On state representations of time-varying nonlinear systems ⋆

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Abstract

This work considers a nonlinear time-varying system described by a state representation, with input u and state x. A given set of functions v, which is not necessarily the original input u of the system, is the (new) input candidate. The main result presents necessary and sufficient conditions for the existence of a local classical state space representation with input v. These conditions rely on integrability tests that are based on a derived flag. As a byproduct, one obtains a sufficient condition of differential flatness of nonlinear systems.

Key words: Nonlinear systems; state representations; differential flatness; differential geometric approach.

1 Introduction

A *classical state representation* is essentially a set of differential equations of the form

$$\dot{z}(t) = g(t, z(t), v(t)) \tag{1}$$

where $z(t) \in I\!R^s$ is the state and $v(t) \in I\!R^q$ is the input. When the state representation depend on the derivatives of the input, that is, when the state equations are of the form

$$\dot{w}(t) = \eta(t, w(t), v^{(0)}(t), v^{(1)}(t), \ldots, v^{(\alpha)}(t)) \tag{2}$$

with η depending on $v^{(\alpha)}$ for $\alpha > 0$, then the state representation (with state w and input v) is said to be *generalized* (see Fliess et al. (1993b)). An output y of (2) given by

$$y = h(t, w(t), v^{(0)}(t), \ldots, v^{(\beta)}(t))$$

is said to be classical with respect to the state representation (2), if y does not depend on $v^{(j)}$ for $j > 0$, that is, y may be written as $y = \hat{h}(t, w(t), v(t))$.

The problem of realization of input/output nonlinear differential equations is the problem of giving classical state representations (1) for input-output equations. This problem was extensively studied in the literature (see for instance van der Schaft (1987); Glad (1988); Crouch and Lamnabhi-Lagarrigue (1988); van der Schaft (1990); Liu and Moog (1994); Moog et al. (2002); Jakubczyk (1986)). A comparison between some of these works can been found in Kotta and Mullari (2005). Closed related to this problem is the one of seeking generalized state transformations $z = \phi(t, w, v^{(0)}, \ldots, v^{(\alpha)})$ in a way that (2) is converted to (1), that is, the input derivatives are eliminated. The time-invariant version of this problem is solved in Delaleau and Respondek (1995).

Recall that, in the behavioral approach of Willems (1992), the input and the output are not chosen *a priori*. The same point of view is shared by the approach of Fliess et al. (1999), and this fact is in accordance of what is found in physical systems. For instance, the DC motor can be represented by the following model [1]

$$Li^{(1)} + Ri + [K(i)]\theta^{(1)} = E, \tag{3a}$$
$$J\theta^{(2)} + B\theta^{(1)} = [K(i)]i - \tau \tag{3b}$$

where θ is the shaft angle, i is the armature current, L is is the armature inductance, E is the external voltage, R is the armature resistance of the motor, τ is the external

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[1] The model (3) may be found in Atay (2000). One may disregard the effect of magnetic saturation, considering that $K(i)$ is a constant. This may be a bad approximation for high current values (see Atay (2000)). The linear version of this example is due to Emmanuel Delaleau.

torque (or the load torque), J is the joint inertia of the armature and shaft, and $K(i)$ is the function that is related to the dependence of the flux on the current. When the input is the voltage E, the disturbance input is the load torque τ, and the output is the shaft angle θ, the device is working as a motor and one may give the following state equations

$$\frac{d}{dt}i = -(R/L)i - (K(i)/L)\theta^{(1)} + (1/L)E, \quad (4a)$$

$$\frac{d}{dt}\theta^{(0)} = \theta^{(1)}, \quad (4b)$$

$$\frac{d}{dt}\theta^{(1)} = (K(i)/J)i - (B/J)\theta^{(1)} - (1/J)\tau, \quad (4c)$$

$$y = \theta^{(0)} \quad (4d)$$

with state $x = (i, \theta^{(0)}, \theta^{(1)})$, input $u = (E, \tau)$ and output $y = \theta^{(0)}$. If one considers that the device is working as a voltage generator, one may take the external torque τ as an input, the current i on the load as a disturbance input, and the voltage E as an output, and one obtains the following state equations

$$\frac{d}{dt}\theta^{(0)} = \theta^{(1)}, \quad (5a)$$

$$\frac{d}{dt}\theta^{(1)} = (K(i)/J)i - (B/J)\theta^{(1)} - (1/J)\tau, \quad (5b)$$

$$y_1 = K\theta^{(1)} + Ri + Li^{(1)} \quad (5c)$$

with state $z = (\theta^{(0)}, \theta^{(1)})$, input $v = (\tau, i)$ and output $y_1 = E$. Note that the dimension of the states of (4) and (5) are different. The equations (3) are the same for the motor and the generator, since they represent the model of the same physical system, but one may choose a different set of inputs and outputs, giving rise to different state equations.

Motivated by this example, one may state the following problem [2]. Given a system that is defined by its state representation

$$\dot{x}(t) = f(t, x(t), u(t)) \quad (6)$$

where $x(t) \in I\!R^n$ is the state, and $u(t) \in I\!R^m$ is the input, when does this system admit another classical state representation (1), with a given input candidate v? Now, v is a new input, and so the original state x and the new state z may have different dimensions. For instance, given the system defined by (4) with input $u = (E, \tau)$ and state $x = (i, \theta, \theta^{(1)})$, one may ask if $v = (\tau, i)$ is a possible input of (4), corresponding to classical state equations. When the output y_1 is disregarded, the expected answer is yes, as (5a-b) is an alternate state representation of the same physical system (3), and this may be confirmed by the application of Theo. 1 of section 2. In section 2, necessary and sufficient conditions for the solution of this problem are given. In section 3 a version of this problem for systems with outputs is considered. As a byproduct, a sufficient condition of differential flatness is shown in section 4. Our approach will follow the infinite dimensional geometric setting introduced in control theory by Fliess et al. (1993a); Pomet (1995); Fliess et al. (1999).

[2] In this section one will only motivate the problem. See the precise statements and definitions in section 2.

2 Notations, Preliminaries and Problem Statement

The standard notations of differential geometry will be considered in the finite and infinite dimensional case. The field of real numbers will be denoted by $I\!R$. The set of natural numbers $\{1, \ldots, k\}$ will be denoted by $\lfloor k \rceil$. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}}$, where z_1 and z_2 are also column vectors, and the transpose of z_i is z_i^{T}, i=1,2. Let $x = (x_1, \ldots, x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$. Some notations and definitions of Fliess et al. (1999); Pereira da Silva and Corrêa Filho (2001) are used along the paper. (*e. g.* the definition of a system as a *diffiety*, and the definition of *state representation* as a local coordinate system). The survey Pereira da Silva (2008a) presents an elementary introduction to this approach. If S is a diffiety with Cartan field $\frac{d}{dt}$, and u is a function defined on S, then the Lie derivative $\frac{d}{dt}u$ of a function u will be denoted by $\dot{u}$ (or $u^{(1)}$) and the k-fold Lie derivative $\frac{d}{dt}^k(u)$ of u will be denoted by $u^{(k)}$. If $u = (u_1, \ldots, u_m)$ is a set of functions defined on S then $u^{(k)} = (u_1^{(k)}, \ldots, u_m^{(k)})$. If $\omega = \sum_{k=1}^{p} \alpha_k d\phi_k$ is a one-form defined on S, then $\dot{\omega}$ stands for $L_{\frac{d}{dt}}\omega = \sum_{k=1}^{p} \dot{\alpha}_k d\phi_k + \alpha_k d\dot{\phi}_k$. A nonsingular codistribution Γ that is generated by a linearly independent set $\{d\omega_1, \ldots, d\omega_r\}$ is *integrable* if $d\omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0$ for $i \in \lfloor r \rceil$. In the rest of this paper, one will refer to system S, as stated in the next definition.

Definition 1 *Consider the state equations* (6)*, where $t \in I\!R$ is the time, $x(t) \in I\!R^n$ is the state, and $u(t) \in I\!R^m$ is the input, and f is smooth with respect to its arguments.* The system S associated to (6) *is the diffiety with (global) coordinates $\{t, x, (u^{(k)} : k \in I\!N)\}$ and the Cartan field $\frac{d}{dt}$ is given by $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k \in I\!N} \sum_{j \in \lfloor m \rceil} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}$. A local state representation (w, v) of S is a set of functions $\{w_1, \ldots, w_s, v_1, \ldots, v_q\}$, defined on some open set of S, such that $\{t, w, (v^{(k)} : k \in I\!N)\}$ is a local coordinate system*[3] *of S. In this*

[3] The approach of Fliess et al. (1999) considers an intrinsic definition of state representation. See Pereira da Silva (2008a) for a comparison with the present definition.

case, the Cartan field will be locally given by $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{s} \eta_i \frac{\partial}{\partial z_i} + \sum_{k\in I\!N} \sum_{j\in\lfloor q\rceil} v_j^{(k+1)} \frac{\partial}{\partial v_j^{(k)}}$ *and the corresponding local state equations*[4] *will be given by* (2)*, where* $\eta = (\eta_1, \dots, \eta_s)$*. A local state representation is said to be* classical *if* $\alpha = 0$ *in* (2)*, that is, if* η_i *does not depend on* $v^{(j)}$ *for* $j > 0$*, and for* $i \in \lfloor q\rceil$*.*

Now, the main problem considered in this paper may be precisely stated.

Problem 1 *Let* $v = (v_1, \dots, v_q)$ *be a set of functions defined on*[5] *the diffiety* S*. The Problem of Classical state Representation with input* v *is solvable around some* $\xi \in S$*, if* S *admits a local classical state representation* (z, v) *defined in an open neighborhood* U *of* ξ*.*

Remark 1 *Note that there is no loss of generality in assuming that the original system* S *is defined by* (6)*. For instance, if the original equations*[6] *were in the form* (2)*, then one may take the dynamic extension* $x = (w, v, \dots, v^{(\alpha-1)})$ *and* $u = v^{(\alpha)}$*. In this case, one will obtain a classical state representation* (6)*.*

The following lemma will be useful in this paper.

Lemma 1 *(Pereira da Silva, 2008a, Lemma 1) Let* (x, u) *and* (z, v) *be two local classical state representations of the system* S *defined on an open neighborhood* U *of* $\xi \in S$*.*
(1) If $b \in I\!N$ *is such that* $\mathit{span}\,\{dv\} \subset \mathit{span}\,\{dt, dx, du^{(0)}, \dots, du^{(b)}\}$*, then* $\mathit{span}\,\{dz\} \subset \mathit{span}\,\{dt, dx, du^{(0)}, \dots, du^{(b-1)}\}$[7]*.*
(2) Let $\beta \in I\!N$ *be the smallest non-negative integer such that there exists an open neighborhood* V *of* ξ *such that*[8] $\mathit{span}\,\{du\} \subset \mathit{span}\left\{dt, dz, dv^{(0)}, \dots, dv^{(\beta)}\right\}$ *on* V*. If* $\mathit{span}\,\{dz, dv\} \subset \mathit{span}\left\{dt, dx, du^{(0)}, \dots, du^{(\gamma)}\right\}$*, then* $\beta \le n + m\gamma$*, where* $n = \dim x$ *and* $m = \dim u$*.*

Now, the main result of the section will be presented.

Theorem 1 *Consider the system* S *defined by* (6)*. Let* $v = (v_1, \dots v_m)$ *be a set of functions locally defined around* $\xi \in S$*. Let* γ *be the least non-negative integer such that* $\mathit{span}\,\{dv\} \subset \mathit{span}\left\{dt, dx, du^{(0)}, \dots, du^{(\gamma)}\right\}$ *(locally). For a given* $\delta \in I\!N$*, one may define derived flag* Γ_k*,* $k \in I\!N$ *on the diffiety* S *by*

$$\Gamma_0 = \mathit{span}\left\{dt, dx, du^{(0)}, \dots, du^{(\gamma)}, dv^{(0)}, \dots, dv^{(\delta)}\right\} \tag{7a}$$
$$\Gamma_k = \mathit{span}\left\{\omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1}\right\} \tag{7b}$$

Then Problem 1 is solvable if and only if there exists an integer δ*, with* $0 \le \delta \le n + \gamma(1 + m)$ *such that*
(i) Γ_k *is nonsingular for* $k = 0, \dots, \delta+1$*, and* $(\dim \Gamma_{k-1} - \dim \Gamma_k) = m$*, for* $k = 1, \dots, \delta + 1$*.*
(ii) $\Gamma_{\delta+1}$ *is integrable.*
(iii) $\Gamma_0 = \Gamma_1 \oplus \mathit{span}\left\{dv^{(\delta)}\right\}$*.*
(iv) The set $\{dv^{(k)}\}$ *is locally linearly independent for* $k = 0, \dots, \delta$*.*

The conditions of Theorem 1 will be briefly discussed before proving this result. Note first that, as one seeks z such that $\{t, z, (v^{(i)}, i \in I\!N)\}$ is a local coordinate system, it is clear that the set $\mathbb{B} = \{dv^{(0)}, \dots, dv^{(k)}\}$ is linearly independent for all $k \in I\!N$. Under certain assumptions[9] the independence of $\mathbb{B}$ for all k would imply that there exits local coordinates $\{t, \zeta, (v^{(i)} : i \in I\!N)\}$. This means that the system would admit a generalized state representation $\dot{\zeta} = F(t, \zeta, v, \dots, v^{(\alpha)})$. Then one would apply, for instance, the conditions of Delaleau and Respondek (1995) to the last state representation, obtaining solvability conditions of Problem 1. This could be a way of proving Theorem 1, but this may lead to some technical difficulties. Note that condition (iv) is weaker than the independence of $\mathbb{B}$ for all $k \in I\!N$. The conditions of Theorem 1 may appear to be technical, and so Examples 4 and 5 will present some situations where such conditions may fail. The proof of necessity of Theorem 1 relies on the next Lemma, whose proof is straightforward by direct computation (see Liu and Moog (1994); Batista (2006)):

Lemma 2 *Let* S *be a system with local classical state representation* (z, v) *and state equations* (1)*. Fix some* $\delta \in I\!N$*. Let* $\Gamma_0 = \mathit{span}\,\{dt,\, dz,\, dv, \dots, dv^{(\delta)}\}$ *and let* $\Gamma_k = \mathit{span}\,\{\omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1}\}$*. Then one locally has* $\Gamma_k = \mathit{span}\left\{dt, dz, dv, \dots, dv^{(\delta-k)}\right\}$ *and* $\Gamma_{\delta+1} = \mathit{span}\,\{dt, dz\}$*.*

The proof of sufficiency is based on the following result.

Lemma 3 *(Pereira da Silva (2008a)) Let* (x, u) *be a local classical state representation of a system* S *around some* $\xi \in S$*, and let* $z = (z_1, \dots, z_s)$ *and* $v = (v_1, \dots, v_q)$ *be sets of functions defined on the diffiety* S *such that the set* $\mathbb{S} = \{dt, dz, dv, \dots, dv^{(\alpha)}\}$ *is (locally) linearly independent. Assume that* $\mathit{span}\,\{dx\} \subset$

[4] The Def. 1 mimics the concept of endogeneous feedback. Note that the dimension of an input v is the differential dimension of S. In particular, $q = m$ (see Fliess et al. (1999)).
[5] Note that each v_i may be a function of $(t, x, u^{(0)}, u^{(1)}, \dots, u^{(\alpha)})$ for some α. If Problem 1 is solvable, one must have $q = m$ (see footnote 4).
[6] If the choice of v of Problem 1 coincides with the v of (2), then one recovers a time-varying version of the Problem stated in Delaleau and Respondek (1995).
[7] When $b = 0$, then span $\{dt, dx, du^{(0)}, \dots, du^{(b-1)}\}$ stands for span $\{dx\}$.
[8] The integer β always exists (locally), since $\{t, z, (v^{(k)}, k \in I\!N)\}$ is a local coordinate system.

[9] See for instance Pereira da Silva (2008a) for several special versions of the inverse function theorem for diffieties.

span $\{dt, dz, dv, \ldots, dv^{(\alpha-1)}\}$, *and span* $\{d\dot{z}, du\} \subset$ *span* $\{dt, dz, dv, \ldots, dv^{(\alpha)}\}$ *around* ξ. *Suppose also that span* $\{dz, dv\} \subset$ *span* $\{dt, dx, du\}$. *Then* (z, v) *is also a local state representation of* S *around* ξ.

Proof of Theorem 1.
Necessity. Let (z, v) be a local classical state representation of system S defined by (6) with state equations (1). Since $\{t, z, (v^{(k)} : k \in I\!N)\}$ is a local coordinate system of the diffiety S, then $x = x(t, z, v^{(0)}, \ldots, v^{(\beta-1)})$ and $u = u(t, z, v^{(0)}, \ldots, v^{(\beta)})$ for $\beta \geq 0$ big enough (see Lemma 1). Hence, $du^{(k)} \in \text{span}\ \{dt, dz, dv^{(0)}, \ldots, dv^{(\beta+k)}\}$ for all $k \in I\!N$. In the same way, one may (locally) write $z = z(t, x, u^{(0)}, \ldots, u^{(r-1)})$ and $v = v(t, x, u^{(0)}, \ldots, u^{(r)})$ for a convenient $r \in I\!N$. Now take $\delta = \beta + r$ and $\gamma = r$. By construction, it follows that $\text{span}\,\{dz\} \subset \text{span}\,\{dt, dx, \ldots, du^{(\gamma-1)}\}$, (or $\text{span}\,\{dz\} \subset \text{span}\,\{dt, dx\}$, if $\gamma = 0$). Hence, $\text{span}\,\{dt, dx, du, \ldots, du^{(\gamma)}\} \subset \text{span}\ \{dt, dz, dv, \ldots, dv^{(\delta)}\}$. In particular, one shows that, for $\gamma = r$ and $\delta = \beta + r$, one has $\Gamma_0 = \text{span}\ \{dt, dz, dv,\ \ldots, dv^{(\delta)}\}$. The proof of necessity then follows from Lemma 2. The fact that $\delta \leq n + \gamma(1 + m)$ follows easily from part 2 of Lemma 1.
Sufficiency. It will be shown first that

$$\text{span}\left\{dv, \ldots, dv^{(\delta-k)}\right\} \subset \Gamma_k, k = 0, \ldots, \delta \tag{8a}$$

$$\Gamma_k = \Gamma_{k+1} \oplus \text{span}\left\{dv^{(\delta-k)}\right\}, k = 0, \ldots, \delta \tag{8b}$$

The property (8a) is a straightforward consequence of the definition (7b). The equation (8b) will be shown by induction. By the assumption (iii) of Theorem 1, (8b) holds for $k = 0$. Assume that it holds for some k, with $0 \leq k < \delta$ in some $V \subset S$. By reduction to the absurd, one assumes that $\Gamma_{k+2} \cap \text{span}\left\{dv^{(\delta-k-1)}\right\}|_\nu \neq \{0\}$ for some $\nu \in V$. By (iv) and (8a), one can construct a local basis of Γ_{k+1} of the form $\{\omega_1, \ldots, \omega_s, dv^{(\delta-k-1)}\}$ around ν. Let $\omega = \sum_{i=1}^{s} \alpha_i \omega_i + \sum_{j=1}^{m} \beta_j dv_j^{(\delta-k-1)}$ be a smooth one form in Γ_{k+2} such that $\omega|_\nu \in \Gamma_{k+2} \cap \text{span}\left\{dv^{(\delta-k-1)}\right\}|_\nu$. This is equivalent to have $\alpha_i|_\nu = 0, i \in \lfloor s \rceil$ and some $\beta_j|_\nu \neq 0$. It follows from (7b) that $\dot{\omega} = \sum_{i=1}^{s} \alpha_i \dot{\omega}_i + \sum_{j=1}^{m} \beta_j dv_j^{(\delta-k)} + \gamma \in \Gamma_{k+1}$, where $\gamma = \sum_{i=1}^{s} \dot{\alpha}_i \omega_i + \sum_{j=1}^{m} \dot{\beta}_j dv_j^{(\delta-k-1)} \in \Gamma_{k+1}$. One concludes that $\sum_{i=1}^{m} \beta_j dv_j^{(\delta-k)}|_\nu \in \Gamma_{k+1} \cap \text{span}\left\{dv^{(\delta-k)}\right\}|_\nu$. This contradicts the induction hypothesis. Now, by (i) and (iv), one shows (8b) from simple dimensional arguments. Now, let $\omega = dt$. Since $\dot{\omega} = 0$, it follows that $dt \in \Gamma_k, k = 0, \ldots \delta + 1$. In particular, $\text{span}\,\{dt\} \in \Gamma_{\delta+1}$. Since $\Gamma_{\delta+1}$ is nonsingular and integrable, from Frobenius theorem, there exists (locally) a set of functions $\{z_1, \ldots, z_p\}$ such that $\Gamma_{\delta+1} = \text{span}\,\{dt, dz\}$. By (8b), it is clear that the set $\mathbb{S} = \{dt, dz, dv, \ldots, dv^{(\delta)}\}$ is (locally) linearly independent and $\Gamma_0 = \text{span}\,\{\mathbb{S}\}$. By (8b) for $k = \delta$, and from (7b), one concludes that $\text{span}\,\{d\dot{z}\} \in \text{span}\,\{dt, dz, dv\}$. By construction, from the fact that $\text{span}\,\{d\dot{x}\} \subset \text{span}\,\{dt, dx, du\} \subset \Gamma_0$, and $\text{span}\left\{du^{(i)}\right\} \subset \Gamma_0$ for $i = 0, \ldots, \gamma$, it follows that $\text{span}\left\{d\dot{x}, du^{(0)} \ldots, du^{(\gamma)}\right\} \subset \Gamma_0$. So, by Lemma 3 applied to the classical state representation $(\tilde{x}, \tilde{u})$, where $\tilde{x} = (x, u, \ldots, u^{(\gamma-1)})$ and $\tilde{u} = u^{(\gamma)}$, it follows that (z, v) is a local state representation for the system. Since $\text{span}\,\{d\dot{z}\} \in \text{span}\,\{dt, dz, dv\}$, this local state representation is classical. From part 2 of Lemma 1, it follows that $\delta \leq n + \gamma(1 + m)$. □

3 Systems with outputs

Consider system S defined by (6) with output y given by [10]

$$y(t) = h(t, x(t), u(t)) \tag{9}$$

Given a set of functions v defined on S, one may state the following problem.

Problem 2 *When there exists a local classical state representation* (z, v) *with state equations* (1), *and the output is also classical (with respect to* (z, v)*)?*

Recall that the output is classical (with respect to (z, v)) if one may locally write

$$y = \tilde{h}(t, z(t), v(t))$$

The following theorem answers this question.

Theorem 2 *Consider system* S *defined by* (6) *with output* y *given by* (9). *Problem 2 is solvable if and only if the assumptions of Theorem 1 hold for some* $\delta \in I\!N, \delta \leq n + \gamma(1 + m)$, *and furthermore, span* $\{dy\} \subset \Gamma_{\delta+1}$.

Proof. Straightforward from the proof of Theo. 1. □

4 A sufficient condition of flatness

In this section one will show that, under some conditions, one may choose a (virtual) input v in a way that there exists a local classical state representation (z, v) such that (1) is linearizable by regular static-state feedback $v = \psi(t, z, \mu)$, where μ is the new input. Since v is not the actual input of the system, this state feedback is endogenous, but is not really a static-state feedback.

[10] There is no loss of generality in considering that (x, u) is a classic state representation and y is a classic output, that is, y does not depend on $u^{(j)}$ for $j > 0$ (see Remark 1).

In particular, the system is linearizable by endogenous feedback, and so it is flat (see Fliess et al. (1999)). Although v is not the actual input of the system, one may develop a flatness based control. Now, consider some results and definitions of Pereira da Silva (2008c). Let S be the system defined by (6). A *time-varying regular static-state feedback (TVRSSF)* is a local diffeomorphism ϕ such that $(t,x,u) \mapsto (t,z,v)$, where $(t,x) \mapsto (t,z)$ is also a local diffeomorphism, called state transformation. The closed loop equations are given by

$$\dot{z}(t) = \tilde{f}(t, z(t), v(t)) \tag{10}$$

where $\tilde{f}(t,z,v) = \left[\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(t,x,u)\right]\Big|_{(t,x,u)=\phi^{-1}(t,z,v)}$. *The linearization problem by TVRSSF* seeks a TVRSSF such that the closed loop system locally reads

$$\dot{z}(t) = Az(t) + Bv(t)$$

where this last system is time-invariant, linear and controllable (and hence it is flat, as shown in Fliess et al. (1999)). This problem is solved for instance in Pereira da Silva (2008c), where it is shown that Prop. 1 generalizes the results of (Aranda-Bricaire et al. (1995); Jakubczyk and Respondek (1980)).

Proposition 1 *Let $\Delta_0 = span\,\{dt, dx\}$ and let $\Delta_k = span\ \{\omega \in \Delta_{k-1} \,|\, \dot{\omega} \in \Delta_{k-1}\}$. Then the linearization problem by TVRSSF is locally solvable around $\xi \in S$ if and only if:*
(1) The codistributions Δ_k are nonsingular at ξ for $k \in I\!N$.
(2) There exists $k^* \in I\!N$ big enough, such that $\Delta_{k^*} = span\,\{dt\}$.
(3) The codistributions Δ_k are locally integrable around ξ, for $k \in I\!N$.

The following result is a sufficient condition of flatness of a nonlinear system.

Theorem 3 *Consider the system S defined by* (6)*. Let $v = (v_1, \ldots, v_m)$ be a collection of functions defined on S. Let δ and γ, be integers, and let Γ_0 be the codistribution defined in Theorem 1. Assume that the conditions of Theorem 1 hold around some $\xi \in S$. Let $\Delta_0 = \Gamma_{\delta+1}$ and let $\Delta_k = span\,\{\omega \in \Delta_{k-1} | \dot{\omega} \in \Delta_{k-1}\}, k \in I\!N$. Then if the conditions 1, 2 and 3 of Proposition 1 are satisfied for $\Delta_k, k \in I\!N$, then the system is locally flat around ξ.*

Proof. By the proof of Theorem 1, system S admits a local classical state representation (z, v) around ξ and $\Delta_0 = \text{span}\,\{dt, dz\}$. In particular, the conditions of Proposition 1 are satisfied for the state representation (z, v). In particular, the system admits a flat output y that is a function of z and t. Note that a given system S is not necessarily static-feedback linearizable with respect to the original state representation (x, u) (see example 2). □

5 Examples

In order to illustrate the results of this work, five very simple examples are given.

Example 1. Consider the system $\dot{x}_1 = x_2$, $\dot{x}_2 = t - x_2 x_3 - x_1 u$ and $\dot{x}_3 = u$. Let $v = x_3$. Take $\Gamma_0 = \text{span}\,\{dt, dx_1, dx_2, dx_3, du, dv, d\dot{v}\} = \text{span}\,\{dt, dx_1, dx_2, dv, d\dot{v}\}$. Simple calculations give $\Gamma_1 = \text{span}\ \{dt, dx_1, dx_2, dv\}$ and $\Gamma_2 = \text{span}\,\{dt, dx_1, dx_2 + x_1 dv\}$. Hence, the conditions of Theorem 1 hold. Taking $z_1 = x_1$ and $z_2 = x_2 + x_1 v$, it follows that $\Gamma_2 = \text{span}\,\{dt, dz_1, dz_2\}$ and so (z, v) is a local state representation for the system, where $z = (z_1, z_2)$. The corresponding state equations are given by $\dot{z}_1 = z_2 - z_1 v$ and $\dot{z}_2 = t$.
Example 2. Consider the system S given by $\dot{x}_1 = e^{x_3} u_1$, $\dot{x}_2 = u_1$ and $\dot{x}_3 = u_2$. For this system one may take $v = (v_1, v_2)$, where $v_1 = u_1$ and $v_2 = e^{x_3} u_1$. Considering $u_1 \neq 0$, it is easy to show that, by taking $\gamma = 0$ and $\delta = 1$, one finds $\Gamma_0 = \text{span}\left\{dt, dx, du, dv, dv^{(1)}\right\}$, $\Gamma_1 = \text{span}\,\{dt, dx, du_1\}$, $\Gamma_2 = \text{span}\,\{dt, dx_1, dx_2\}$, and $\Gamma_3 = \text{span}\,\{dt\}$. It is easy to see that the conditions of theorems 1 and 3 hold. In this case it is easy to integrate Γ_2, furnishing the state representation $\dot{z}_1 = v_2, \dot{z}_2 = v_1$, where $z_1 = x_1$ and $z_2 = x_2$. It is then clear that $y = (x_1, x_2)$ is a flat output of S. Note that S is a classical example of a system that is not static-feedback linearizable (with respect to the original state representation (x, u)), but it becomes static-feedback linearizable when one puts one integrator in series with the first input u_1. This last situation may be regarded in the present approach, by choosing $v = (\dot{u}_1, u_2)$ and applying Theorems 1 and 3.
Example 3. Consider now equation (5). Following the idea of Remark 1, one considers a dynamic extension in order to get a classical state representation (x, u), since y_1 depends on the derivative of the input i. Define $x = (x_1, x_2, x_3) = (\theta^{(0)}, \theta^{(1)}, i)$ and $u = (\tau, i^{(1)}) = (u_1, u_2)$, obtaining the following example of equation (6): $\dot{x}_1 = x_2$, $\dot{x}_2 = (K(x_3)/J)x_3 - (B/J)x_2 - (1/J)u_1$, $\dot{x}_3 = u_2$, $y_1 = K(x_3)x_2 + Rx_3 + Lu_2$. Let $v = (\tau, i) = (v_1, v_2)$. One obtains $\Gamma_0 = \text{span}\ \{dt, dx, du, dv, \ldots, dv^{(\delta)}\}$, and for $k \in \lfloor \delta \rceil$, $\Gamma_k = \text{span}\{dt, dx, du_1, \ldots, du_1^{(\delta-k+1)}, du_2, \ldots, du_2^{(\delta-k)}\}$, and $\Gamma_{\delta+1} = \text{span}\ \{dt, dx, du_1^{(0)}\}$, for a given $\delta \in I\!N$. Since $\text{span}\,\{dy_1\} \not\subset \Gamma_{\delta+1}$, by Theorem 2, Problem 2 is not solvable with input v and output y_1.
Example 4. Let $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$, $\dot{x}_3 = u$. Choose $v = x_1$. As this example is linear, the nonsingularity and integrability conditions of Theorem 1 hold. Note that $\gamma = 0$. Given $\delta \in I\!N$, let $\Gamma_0 = \text{span}\left\{dt, dx, du, dv, \ldots, dv^{(\delta)}\right\}$. Simple calculations show that, $\Gamma_0 = \text{span}\,\{dt, dx_1, dx_2, dx_3, du\}$, $\Gamma_1 = \text{span}\,\{dt, dx_1, dx_2, dx_3\}$, $\Gamma_2 = \text{span}\,\{dt, dx_1, dx_2\} = \Gamma_k, k \in I\!N$. In particular, condition (iii) does not hold, since $dv^{(\delta)} \in \text{span}\,\{dx\} \subset \Gamma_1$. In fact, as x_1 is a solution of the differential equation $\ddot{x}_1 + x_1 = 0 = \dot{x}_2 + x_1$, the

function x_1 cannot be an input for this system. In this example, the condition that $\dim \Gamma_k - \dim \Gamma_{k+1} = m$ does not hold for $k > 2$.

Example 5. Let $\dot{x}_1 = x_2, \dot{x}_2 = u, \dot{x}_3 = x_2^2$. Choose $v = x_1$. It is easy to show that, for any $\delta \in I\!N$, then $\Gamma_{\delta+1} = \text{span}\,\{dt, dx_3 - 2x_2 dx_1\}$ which is not integrable. By Theorem 1, Problem 1 is not solvable. However, in this case the system admits a state representation $\dot{x}_3 = \dot{v}^2$, which is not classical.

6 Conclusions

The main result of this paper checks if there exists another classical state representation (z, v) of (6), considering v as the new input. A negative answer will be obtained if v is not the input of any state representation, classical or not. This will happen for instance, if there exists a differential equation linking the components of v (see Example 4). A simple trick, namely, to include the differential dt in Γ_0, allow to consider time-varying systems in a very natural way. Our results also contain, with a proper choice of v, the linearization by extending the state with integrators (see Examples 1 and 2). Note that $\dim z$ may be smaller than $\dim x$ (see Example 1), and $\dim z$ may be greater than $\dim x$ (see Example 2, in the case where $v = (\dot{u}_1, u_2)$). The results of this paper may be generalized for input-output equations and implicit systems Pereira da Silva (2008b).

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