Γ such a construction defines the same field τ defined by (19).

One can show the following result that means that the solutions of the auxiliary system (18) converges to the solution of the implicit system (1).

Theorem 1. The following statements holds:
(i) Let $\zeta(t) = (x(t),u(t))$ be a solution of (18) with $\zeta(t_0) \in \Gamma$. Then $x(t) = \pi_x(\zeta(t))$ is a solution of (1) with input $u(t)$. Conversely, if $x(t)$ is a smooth solution of (1), then $x(t)$ is equal to $\pi_x(\zeta(t))$ for some solution of (18) with $(\zeta(t_0),u(t_0)) \in \Gamma$.

(ii) Let $\zeta(t) = (x(t),u(t)$ be a solution of (18) with initial condition ζ_0. Assume that $\zeta(t)$ is well defined for $t \in ([t_0,t_1]$, then $\|Y^{\langle \rho \rangle}\| \le e^{-\gamma t}\|Y^{\langle \rho \rangle}\|$ for all $t \in [t_0,t_1]$.

(iii) Let $L \subset \Sigma = I\!R^n \times I\!R^m$ be a compact set. Let $L_1 = \{\mu \in \Sigma \mid \mathrm{dist}(\mu,L) < \epsilon_1\}$ for a given $\epsilon_1 > 0$. Assume that every solution $\zeta(t)$ of (18) with initial condition $\zeta(t_0) \in L_1$ is such that $\zeta(t)$ is well defined and is inside a compact set $R \subset \Sigma$ for every $t \in [t_0,t_1]$. Then there exists $\epsilon > 0$ such that, if $\zeta(t), t \in I = [t_0,t_f]$ is a solution of (18) with initial condition inside L, and $\|Y^{\langle \rho \rangle}(t_0)\| < \epsilon$, then there exist $\kappa_1,\kappa_2 > 0$ and a solution $x(t)$ of (1) such that $\|\pi_x(\zeta(t)) - x(t)\| \le \kappa_1\|Y^{\langle \rho \rangle}(t_0)\|e^{\kappa_2(t-t_0)}$ for all $t \in [t_0,t_f]$.

4. NUMERICAL EXPERIMENTS

In the following numerical experiments we have used the initial condition $z_0 = T_{-1}x_0$, with $x_0 = (2, 1e-4, 1e-4, 1e-4)^T$ Note that (x_0,u_0), with small u_0 is close to the the submanifold Γ.

Fig. 1 plots $x_1(t)$ obtained with Method 1, $\gamma = 1$, Matlab Simulink option = ode45, tolerance = 1e-04. The solution diverges.

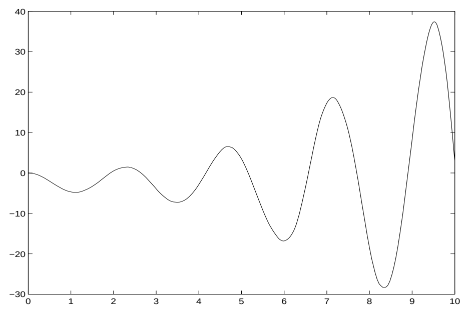

Fig. 1.

Fig. 2 depicts $x_1(t)$, simulated under Method 1, $\gamma = 10$, Matlab Simulink option = ode23, tolerance = 1e-04. The solution still diverges.

On Fig. 3 one can see $y(t)$ simulated with Method 2, $\gamma = 1$, Matlab Simulink option = ode45, tolerance = 1e-04.The ideal value of $y(t)$ is zero, but a numerical error of about 1×10^{-6} is observed.

Finally, Fig. 4 shows the result of the simulation of $x_1(t) - 2$, obtained with Method 2, $\gamma = 1$, Matlab Simulink option = ode45, tolerance = 1e-04. The ideal value is constant, and a numerical error of only 2×10^{-4} is observed.

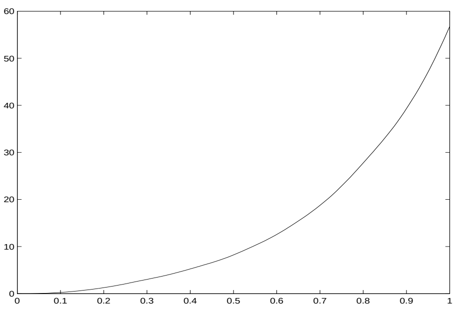

Fig. 2.

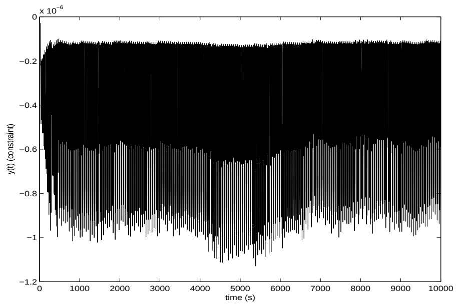

Fig. 3.

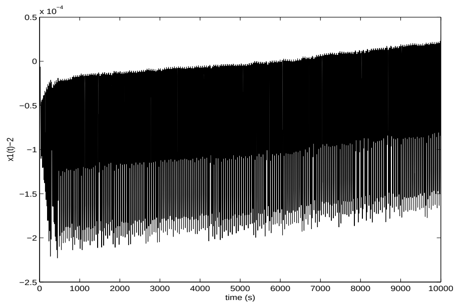

Fig. 4.

5. CONCLUSIONS

This paper have established an efficient method for integrating DAEs numerically. This method can be easily extended to time-varying systems and systems with inputs, *i. e.* , systems that are not completely determined. Many numerical experiments have been performed for both systems, with linear and nonlinear systems and using various Matlab Simulink options. By reasons of space, only a small number of such experiments are shown in the appendix.

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