SOME GEOMETRIC PROPERTIES OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. The aim of this paper is the study of some geometric properties of a class of nonlinear differential-algebraic equations (DAEs). A canonical state representation of this class of DAEs is introduced. When the input of the system is a canonical input, it is shown that the DAE does not have impulsive behavior. The solvability of this class of DAEs is studied. A definition of differential index is introduced. This new definition is compatible with underdetermined DAEs, i. e., DAEs that represent control systems. The standard definition of differential index is then compared with the proposed definition, showing the equivalence of this notions when the system is completely determined. A class of index-zero implicit systems, called pseudo-explicit, is introduced. The solutions of a pseudo-explicit system are the solutions of an explicit system with initial conditions that lies on a invariant manifold Γ. It is shown that Γ can be stabilized by a convenient modification of the explicit system without modifying the dynamics over Γ. The relationship between the dynamic extension algorithm and the transformation of an implicit system into a pseudo-explicit form is discussed. This would led to a symbolic method of index reduction, and of stabilization of the corresponding invariant manifold, but this method seems to have some practical disadvantages and numerical difficulties. The main result of the paper shows that one can construct an explicit system whose solutions converge to the ones of a given implicit system. This would led to a second method that can be useful for the numerical integration of a class of higher-index DAEs.

1. Introduction

Implicit systems, Singular Systems, Descriptor Systems, or Differential-Algebraic Equations (DAES's) are deeply studied in the literature. Linear descriptor systems are an important class of control systems and many papers and books on this subject are found in the literature [8, 30, 9]. Solvability of DAEs is considered in [41]. The numerical integration of DAEs is the subject of two excellent books [6, 4], but this matter is still an active area of research, specially for higher-index DAEs.

Our paper is an attempt to study DAEs throw a geometric approach, as done earlier for instance by [40, 23, 39, 18]. In previous works, the connections between DAEs and the relative degree and zero dynamics was already pointed out [42, 26] (see also [27, 7]). In the literature it is shown that, when the index is reduced by

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symbolic operations, one may transform the system into an explicit system with an invariant manifold. In this case it may be important to stabilize this invariant manifold, otherwise the numerical integration methods may introduce a drift in the constraints [40, 3, 1, 2].

Our work studies these aspects further and is mainly based on the infinite dimensional geometric approach of [20] and on some results of [35] that will be briefly recalled in sections 2 ans 3.

In this paper we introduce the notion of canonical state representation of a DAE, which has a geometric meaning and assures that the system does not possess impulsive behavior of its response. We improve some solvability results obtained in [35] showing that, under some regularity assumptions, a DAE is a control system that admits local state representations around any point. Hence, if one fixes the input and choose a compatible initial condition, then the solvability of this class of DAEs is assured in the same way as the solvability of a standard control system.

Assume that one regards the constraints $y = 0$ of a DAE as outputs y of an explicit system S and this explicit system can be decoupled by static-state feedback. Then the index is directly related to the relative degree. In this first situation the system may be reduced to an index-zero system by the application of the decoupling feedback law [42, 25, 26]. The more general situation, considered in this paper, arises when system S is not decoupable by static-state feedback. In this second situation, the standard way of reducing the index simply by adding the derivatives of constraints[1] becomes more complicated because the derivatives of the constraints will depend on the algebraic variables (it is the case of the example of section 7).

When the explicit system S is not decoupable by static-state feedback, we show that the general method of index-reduction is directly related to the dynamical extension algorithm and we also prove that the differential index can be deduced from the algebraic structure at infinity defined in [15]. A notion of differential index that consider also underdetermined DAEs (*i. e.*, control systems defined by DAEs) is then introduced and compared with the standard notion of differential index. It is important to stress that the standard definition of differential index is suitable for completely determined DAE's. In order to consider this definition for implicit control systems one must choose a particular input. However, our definition of differential index makes sense even if the system is underdetermined, without the need of choosing an input function.

We give a new insight to the problem of index reduction showing that the DAEs of this class can be (locally) transformed, by computable symbolic operations, into a a class of index-zero DAEs called here pseudo-explicit systems. A system E of such class is equivalent to an explicit system S with an invariant manifold Γ in a way that the solutions of the implicit system E are the solutions of S with initial conditions in Γ. We also show that pseudo-explicit systems can be modified, without changing the dynamics over Γ, in a way that Γ becomes (locally) stable, combining previous ideas of [5, 18]. Based on these ideas one could establish a numerical integration method that combines simultaneous symbolic and numerical manipulations. Although such method may work well in some particular cases, we believe that it is not reliable in general and may have many practical problems for its implementation. This belief is justified bellow and motivates the need of an alternative method. This second method is based in the main result of the paper,

[1] See chapter 2 and the example of equations equations (6.2.9)-(6.2.10) of [6].

namely, theorem 2, which states that given an implicit system Γ one can construct an explicit system S (using some symbolic differentiations of the constraints y that are executed for once and for all), in a way that the solutions of S converge in to the solutions of Γ. In fact, under some regularity assumptions we will show that there exists a explicit control system[2] with input $\widehat{u}$ given by

$$\left(\begin{array}{c}\dot{x}\\ \dot{\nu}\end{array}\right) = \tau(t, x(t), \nu(t), \widehat{u}(t), \widehat{u}^{(1)}(t)) \tag{1}$$

where $\bar{u}$ is a subvector of ν, $u = (\bar{u}, \widehat{u})$ and system (1) has the property that $x(t)$ of the solutions of (1) converge (globally) to the solutions of the implicit system. Note that the parameterized field τ may be constructed using the symbolic derivatives $y^{(k)}$ and their differentials for $k = 0, \ldots k^*$, where k^* is the differential index. Note that this second method is very different from the reduction of index using symbolical operations described in the first method, as it will be clear from section 6. In both methods one has to compute symbolic derivatives of the constraints. For the second method, all the other computations may be performed numerically, but for the first method however, the implementation of the dynamic extension algorithm needs also symbolic matrix inversions and the rank computation of symbolical matrices, which is a hard task. Note that, if the equations of the system are sparse, the matrix inversions may destroy this property, whereas the symbolic derivations of restrictions will preserve sparsity.

If the implicit system is completely determined ($\widehat{u} = \emptyset$), then $\widehat{u}$ and $\widehat{u}^{(0)}$ are not present in equation 1 (see the example of section 7). When the implicit system is a control system, then the presence of the derivative of the input may be regarded as a disadvantage of the method. This difficulty may avoided if the input of the implicit control system is driven by a control system of the form

$$\begin{aligned}\dot{z}(t) &= \psi(x(t), z(t), v(t))\\ \widehat{u}(t) &= \alpha(x(t), z(t), v(t))\end{aligned}$$

where $z(t)$ is the state of the controller and $v(t)$ is the new external input. Then

$$\widehat{u}^{(1)} = \frac{\partial \alpha}{\partial x}[f(t,x) + g(t,x)u] + \frac{\partial \alpha}{\partial z}\psi(x(t), z(t), v(t)) + \frac{\partial \alpha}{\partial v}v^{(1)}$$

If α does not depend on $v(t)$, *i. e.*, there is no direct feedthrough, then $\widehat{u}^{(1)}$ does not depend on $v^{(1)}$. Otherwise, if the new external input $v(t)$ is known *a priori*, then one may assume that $v^{(1)}(t)$ is also known.

In the present paper we do not perform any numerical analysis. Our main result is a strong indication that our geometric results may be a starting point for establishing numerical methods for the integration of higher-index DAEs. However, in order to develop a reliable numerical integration method based on the ideas of our main result, it is necessary to look these issues in a deeper way yet, adapting our algorithms to the needs of stable numerical calculus (for instance, working with orthonormal basis and orthogonal matrices, QR factorizations etc.).

The paper is organized as follows. In section 2 the preliminary remarks and notations are introduced. A brief overview of the infinite dimensional differential geometric approach of [20] is also presented. Some geometric results about the

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[2] Note that this system is nonclassical in the sense that it is affected by the derivatives $\widehat{u}^{(1)}(t)$ of the inputs.

solvability and state space representations of implicit systems are presented in section 3. A new definition of differential the index (k^*) is presented and is compared with the standard one (ν_d) in section 4. In section 5, the dynamic extension algorithm is shown to be a technique of index-reduction by symbolic manipulations. The stabilizability of the invariant manifolds associated to the index-zero system obtained by this method is also studied in this section. The main result is obtained in section 6 and a example of its application is presented in section 7 along with numerical experiments. Finally, some conclusion remarks are stated in section 8. Some complementary material and some proofs are presented in the appendices.

2. Preliminaries and notation

The field of real numbers will be denoted by $\mathbb{R}$. The set of real matrices of n rows and m columns is denoted by $\mathbb{R}^{n\times m}$. The matrix M^T stands for the transpose of M. The set of natural numbers $\{1,\ldots,k\}$ will be denoted by $\lfloor k \rceil$. Our approach will follows the infinite dimensional geometric setting introduced in control theory by [17, 38, 20] in combination with the ideas presented in [35]. We will use the standard notations of differential geometry in the finite and infinite dimensional case. A brief overview of the infinite dimensional approach of [20] is presented in section 2.1. Some notations and definitions of section 2.1 are used along the paper (*e. g.* the definition of system as a *diffiety*, and the definition of *state representation* as a local coordinate system).

For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x = (x_1,\ldots,x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1,\ldots,dx_n\}$. If L is a set in a metric space M, with metric $\text{dist}(\cdot,\cdot)$, and $\mu \in M$, then $\text{dist}(\mu, L) = \inf_{a\in L} \text{dist}(\mu, a)$. Given a control system evolving on a manifold S, we say that a submanifold Γ is invariant if, given initial conditions over Γ, then all the corresponding solutions are always contained in Γ.

2.1. Diffieties and Systems.

The aim of this section is to introduce a brief overview of the approach of [20]. The presentation will follow the lines of [35].

$\mathbb{R}^A$-Manifolds. Let A be a countable set. Denote by $\mathbb{R}^A$ the set of functions from A to $\mathbb{R}$. One may define the coordinate function $x_i : \mathbb{R}^A \to \mathbb{R}$ by $x_i(\xi) = \xi(i), i \in A$. This set can be endowed with the Fréchet topology (see [20]). A function $\phi : \mathbb{R}^A \to \mathbb{R}$ is smooth if $\phi = \psi(x_{i_1},\ldots,x_{i_s})$, where $\psi : \mathbb{R}^s \to \mathbb{R}$ is a smooth function. Only the dependence on a finite number of coordinates is allowed.

From this notion of smoothness, one can easily state the notions of vector fields and differential forms on $\mathbb{R}^A$ and smooth mappings from $\mathbb{R}^A$ to $\mathbb{R}^B$. The notion of $\mathbb{R}^A$-manifold can be also established easily as in the finitely dimensional case.

Given an $\mathbb{R}^A$-manifold $\mathcal{P}$, $C^\infty(\mathcal{P})$ denotes the set of smooth maps from $\mathcal{P}$ to $\mathbb{R}$. Let $\mathcal{Q}$ be an $\mathbb{R}^B$-manifold and let $\phi : \mathcal{P} \to \mathcal{Q}$ be a smooth mapping. The corresponding tangent and cotangent mapping will be denoted respectively by $\phi_* : T_p\mathcal{P} \to T_{\phi(p)}\mathcal{Q}$ and $\phi^* : T^*_{\phi(p)}\mathcal{Q} \to T^*_p\mathcal{P}$. The map $\phi : \mathcal{P} \to \mathcal{Q}$ is called an *immersion* if, around every $\xi \in \mathcal{P}$ and $\phi(\xi) \in \mathcal{Q}$, there exist local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates $\phi(x) = (x, 0)$. The map ϕ is called a *submersion* if, around every $\xi \in \mathcal{P}$ and $\phi(\xi) \in \mathcal{Q}$, there exist local charts of $\mathcal{P}$ and $\mathcal{Q}$ such that, in these coordinates, $\phi(x, y) = x$. Contrarily to the finite dimensional case, immersions and submersions cannot be characterized by the injectivity or the surjectivity of the corresponding tangent mappings. In fact, the inverse function, implicit function and rank theorems do not hold in this context [45].

Diffieties. A *diffiety* M is a $\mathbb{R}^A$ manifold equipped with a distribution Δ of finite dimension r, called *Cartan distribution.* A section of the Cartan distribution is called a *Cartan field.* An *ordinary diffiety* is a diffiety for which $\dim \Delta = 1$ and a Cartan field ∂_M is distinguished and called *the Cartan field.* In this paper we will only consider ordinary Diffieties that will be called simply by *Diffieties.*

A Lie-Bäcklund mapping $\phi : M \mapsto N$ between Diffieties is a smooth mapping that is compatible with the Cartan fields, *i. e.*, $\phi_* \partial_M = \partial_N \circ \phi$. A *Lie-Bäcklund immersion* (respectively, *submersion*) is a Lie-Bäcklund mapping that is an immersion (resp., submersion). A Lie-Bäcklund isomorphism between two diffieties is a diffeomorphism that is a Lie-Bäcklund mapping. Context permitting, we will denote the Cartan field of an ordinary diffiety M simply by $\frac{d}{dt}$. Given a smooth object ϕ defined on M (a smooth function, field or form), then $\dot{\phi}$ stands for $L_{\frac{d}{dt}} \phi$ and $L^n_{\frac{d}{dt}} \phi = \phi^{(n)}, n \in \mathbb{N}$.

Systems. The set of real numbers $\mathbb{R}$ have a trivial structure of diffiety with the Cartan field $\frac{d}{dt}$ given by the operation of derivation of smooth functions. A *system* is a triple $(S, \mathbb{R}, \tau)$ where S is a diffiety equipped with Cartan field ∂_S and $\tau : S \mapsto \mathbb{R}$ is a Lie-Bäcklund submersion. The global coordinate function t of $\mathbb{R}$ represents *time*, that is chosen for once and for all. A Lie-Bäcklund mapping between two systems $(S, \mathbb{R}, \tau)$ and $(S', \mathbb{R}, \tau')$ is a time-respecting Lie-Bäcklund mapping $\phi : S \mapsto S'$, *i. e.*, $\tau' = \tau \circ \phi$. Context permitting, the system $(S, \mathbb{R}, \tau)$ is denoted simply by S.

State Representation and Outputs. A local state representation of a system $(S, \mathbb{R}, \tau)$ is a local coordinate system, $\psi = \{t, x, U\}$ where $x = \{x_i, i \in \lfloor n \rceil\}$, $U = \{u_j^{(k)} \mid j \in \lfloor m \rceil, k \in \mathbb{N}\}$ where $\tau \circ \psi^{-1}(t, x, U) = t$. The set of functions $x = (x_1, \ldots, x_n)$ is called *state* and $u = (u_1, \ldots, u_m)$ is called *input.* As a consequence of the last definition, in these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in \mathbb{N}, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{2}$$

A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by (x, u). An *output* y of a system S is a set of functions defined on S. The state representation (x, u) is said to be *classic* if the functions f_i depend only on (t, x, u) for $i = 1, \ldots, n$. The output y is said to be *classic* if y depends only on (t, x, u).

System associated to differential equations. Now assume that a control system is given by a set of equations

$$\begin{array}{rcl} \dot{t} & = & 1 \\ \dot{x}_i & = & f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n \rceil \\ y_j & = & \eta_j(x, u, \ldots, u^{(\beta_j)}),\ j \in \lfloor p \rceil \end{array} \tag{3}$$

One can always associate to these equations a diffiety S of global coordinates $\psi = \{t, x, U\}$ and Cartan field given by (2).

Endogenous feedback. In this section we state a simplified notion of endogenous feedback based on coordinate changes. This definition is convenient for our purposes, but it is not suitable for studying feedback equivalence (see [20] for a notion of endogenous feedback that is an equivalence relation between systems).

Two local state representations (x,u) and (z,v) of S induces a local coordinate change map called *endogenous feedback.* If we have span $\{dt,dx\}$ = span $\{dt,dz\}$ and span $\{dt,\, dx,\, du\}$ = span $\{dt,\, dz,\, dv\}$, then we locally have diffeomorphisms $(t,x)\mapsto(t,z)$ and $(t,x,u)\mapsto(t,z,v)$ called static-state feedback. The extension of state by integrators is another particular example of endogenous feedback. For instance, putting integrators in series with the first k inputs of the state representation (x,u) gives $z=(x,u_1,\ldots,u_k)$ and $v=(\dot u_1,\ldots,\dot u_k,u_{k+1},\ldots,u_m)$. Note that the local coordinate functions of S in this case are the same, but they are joined together in a different way, giving rise to (x,u) and (z,v), which are related by an endogenous feedback.

2.2. **Regular implicit systems.** Let Γ be an smooth implicit system of the form

$$\dot x(t) = f(t,x(t))+g(t,x(t))u(t) \tag{4a}$$
$$y(t) = a(t,x(t))+b(t,x(t))u(t)=0 \tag{4b}$$

where $x(t)\in I\!\!R^n$ is the pseudo-state of the system, $u(t)\in I\!\!R^m$ is the pseudo-input[3], and $y_i\equiv 0, i=1,\ldots,r$ are the constraints.

One may associate to the implicit system Γ, the **explicit** system S given by

$$\dot x(t) = f(t,x(t))+g(t,x(t))u(t) \tag{5a}$$
$$y(t) = a(t,x(t))+b(t,x(t))u(t) \tag{5b}$$

Now consider the system S with Cartan field $\frac{d}{dt}$ given by (2) and output y, in the framework of [20] (see section 2.1). Then $y^{(k)}$ stands for the function $\frac{d^k}{dt^k}y$ defined on S, which may depend $x,u^{(0)},u^{(1)},\ldots$.

Definition 1. *In the sequel we shall consider the following codistributions defined on S*

$$\mathcal{Y}_{-1}=\text{span}\,\{dt,dx\}\,,\qquad \mathcal{Y}_k=\text{span}\left\{dt,dx,dy,\ldots,dy^{(k)}\right\}\ \textit{for all } k\in I\!\!N \tag{6a}$$
$$Y_{-1}=\text{span}\,\{dt\}\,,\qquad Y_k=\text{span}\left\{dt,dy,\ldots,dy^{(k)}\right\}\ \textit{for all } k\in I\!\!N \tag{6b}$$
$$\mathbb{Y}_{-1}=\{0\},\qquad \mathbb{Y}_k=\text{span}\left\{dy,\ldots,dy^{(k)}\right\}\ \textit{for all } k\in I\!\!N \tag{6c}$$

♠

In [35] it is shown that we may identify (canonically) the implicit system Γ defined by (4) with the subset of S defined by[4]

$$\Gamma=\{\xi\in S\mid y^{(k)}=0,k\in I\!\!N\}. \tag{7}$$

Definition 2. *Let $U\subset S$ be the open and dense set of regular points of all the codistributions $\mathcal{Y}_k,Y_k$ and $\mathbb{Y}_k$, $k=0,\ldots n$. The implicit system Γ given by (4) is said to be regular if $\Gamma\subset U$, $\Gamma\neq\emptyset$ and there exists a set of (fixed) integers $\{\sigma_0,\ldots,\sigma_n\}$, such that $\sigma_k=\dim\mathcal{Y}_k(\xi)-\dim\mathcal{Y}_{k-1}(\xi)$ for every $\xi\in\Gamma$. In this case, the sequence $\{\sigma_0,\ldots,\sigma_n\}$ is called* the structure at infinity of the implicit system[5].
♠

[3] Note that u is not a differentially independent input for Γ, since the constraints $y\equiv 0$ induce differential relations linking the components of u. By the same reasons, x is not a state of Γ.

[4] The Prop. 1 will show that Γ is a immersed submanifold of S.

[5] This sequence is in fact the algebraic structure at infinity of the explicit system S defined by (5) [15, 12] (see also [34]).

The class of nonlinear implicit systems (4) is not particular as it appears at a first glance. In fact, the next remark shows that general nonlinear differential-algebraic equations (DAEs) can be always converted into an equivalent system of the form (4). For this, consider the following differential equations:

$$\phi_i(t, w_1, \ldots, w_1^{(\alpha_{1i})}, \ldots, w_s, \ldots, w_s^{(\alpha_{si})}) = 0, i \in \lfloor r \rceil \tag{8}$$

Let β_j be the greater order of derivative of ω_j that may appear in (8) for $i = 1, \ldots, r$. Then $\beta_j = \max\{\alpha_{ji}, i \in \{1, \ldots, r\}\}, j \in \{1, \ldots, s\}$. Consider the system S with state $x = (w_1, \ldots, w_1^{(\beta_1 - 1)}, \ldots, w_s, \ldots, w_s^{(\beta_s - 1)})$, input $u = (u_1, \ldots, u_s)$, where $u_j = w_j^{(\beta_j)}$, and output y defined by equations[6]:

$$\left\{ \begin{array}{rcl} \dot{w}_j^{(0)} & = & w_j^{(1)} \\ & \vdots & \qquad j \in \lfloor s \rceil \\ \dot{w}_j^{(\beta_j - 1)} & = & u_j \end{array} \right. \tag{9}$$
$$y_i = \phi_i(t, w_1, \ldots, w_1^{(\alpha_{1i})}, \ldots, w_s, \ldots, w_s^{(\alpha_{si})}), i \in \lfloor r \rceil$$

It is clear that the system (8) is represented by the system (9) with the constraints $y_i = 0$, which is in the form (4a)-(4b). So, all the results developed here may be applied to a set of DAEs of arbitrary order. From the results of this paper it will be clear that a (local) state $\widehat{x}$ of the implicit system can be always chosen as a subset of $x = (w_1, \ldots, w_1^{(\beta_1 - 1)}, \ldots, w_s, \ldots, w_s^{(\beta_s - 1)})$ and an input $\widehat{u}$ can be always a subset of u.

For instance, consider a DAE in descriptor form:

$$E(z)\dot{z} = A(z) + B(z)v$$

and note that it can be transformed into the implicit system

$$\begin{array}{rcl} \dot{z} & = & \mu + \mathbf{0}\, v \\ y & = & E(z)\mu - A(z) - B(z)v = 0 \end{array}$$

which is in the form (4a)–(4b) where $x = z$ and $u = (v, \mu)$. Note that y is affine in u.

2.3. **Affine systems and unbounded coordinates.** We now define a notion of *unbounded* coordinates on $I\!R^A$-manifolds (see section 2.1) and some related results. Roughly speaking, a set of coordinates is unbounded if one may choose the value of these set of coordinates arbitrarily. In other words, given coordinates (x, w) with w unbounded, if (x_0, w_0) is admissible, then (x_0, w) is also admissible for any w.

Definition 3. *Let $\phi : U \to I\!R^A \times I\!R^B$, where $\phi = (X, Y)$ is a local chart of a manifold S defined on the open set $U \subset S$ where $X : U \to I\!R^A$ and , $Y : U \to I\!R^B$ are smooth maps. Then, the coordinates Y are said to be unbounded if $V := \phi(U) = W \times I\!R^B$, where W is an open subset of $I\!R^A$. Let $\Psi : H \to H_1$ be a diffeomorphism between open subsets H and H_1 of $I\!R^A \times I\!R^B$, such that $(X, Y) \mapsto (X_1, Y_1)$. Then Ψ is said to be unbounded in Y_1 if the image of Ψ is of the form $V \times I\!R^B$ with V an open subset of $I\!R^A$* ♠

[6]Note that y_i ($i \in \lfloor r \rceil$) may depend on some u_j ($j \in \lfloor s \rceil$).

Remember that a state representation (x, u) is a special coordinate chart for a system S (see section 2.1). Given an explicit system (5) one may construct a system S with global coordinates $\{t, x, U\}$, where $U = (u^{(k)} : k \in I\!N)$ and Cartan field (2). It is a simple exercise to show that an affine locally regular static-state feedback, *i. e.*, $u = \alpha(x) + \beta(x)v$, for β that is locally nonsingular, defines a new state representation (x, v) and the chart $\{t, x, V\}$, where $V = (v^{(k)} : k \in I\!N)$ is unbounded in V (see the proof of lemma 2).

We may state the following auxiliary results.

Lemma 1. *If* (Z_1, W) *is a local chart with unbounded* W, $\operatorname{card} Z_1 = \operatorname{card} Z_2$ *is finite and* $\operatorname{span}\{dZ_1\} = \operatorname{span}\{dZ_2\}$, *then* (Z_2, W) *is also a local coordinate system with unbounded* W.

Proof. By the (finite dimensional) inverse function theorem it follows that $Z_1 \mapsto Z_2(Z_1)$ is a local diffeomorphism defined in some open subset $V \subset I\!R^A$ with image $U \subset I\!R^A$. Hence $(Z_1, W) \mapsto (Z_2(Z_1), W)$ is also a local diffeomorphism defined in some open subset $V \times I\!R^B \subset I\!R^A \times I\!R^B$ with image $U \times I\!R^B$. It follows easily that (Z_2, W) is a local chart with unbounded W. □

The next lemma is directly related to the steps of the dynamical extension algorithm (see section 2.4). Part (i) refers to a regular static-state feedback and part (ii) to a dynamic extension.

Lemma 2. *Let* S *be a system, and let* (x, u) *be a state representation of* S. *Assume that* $\{t, x, u^{(0)}, \ldots, u^{(k)}, W\}$ *is a local chart with unbounded* $\{u^{(0)}, \ldots, u^{(k)}, W\}$, *and let* (x, v) *be defined by an affine locally regular static-state feedback* $u = \alpha(x) + \beta(x)v$.

(i) *Then* $\{t, x, v^{(0)}, \ldots, v^{(k)}, W\}$ *is a local chart with unbounded* $\{v^{(0)}, \ldots, v^{(k)}, W\}$ *and* $\operatorname{span}\left\{dt, dx, dv^{(0)}, \ldots, dv^{(k)}\right\} = \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(k)}\right\}$.

(ii) *If* v *is partitioned as* $v = (\bar{v}, \widehat{v})$, $\xi = (x, \bar{v})$ *and* $\mu = (\bar{v}^{(1)}, \widehat{v})$, *then* $\{t, \xi, \mu^{(0)}, \ldots, \mu^{(k-1)}, \widehat{v}^{(k)}, W\}$ *is a local chart with unbounded* $\{\mu^{(0)}, \ldots, \mu^{(k-1)}, \widehat{v}^{(k)}, W\}$ *and such that* $\operatorname{span}\left\{dt, d\xi, d\mu^{(0)}, \ldots, d\mu^{(k-1)}, d\widehat{v}^{(k)}\right\} = \operatorname{span}\left\{dt, dx, du^{(0)}, \ldots, du^{(k)}\right\}$.

Proof. Note that the equations

$$\begin{array}{rcl} v & = & -\beta^{-1}(x)\alpha(x) + \beta^{-1}(x)u \\ \dot{v} & = & \phi_1(x, u) + \beta^{-1}\dot{u} \\ & \vdots & \\ v^{(k)} & = & \phi_k(x, u, \ldots, u^{(k-1)}) + \beta^{-1}u^{(k)} \end{array}$$

for $k \in I\!N$ define a local diffeomorphism Ψ_A such that $(t, x, U) \mapsto (t, x, V)$, where $U = \{u^{(k)} : k \in A\}$, $V = \{v^{(k)} : k \in A\}$ with $A = I\!N$ or with $A = \{0, \ldots, k\}$. This diffeomorphism is easily seen to be unbounded in V (with inverse unbounded in U). Taking $A = \{0, \ldots, k\}$ note that the map $(t, x, U, W) \mapsto (t, x, V, W)$, where $(t, x, V, W) = (\Psi_A(t, x, U), W)$ is a local morphism which is unbounded in (V, W). This shows (i). To show (ii) it suffices to see that $\{t, \xi, \mu^{(0)}, \ldots, \mu^{(k-1)}, \widehat{v}^{(k)}, W\} = \{t, (x, \bar{v}^{(0)}), (\bar{v}^{(1)}, \widehat{v}^{(0)}), \ldots, (\bar{v}^{(k)}, \widehat{v}^{(k-1)}), \widehat{v}^{(k)}, W\}$, *i. e.*, these sets of coordinates coincide up to a renaming of the variables. □

2.4. **Dynamic extension algorithm (DEA).** The DEA is a well known algorithm in nonlinear control theory and it is essentially a tool for computing system right-inverses and the output rank [16]. It is strongly related to the problem of input-output decoupling [14, 32], disturbance decoupling [33, 13] and input-output linearization [13, 36]. The dynamic extension algorithm for a system (5) has an intrinsic interpretation [15]. This interpretation was considered further for the study of quasi-static feedback in nonlinear control theory [12].

We will see that the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators. According the the ideas of the end of section 2.1, one sees that this algorithm can be regarded as the choice of a new local state representation of system S. Now we state a slightly different version of DEA that is useful for our purposes. Let S be the system (5) with Cartan field $\frac{d}{dt}$ defined by (2), classical state representation (x, u) and classical output y. Assume that $y^{(0)} = y = a_0(t,x) + b_0(t,x)u$ and denote $x_{-1} = x$, $u_{-1} = u$, $f_{-1}(t,x) = f(t,x)$, $g_{-1}(t,x) = g(t,x)$. The step k of this algorithm ($k = 0, 1, \ldots$) is described below:

Step k. In the step $k-1$ we have constructed state equations

$$\dot{x}_{k-1} = f_{k-1}(t, x_{k-1}) + g_{k-1}(t, x_{k-1})u_{k-1} \tag{10}$$

$$y^{(k)} = a_k(t, x_{k-1}) + b_k(t, x_{k-1})u_{k-1} \tag{11}$$

where $x_{k-1} = (x, \bar{v}_0, \ldots, \bar{v}_{k-1})$. Assume that $(\bar{t}, \bar{x}_{k-1})$ is a regular point for the matrix $b_k(t, x_{k-1})$ and let σ_k be the rank of b_k around $(\bar{t}, \bar{x}_{k-1})$. There exist a partition[7] $y^{(k)} = (\bar{y}_k^{(k)}, \widehat{y}_k^{(k)})$ of $y^{(k)}$ such that $\dim \bar{y}_k^{(k)} = \sigma_k$ and we may define a (locally) regular static-state feedback (see appendix B):

$$u_{k-1} = \alpha_k(t, x_{k-1}) + \beta_k(t, x_{k-1})v_k \tag{12}$$

where $v_k = (\bar{v}_k, \widehat{v}_k)$ is such that

$$\begin{aligned} \bar{y}_k^{(k)} &= \bar{v}_k \\ \widehat{y}_k^{(k)} &= \widehat{y}_k^{(k)}(t, x_{k-1}, \bar{v}_k) \end{aligned} \tag{13}$$

Add the dynamic extension:

$$\begin{aligned} \bar{u}_k &= \dot{\bar{v}}_k \\ \widehat{u}_k &= \widehat{v}_k \end{aligned} \tag{14}$$

and define $u_k = (\bar{u}_k, \widehat{u}_k)$. This defines a new set of state equations:

$$\dot{x}_k = f_k(t, x_k) + g_k(t, x_k)u_k \tag{15}$$

where $x_k = (x_{k-1}, \bar{y}_k^{(k)})$ and $u_k = (\bar{y}_k^{(k+1)}, \widehat{u}_k)$. By construction we have $y^{(k)} = y^{(k)}(t, x_k)$. Hence we may compute

$$\begin{aligned} y^{(k+1)} &= \frac{\partial y^{(k)}}{\partial t} + \frac{\partial y^{(k)}}{\partial x_k}(f_k + g_k u_k) \\ &= a_{k+1}(t, x_k) + b_{k+1}(t, x_k)u_k \end{aligned} \tag{16}$$

♦

The following lemma summarizes the main geometric properties of the DEA for time-invariant nonlinear systems. This lemma is a geometric version of previous results stated in [15, 31, 12, 35] and it improves some results obtained in [34]. We stress that the list of integers $\{\sigma_0, \ldots, \sigma_n\}$, where $n = \dim x$, is the geometric counterpart of the *algebraic structure at infinity* (see [15]) and the integer $\rho = \sigma_n$

[7] Including possibly a reordering of its elements.

is called *output rank* at ξ. We stress that, with the exception of part 9, this result is valid for nonaffine systems[8].

Lemma 3. *Let S be the system* (5) *with Cartan field $\frac{d}{dt}$ defined by* (2)*, classical state representation (x, u) and classical output y. Let V_k be the open and dense set of regular points of the codistributions Y_i and $\mathcal{Y}_i$ for $i = 0, \ldots, k$ defined in* (6a) *and* (6b)*. Let $\xi \in V_k$. In the* kth *step of the dynamic extension algorithm, one may construct a new local classical state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_k^{(k)})$, input $u_k = (\dot{\bar{y}}_k^{(k)}, \widehat{u}_k)$ and output $y^{(k+1)} = h_k(t, x_k, u_k)$ defined in an open neighborhood U_k of ξ, such that*

(i) $\operatorname{span}\{dt, dx_k\} = \operatorname{span}\{dt, dx, dy, \ldots, dy^{(k)}\} = \mathcal{Y}_k$.
(ii) $\operatorname{span}\{dt, dx_k, du_k\} = \operatorname{span}\{dt, dx, dy, \ldots, dy^{(k+1)}, du\} = \mathcal{Y}_k + \operatorname{span}\{du\}$.
(iii) *It is always possible to choose $\bar{y}_{k+1}^{(k+1)}$ in a way that $\bar{y}_k^{(k+1)} \subset \bar{y}_{k+1}^{(k+1)}$.*
(iv) *It is always possible to choose $\widehat{u}_{k+1} \subset \widehat{u}_k$.*
(v) *Let $\xi \in V_n$. Let S_k be the greater open neighborhood of ξ such that the dimensions of $Y_j, \mathcal{Y}_j$ $j \in \{0, \ldots, k\}$ are constant inside S_k. The sequence $\sigma_k = \dim(\mathcal{Y}_k|_\xi) - \dim(\mathcal{Y}_{k-1}|_\xi)$ is nondecreasing, the sequence $\rho_k = \dim(Y_k|_\xi) - \dim(Y_{k-1}|_\xi)$ is nonincreasing, and both sequences converge to the same integer ρ, called the* output rank at ξ*, for some $k^* \leq n = \dim x$.*
(vi) *$S_k = S_{k^*}$ for $k \geq k^*$.*
(vii) *$Y_k \cap \operatorname{span}\{dx\}|_\nu = Y_{k^*-1} \cap \operatorname{span}\{dx\}|_\nu$ for every $\nu \in S_{k^*}$ and $k \geq k^*$.*
(viii) *For $k \geq k^*$, one may choose $\bar{y}_k = \bar{y}_{k^*}$ in U_{k^*}. Furthermore, $Y_{k+1} = Y_k + \operatorname{span}\left\{\bar{y}_k^{(k+1)}\right\}$ for $k \geq k^*$.*
(ix) *If the system is affine,* i. e.*, it is of the form* (5)*, then the state representation (x_k, u_k) obtained in step k induces a local chart $\{t, x_k, (u_k^{(j)} : j \in I\!N)\}$ of S that is unbounded in $W = \{u_k^{(j)} : j \in I\!N\}$.*

Proof. See appendix A. □

Remark 1. *Note that* $\dim \mathcal{Y}_k = 1 + \dim x_k = 1 + n + \sum_{i=0}^{k} \sigma_i$*,* $\dim \bar{y}_k = \sigma_k$*,* $\dim u_k = m$ *and* $\dim \widehat{u}_k = m - \sigma_k$*, where $n = \dim x$, and $m = \dim u$.*

3. Solvability

In this section we study the solvability and state space representations of regular DAEs. We improve some previous results of [35] and introduce the notion of canonical state representation. The results of this section are closely related to some ideas of [19].

3.1. **Regular implicit systems are immersed submanifolds.** It is shown in [35] a regular DAE defined by (4a)–(4b) can be regarded as an immersed system in the explicit system S defined by (5).

Proposition 1. [35] *Let S be the system associated to* (5) *in the sense of* [20] *(*i. e.*, a diffiety with Cartan field* (2) *and a time notion $\tau : S \to I\!R$). Let Γ be the subset of S defined by $\Gamma = \{\xi \in S \mid y^{(k)}(\xi) = 0, k \in I\!N\}$. Suppose that:*
(A1) Γ is nonempty and every $\xi \in \Gamma$ is a regular point of the codistributions $Y_k, \mathcal{Y}_k, k = 0, \ldots, n$ (see (6a)-(6b)*).*

[8] In this case the steps (S1) and (S2) of appendix A may be regarded as the description of the DEA and the calculations may depend on the implicit function theorem [29].

(A2) For every $\xi \in \Gamma$ and every open neighbourhood $U \subset S$ of ξ, there exists some $\epsilon \geq 0$ such that $\tau(\Gamma \cap U)$ contains an open interval $(\tau(\xi) - \epsilon, \tau(\xi) + \epsilon)$.
Then the subset $\Gamma \subset S$ has a canonical structure of immersed (embedded) submanifold of S such that the canonical insertion is a Lie-Bäcklund immersion. Furthermore Γ admits a local state representation around every point $\xi \in \Gamma$.

The proof of proposition 1 is based on the following result (compare with [35, Thm. 4.3]):

Lemma 4. *Let S be the system (5). Let $n = \operatorname{card} x$. Let $U \subset W$ be the set of regular points of the codistributions $Y_k, \mathcal{Y}_k, k \in \lfloor n \rceil$, defined on S and let $\xi \in U$. Set k^* as in lemma 3. Let z_a, v_a, z_b, v_b be sets of functions such that $\{dt, dz_a\}$, $\{dt, dz_a, dv_a\}$, $\{dt, dz_a, dz_b\}$ and $\{dt, dz_a, dz_b, dv_a, dv_b\}$ are respectively local basis of Y_{k^*-1}, Y_{k^*}, $\mathcal{Y}_{k^*-1}$ and $\mathcal{Y}_{k^*} + \operatorname{span}\{du\}$ around ξ.*

Then, there exists an open neighbourhood V_ξ of ξ such that $(z, v) = ((z_a, z_b), (v_a, v_b))$ is a local classical *state representation of the system S that is defined on V_ξ and is such that the (local) state equations are of the form:*

$$\dot{z}_a = f_a(t, z_a, v_a) \tag{17a}$$

$$\dot{z}_b = f_b(t, z_a, z_b, v_a, v_b). \tag{17b}$$

and $\operatorname{span}\{dt, dz_a, (dv_a^{(k)} : k \in I\!N)\} = \operatorname{span}\{dt, dy^{(k)} : k \in I\!N\}$. *Let $\mathcal{Z} = \{z_a, (v_a^{(k)} : k \in I\!N)\}$ and $\mathcal{Y} = \{y_j^{(k)} : j \in \lfloor p \rceil, k \in I\!N\}$. Then we may choose $\mathcal{Z} \subset \mathcal{Y}$. Furthermore, the state representation (z, v) induces a local chart $\{t, z, (v^{(k)} : k \in I\!N)\}$ that is unbounded in $\{v^{(k)} : k \in I\!N\}$.*

Proof. The idea of the proof of this lemma is to execute the dynamic extension algorithm considering the explicit system S given by equation (5) with output y. The conditions of Def. 2 assures (according lemma 3) that the dynamic extension algorithm may be executed without any local singularities. In the step $k^* - 1$ of this algorithm we have computed a new state representation $(\tilde{x}, \tilde{u})$ where $\tilde{x} = (x, \bar{y}_1^{(1)}, \ldots, \bar{y}_{k^*}^{(k^*-1)})$ and $\tilde{u} = (\omega, \mu)$, where $\omega = \bar{y}_{k^*-1}^{(k^*)}$ and $\mu = \widehat{u}_{k^*-1}$. Note that the new state equations are affine, *i. e.*, they are of the form

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}) + \bar{g}(t, \tilde{x})\omega + \widehat{g}(t, \tilde{x})\mu \tag{18}$$

By parts 1 and 2 of lemma 3 we have $\operatorname{span}\{dt, d\tilde{x}\} = \operatorname{span}\{dt, dx, dy, \ldots, dy^{(k^*-1)}\}$ and $\operatorname{span}\{dt, d\tilde{x}, d\tilde{u}\} = \mathcal{Y}_{k^*}$. Note that, by construction we have $\operatorname{span}\{dt, d\tilde{x}\} = \operatorname{span}\{dt, dz\}$ and $\operatorname{span}\{dt, d\tilde{x}, d\tilde{u}\} = \operatorname{span}\{dt, dz, dv\}$, which defines a relation of local static-state feedback between the state representations $(\tilde{x}, \tilde{u})$ and (z, v) (see the end of section 2.1). The other properties are easy consequences of parts 5, 8 and 9 of lemma 3 and of lemma 1. □

The idea[9] of the proof of Prop. 1 is to apply Lemma 4 and to show that (z_b, v_b) is a local state representation of Γ and in the coordinates $\{t, z_a, V_a\}$ for Γ and $\{t, z_a, z_b, V_a, V_b\}$ for S, the immersion ι is given by $\iota(t, z_b, V_b) = (t, 0, z_b, 0, V_b)$. Furthermore, the state representation (z_b, v_b) induces state equations of Γ given by

$$\dot{z}_b = f_b(t, 0, z_b, 0, v_b) \tag{19}$$

[9]An important detail that is overlooked in the present sketch of the proof of Prop. 1 is the construction of the smooth atlas of Γ.

where f_b is given by (17b). Note that, in Lemma 4, the state z_b of Γ may be chosen as a convenient subset of x and the input v_b may be chosen as a convenient subset of u. In other words, v_b is the differentially independent part of u and the constraints $y \equiv 0$ induce differential relations linking the other components of u. This explains why we call x by "pseudo-state" and u by "pseudo-input" of Γ.

Now we will show that assumptions (A1)–(A2) of proposition 1 are implied by the assumptions of definition 2:

Lemma 5. *Let S be an explicit system* (5) *such that the corresponding implicit system* (4) *is regular. Let $n = \operatorname{card} x$. Then the following affirmations holds:*

(i) *Let $\mathbb{Y}_k = \operatorname{span}\{dy, \ldots, dy^{(k)}\}$. Then $\operatorname{span}\{dt\} \cap \mathbb{Y}_k|_\xi = \{0\}$ in an open neighborhood of every point $\xi \in \Gamma$, for $k \in I\!N$.*

(ii) *Consider the state representation* (17) *of lemma 4. Then $\operatorname{span}\{dz_a, (dv_a^{(k)} : k \in I\!N)\} = \operatorname{span}\{dy^{(k)} : k \in I\!N\}$ around every $\xi \in \Gamma$.*

(iii) *The assumptions of definition 2 implies* (A1) *and* (A2) *of proposition 1.*

Proof. We show first that $\operatorname{span}\{dt\} \cap \mathbb{Y}_k = \{0\}$ for every point of Γ. In fact, let $\xi \in \Gamma$ and let $\eta = \sum_{i=0}^{k} \sum_{j=1}^{r} \alpha_{ij} dy_j^{(i)}|_\xi = \beta dt|_\xi$. Then $\langle \eta; \frac{d}{dt} \rangle = \sum_{i,j} \langle \alpha_{ij} dy_j^{(i)}|_\xi; \frac{d}{dt} \rangle = \sum_{i,j} \alpha_{ij} \langle dy_j^{(i)}|_\xi; \frac{d}{dt} \rangle = \sum_{i,j} \alpha_{ij} y_j^{(i+1)}|_\xi = 0 = \beta|_\xi$. Note that, for all $\nu \in S$ we have

$$\dim Y_k|_\nu = \dim\left(\operatorname{span}\{dt\}|_\nu\right) + \dim \mathbb{Y}_k|_\nu - \dim(\operatorname{span}\{dt\}|_\nu \cap \mathbb{Y}_k|_\nu)$$

then the nonsingularity of $\operatorname{span}\{dt\}$, of Y_k and of $\mathbb{Y}_k$ implies that $\operatorname{span}\{dt\} \cap \mathbb{Y}_k$ is nonsingular around $\xi \in \Gamma$ and hence (i) holds. To show (ii), let $\eta \in \mathbb{Y}_k$. Since $\operatorname{span}\{dt, dz_a, (dv_a^{(k)} : k \in I\!N)\} = \operatorname{span}\{dt, dy^{(k)} : k \in I\!N\}$, it follows that $\eta = \beta dt + \sum_i \gamma_i dz_{a_i} + \sum_{i,j} \delta_{ij} dv_{a_j}^{(i)}$. By lemma 4 we have $\mathcal{Z} = \{z_a, (v_a^{(k)} : k \in I\!N)\} \subset \mathcal{Y} = \{y^{(k)} : k \in I\!N\}$. From (i), it follows that β must be zero and (ii) holds. Now, since $\{t, z_a, z_b, (v_a^{(k)}, v_b^{(k)}, k \in I\!N)\}$ is a local coordinate system, by (ii) it follows that in these coordinates, $y^{(k)}$ do not depend on t, but only on $\{z_a, z_b, (v_a^{(k)}, v_b^{(k)}, k \in I\!N)\}$. In particular, if $(\bar{t}, \bar{z}_a, \bar{z}_b, \bar{V}_a, \bar{V}_b) \in \Gamma$ then $(\bar{t} + \epsilon, \bar{z}_a, \bar{z}_b, \bar{V}_a, \bar{V}_b) \in \Gamma$ for every $|\epsilon|$ small enough, showing that (A1) and (A2) are implied by the assumptions of definition 2. □

3.2. **Canonical state representation of implicit systems.**

The following proposition characterizes special state representations of implicit systems which have a canonical meaning.

Proposition 2. *Let k^* be the integer defined in Lemma 3. Consider now the explicit system S defined by* (5) *and let $\widehat{x}, \widehat{u}$ be sets of functions defined on S such that (locally) the canonical projections of $\{d\widehat{x}\}$ on $\mathcal{Y}_{k^*}/Y_{k^*}$ and the canonical projections of $\{d\widehat{u}\}$ on $(\mathcal{Y}_{k^*} + \operatorname{span}\{du\})/\mathcal{Y}_{k^*}$ are both basis. Then $(\widehat{x}, \widehat{u})$ is a local state representation of the implicit system* (4) *called* canonical state representation.

Proof. From lemma 3 part 7 it is easy to show that $\dim \mathcal{Y}_{k^*}/Y_{k^*} = \dim \mathcal{Y}_{k^*-1}/Y_{k^*-1}$. From this, it follows easily that the construction of the state representation (z_b, v_b) of Γ given by the equation (19) (see lemma 4) is equivalent to the statement of the present proposition with $z_b = \widehat{x}$ and $v_b = \widehat{u}$. □

Remark 2. *Note that the state representation* (19)*, obtained by the choice of a canonical state representation, is classical,* i. e., *the derivative of the state is a function of the state and the input. In particular such a state representation does not* *have impulsive response, which may arise with other choices of state representation. For instance consider the DAE* $\dot{x}_1 = x_2 + u_1$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u_2$, $y = x_1 = 0$. *It is easy to verify that* $(\widehat{x}, \widehat{u})$, *where* $\widehat{x} = (x_2, x_3)$ *and* $\widehat{u} = u_2$ *is a canonical state representation with state equations* $\dot{x}_2 = x_3$ *and* $\dot{x}_3 = u_2$. *If the input of the implicit system is* u_1 *then one may choose a non-canonical state representation* $(\widehat{x}, \widehat{u})$, *where* $\widehat{x} = x_2$ *and* $\widehat{u} = u_1$, *corresponding to the state equations* $\dot{x}_2 = \dot{u}_1$. *Note that the state representation* $(\widehat{x}, \widehat{u})$ *is not classic.*

4. Differential Index

The index of general DAEs was studied for instance in [10, 28]. In this work we deal with the differential index[10]. In [18] a geometrical definition of the index of DAEs was given. It was shown that this definition was compatible with another definition given earlier in [22] for linear systems, since the linear definition of index applied to the linearized system coincides with the nonlinear definition.

In this section we will give a new geometrical definition of the *differentiation index* of DAEs of the form (4). This geometric definition will be compared with the classical notion of index and will be shown to be independent of the canonical state representation chosen. In particular, the index of a system does not depend on the canonical input chosen among the components of the pseudo-input u. We stress again that our definition is compatible with underdetermined DAEs, whereas the standard definition of index is not. Our regularity conditions of definition 2 assure that the index is an invariant of the system.

4.1. A new definition of differential index.

Definition 4. *Let* (4) *be a regular implicit system and let* $\{\sigma_0, \ldots, \sigma_n\}$ *be the algebraic structure at infinity of this system (see Def. 2). Let* k^* *be the least integer such that* $\sigma_{k^*} = \max\{\sigma_0, \ldots, \sigma_n\}$. *The integer* k^* *is called the* differentiation index *of the regular implicit system* (4). ♠

The following proposition links our last definition with the "classical" notion of index, that states that this integer is the least order of derivation of constraints that are necessary to compute $\dot{x}$ [10]. It means that the index is the least order of derivation of the constraints y in a way that $\dot{x}$ may be computed as a function of a canonical state and of a canonical input.

Proposition 3. *Assume that* $(\widehat{x}, \widehat{u})$ *is a canonical state representation of the regular implicit system* (4). *Assume that the explicit system* S *defined by* (5a) *is well-formed*[11], i. e., $\text{span}\,\{dt, dx, d\dot{x}\} = \text{span}\,\{dt, dx, du\}$. *Then the index* k^* *is the least integer* k^* *such that* $\dot{x}$ *may be computed as a function of* $\{t, \widehat{x}, \widehat{u}, y, \ldots, y^{(k^*)}\}$. *In other words, the following condition holds for the explicit system* S *defined by* (5) *for* $k = k^*$:

$$\text{span}\,\{d\dot{x}\} \subset \text{span}\left\{dt, d\widehat{x}, d\widehat{u}, dy, \ldots, dy^{(k)}\right\} \subset \mathcal{Y}_k + \text{span}\,\{d\widehat{u}\} \tag{20}$$

but the same condition do not hold for $k < k^*$.

[10] Which may differ from the perturbation index or other notions of index, as shown in [10].

[11] This is equivalent to say that $g(t, x)$ of (5) has full column rank [43].

Proof. By construction, we have that

$$\text{span}\,\{dt, dx, du\} \subset \text{span}\left\{dt, d\widehat{x}, d\widehat{u}, dy, \ldots, dy^{(k^*)}\right\} \tag{21}$$

As S is classic, it follows that (20) is satisfied. Now assume that (20) holds for some $k \leq k^*$. By Lemma 3 part 2 it is easy to see that $\dim \widehat{u}$ is given by $m - \sigma_{k^*}$ and that[12]

$$\dim\left(\mathcal{Y}_k \oplus \text{span}\,\{d\widehat{u}\}\right) = (1 + n + \sigma_0 + \ldots + \sigma_k) + (m - \sigma_{k^*}) \tag{22}$$

From (20) we see that $\text{span}\,\{dt, dx, du\} = \text{span}\,\{dt, d\dot{x}, dx\} \subset \mathcal{Y}_k \oplus \text{span}\,\{d\widehat{u}\}$. In particular, it follows that

$$\mathcal{Y}_k \oplus \text{span}\,\{d\widehat{u}\} = \mathcal{Y}_k + \text{span}\,\{du\} \tag{23}$$

By Lemma 3 parts 1 and 2 it follows that $\mathcal{Y}_k + \text{span}\,\{du\} = \text{span}\,\{dx_{k-1}, du_{k-1}\}$. Hence,

$$\dim\left(\mathcal{Y}_k + \text{span}\,\{du\}\right) = 1 + n + \sigma_0 + \ldots + \sigma_{k-1} + m \tag{24}$$

It follows from (22), (23) and (24) that $\sigma_k = \sigma_{k^*}$ and so $k = k^*$. □

Remark 3. *The following points must be stressed out:*

(i) *One may re-state a new version of proposition 3 by replacing* (20) *by the condition:*

$$\text{span}\,\{d\dot{x}\} \subset \text{span}\left\{dt, dx, d\widehat{u}, dy, \ldots, dy^{(k)}\right\} = \mathcal{Y}_k + \text{span}\,\{d\widehat{u}\} \tag{25}$$

In this version, the index is the least integer k^* such that (25) *holds for $k = k^*$.*

The proof is similar and is left to the reader.

(ii) *When* (5a) *is not well-formed, one may replace condition* (20) *by*

$$\text{span}\,\{dx, du\} \subset \text{span}\left\{dt, d\widehat{x}, d\widehat{u}, dy, \ldots, dy^{(k)}\right\}$$

(iii) *When $\sigma_{k^*} = m$, then the regular implicit system is completely determined,* i. e.*, there is no input ($\widehat{u} = \emptyset$). In this case, from* (25) *we see that the index k^* is the least integer k such that* $\text{span}\,\{d\dot{x}\} \subset \text{span}\,\{dt, dx, dy, \ldots, dy^{(k)}\}$*, which is similar to the usual definition of differential index (see section 4.2 for a complete comparison).*

(iv) *It can be shown that Def. 4 is equivalent to the the one of* [18]*. In particular, assume that the regular implicit system* (4) *is influenced by a disturbance $w(t) \in \mathbb{R}^r$ according the following equations:*

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t) \tag{26a}$$

$$y(t) = h(t, x(t), u(t)) = w(t) \tag{26b}$$

Then the index k^* is the greater order of time-derivative of $w(t)$ that influences the response of system (26).

[12] Note that direct sum of equation 22 is a consequence of the fact that the canonical projections of $\{d\widehat{u}\}$ form a basis of $(\mathcal{Y}_{k^*} + \text{span}\,\{du\})/\mathcal{Y}_{k^*}$.

4.2. **Comparison.** In this section we will compare our definition of differential index with the standard one. For this, recall the definition of differential index given in [10, 6, 4]. Given a (solvable) nonlinear DAE:

$$F(t, \dot{x}(t), x(t)) = 0$$

where $x \in I\!R^n$ and consider the *derivative array equations*:

$$\begin{bmatrix} F(t,\dot{x},x) \\ F_t + F_{\dot{x}}(t,\dot{x},x)\ddot{x} + F_x(t,\dot{x},x)\dot{x} \\ \vdots \\ \frac{d^k}{dt^k}F(t,\dot{x},x) \end{bmatrix} = F_k(t,\dot{x},x,w) = 0 \tag{27}$$

where $w = (x^{(2)}, \ldots, x^{(k+1)})$. Roughly speaking, the differential index ν_d of the DAE is the least integer k such that $\dot{x}$ is uniquely determined by (t, x) and the equations (27). It is clear that this definition is not suitable when the DAE represents a underdetermined system. In this case, one must choose an input function $\widehat{u}(\cdot)$ in a way that the DAE becomes completely determined. Let us assume that this is the case. Consider the system:

$$\dot{x} = u \tag{28a}$$

$$y = F(t, u, x) = 0 \tag{28b}$$

which is in the form (4). Assuming that this DAE is regular according definition (2), and that the DAE is completely determined, i.e., $\sigma_{k^*} = n$, then the part (iii) of remark 3 implies that our definition of index applied to system (28) gives the standard definition of index. Considered in this way, the relationship between the indices ν_d and k^* is $\nu_d = k^* + 1$ (see section 4 of [10]). To see this, take for instance equation (6.2.9) of [6, p.154][13] which is given by

$$\dot{x}_1 = x_3 \tag{29a}$$

$$\dot{x}_2 = x_4 \tag{29b}$$

$$\dot{x}_3 = -x_1\lambda \tag{29c}$$

$$\dot{x}_4 = -x_2\lambda - g \tag{29d}$$

$$y = x_1^2 + x_2^2 - L^2 = 0 \tag{29e}$$

where L, g are positive real numbers. This system is clearly in the form (4) with $u = \lambda$. Computing the standard differential index one obtains $\nu_d = 3$ when considering $\lambda = x_5$. However, computing the differential index according our definition one obtains $k^* = 2$. The explanation of this difference is the following. In order to integrate this system, it is necessary to determine λ and $\dot{\mathbf{x}}$ but it is not necessary to know $\dot{\lambda}$. So, in order to recover our definition of index from the standard one, we may re-estate its definition in the following way. Given a (solvable) nonlinear completely determined DAE:

$$F(t, \dot{x}(t), x(t), \lambda) = 0$$

where F does depend on $\dot{x}_i$ for all $i = 1, \ldots, n$, then the index is the least integer k such that one may compute $\dot{x}$ as a function of (t, x) and the DAE and their derivatives up to order k. In other words one may distinguish differential variables x from the algebraic variables λ. Note also that, for system (29), one may compute

[13] This DAE is a model of a pendulum.

λ as a function of x and $\ddot{y}$. Hence the system can be integrated with the knowledge of the second order derivative of the constraint. Hence, it seems that the index 2 represents the real difficulty of integrating this DAE, rather than the index 3. Note that, applying the definition of index to an explicit control system (possessing algebraic variables) in the same way that we have done to recover the standard definition of index, one obtains $\nu_d = 1$. However, our definition gives $k^* = 0$.

5. Transforming DAEs into an index-zero form

We now recall some ideas of [42, 25, 21]. Assume that the DAE of (4) is time-invariant and that the explicit system (5) with output $y(t) \in I\!R^r$ is decoupable by static state-feedback, i.e., it admits relative degree and the decoupling matrix $A(x)$ has constant rank r [24]. This means that there exists a regular static-state feedback:

$$u = \alpha(x) + \beta(x)v$$

where $v = (\bar{v}, \widehat{v})$ and the first r components of v are of the form $\bar{v} = (y_1^{(k_1)}, \ldots, y_r^{(k_r)})$ [24]. In this case, it is not difficult to show that our definition of index gives $k^* = \max\{k_1, \ldots, k_r\}$ and the input of the DAE is $\widehat{v}$. The explicit system S obtained by taking $u = \alpha + \beta v$ with $\bar{v} = 0$ has an invariant manifold Γ and the solutions with initial conditions in Γ are the solutions of the original DAE. For instance, consider the system (29), let $\lambda = u$ and note that $y^{(2)} = 2(x_3^2 + x_4^2 - x_2 g) - 2(x_1^2 + x_2^2)\lambda$. Hence, if $y \equiv 0$ then

$$\lambda = (x_3^2 + x_4^2 - x_2 g)/(x_1^2 + x_2^2). \tag{30}$$

The explicit system obtained by substituting (30) back in (29) is such that the submanifold Γ of the state space defined by $y = \dot{y} = 0$ is an invariant manifold. In this work we generalize this ideas when the system S with output y is not static-feedback decoupable. We begin this section with the study of a particular class of DAEs.

5.1. Pseudo-explicit systems.

We now define a class of index-zero implicit control system, called *pseudo-explicit systems*, for which the problem of finding its solutions is equivalent to seeking the solutions of an explicit system, with initial conditions that lies on an invariant submanifold Δ of the state space.

Definition 5. *A regular implicit system* (4) *is called* pseudo-explicit *if, for the explicit system S defined for* (5) *(considered in the sense of the section 2.1) we have:*

- (i) span $\left\{dy^{(k)}, k \in I\!N\right\} \subset$ span $\{dt, dx\}$.
- (ii) *There exists $k^* \in I\!N$ such that*[14] span $\left\{dy^{(k)}\right\} \subset$ span $\{dt, dy, \ldots, dy^{(k^*)}\}$, *for all $k \in I\!N$.*

♠

Note that in this case, the infinite dimensional manifold S defined by system (5) is given by $I\!R \times \mathcal{X} \times \mathcal{U}^\infty$, where $\mathcal{X} = I\!R^n$ and $\mathcal{U} = I\!R^m$ and the corresponding Cartan field is given by (2). The following proposition summarizes the main properties of pseudo-explicit systems:

Proposition 4. *Assume that* (4) *is a pseudo-explicit system. Then:*

[14] We consider that k^* is the least integer with this property.

(i) *Let* $\Gamma = \{\xi \in S \,|\, y^{(k)}(\xi) = 0$ *for all* $k \in \mathbb{N}\}$. *Then* $\Gamma = \Delta \times \mathcal{U}^\infty$ *where* Δ *is an immersed (embedded) invariant submanifold of* $\mathbb{R} \times \mathcal{X}$.

(ii) *The curve* $\xi(t)$ *is a solution of the implicit system* (4) *if and only if* $\xi(t)$ *is a solution of the explicit system* S *given by* (5) *with initial condition* $x(t_0) = x_0$ *with* $(t_0, x_0) \in \Delta$.

Proof. See appendix C. □

The pseudo-explicit system is said to be Γ-stable if, for every initial condition (t_0, x_0) close enough to Δ, the solution of the system tends to Γ. This fact may be important for numerical integration for such a system, since errors, that could arise in the numerical integration, would be corrected by the attractiveness of Γ.

Definition 6. *A pseudo-explicit system is said to be* Γ*-stable if, for* $(t_0, x_0) \in \Delta$ *there exists* $\epsilon > 0$ *such that if the initial condition is* $\bar{x}$ *with* $dist(\bar{x}, x_0) < \epsilon$, *then the solution* $\xi(t)$ *of the explicit system* S *for every*[15] $u(\cdot)$ *converges to* Γ, *i. e.,* $dist(x(t), \Delta) \to 0$. ♠

Now we generalize some ideas of [5] about stabilization of invariant manifolds related to DAEs.

Proposition 5. *Assume that* S *is a pseudo-explicit explicit system with invariant manifold* Γ. *Then, around every* ξ *in* Γ *there exists a* Γ*-stable pseudo explicit system* $\widetilde{S}$ *such that, for every applied input and every initial condition* ξ *on* Γ, *the solutions of* $\widetilde{S}$ *and* S *(locally) coincide.*

Proof. Let $\mathbb{B} = \{dy_1^{(0)}, \ldots, dy_1^{(\rho_1)}, \ldots, dy_r^{(0)}, \ldots, dy_r^{(\rho_r)}\}$ be the basis of $\mathbb{Y}_{k^*}$ of the proof of proposition 4 . Let $y^{\langle\rho\rangle} = (y_1^{(\rho_1)}, \ldots, y_r^{(\rho_r)})^T$.

Write $dy^{\langle\rho\rangle}$ as row vectors in local coordinates, obtaining the $r \times n$ matrix $M(x)$. Let $\widetilde{g}(t,x) = M^T(MM^T)^{-1}$. By construction, $\widetilde{g}$ may be regarded as a set of r (column) fields such that $A(t,x) = \langle dy^{\langle\rho\rangle}; \widetilde{g}\rangle = I_r$, where I_r is the $r \times r$ identity matrix. We show now, using the same ideas of decoupling theory [24], that there exist $v = \phi(t,x)$ such that the system $\dot{x} = f + gu + \widetilde{g}v$ in closed loop with $v = \phi(t,x)$ (locally) gives the system $\widetilde{S}$ with the desired properties. Let $a(t,x) = \langle dy^{\langle\rho\rangle}; f + gu\rangle$. Take $v = (\phi_1, \ldots, \phi_r)$, where $\phi_j = -a_j - \sum_{i=0}^{\rho_j} b_{ij} y_j^{(i)}$ and the constants $b_{ij} \in \mathbb{R}$ are chosen in a way that the linear differential equations $y_j^{(\rho_j)} + \sum_{i=0}^{\rho_j} b_{ij} y_j^{(i)} = 0$ are asymptotically stable for $j \in \lfloor r \rceil$. Since $\phi(t,x) = 0$ for all $(t,x) \in \Delta$, the desired result follows.

Note that, if $\mathbb{B}$ is globally a basis of Y_{k^*}, then the construction of $\widetilde{S}$ may be also global (the integers ρ_i may change from point to point and this is the obstruction to the globalization of the result). □

In [18] it was shown that implicit systems may be locally put in pseudo-explicit form. The next result shows that this can be done using computable (effective) algebraic operations and the invariant manifold is always stabilizable.

Theorem 1. *A regular implicit system* Γ *defined by* (4) *can be locally transformed into a* Γ*-stable pseudo-explicit system around every point of* Γ. *Furthermore, if the restrictions are affine, i. e., if* $y = a(x) + b(x)u$, *then the requires transformation is a composition of a sequence of (effective) symbolic operations.*

[15] We assume that $u(\cdot)$ is such that the solution of S exists for $t \in [t_0, \infty)$.

Proof. See appendix D. □

6. Main result

In this section we show that, given an implicit system, one may construct a explicit system such that some components of the solutions of the explicit system converges to the solutions of the implicit system. For this, one has to compute symbolically (for once and for all) the derivatives of the constraints up to order k^*, where k^* is the differential index. We first study this question only locally. After that, we discuss how to establish a global version of this result.

6.1. Special coordinates.

In this subsection we will establish the existence of special coordinates of system S that are instrumental for establishing our main results.

Proposition 6. *Let S be a system with global state representation* (5). *Let $\xi \in S$ be a regular point of $Y_k, \mathcal{Y}_k, k \in \{0, \ldots, k^*\}$ defined in* (6a) *and* (6b). *Consider the notation of lemma 3 and let $\{\sigma_0, \ldots, \sigma_n\}$ be the algebraic structure at infinity of this system, let dx_k be a local basis of $\mathcal{Y}_k$ around ξ and let k^* be the convergence index of the structure at infinity. Choose a nested family of subsets of the input $u \supset \widehat{u}_0 \supset \ldots \supset \widehat{u}_{k^*}$ with $\operatorname{card}(\widehat{u}_k) = m - \sigma_k$ and in a way that $(dt, dx_k, d\widehat{u}_k)$ is a local basis of $\mathcal{Y}_k + \operatorname{span}\{du\}$ around ξ. Then the functions*

$$\{t, x_{k^*}, \widehat{u}_{k^*}^{(0)}, \ldots, \widehat{u}_0^{(k^*)}, Z\} \tag{31}$$

where $Z = \{u^{(k*+k+1)} : k \in I\!N\}$, form a local coordinate system around ξ which is unbounded in $W = (\widehat{u}_{k^*}^{(0)}, \ldots, \widehat{u}_0^{(k^*)}\ Z)$ and is such that $\{dt, dx_{k^*}, d\widehat{u}_{k^*}^{(0)}, \ldots, d\widehat{u}_0^{(k^*)}\}$ is a basis of $\operatorname{span}\{dt, du, \ldots, du^{(k^*)}\}$. In particular, if $\{d\bar{y}\}$ is a local basis of Y_{k^*} and $(\widehat{x}, \widehat{u})$ are defined as in proposition 2, then

$$\{t, \widehat{x}, \bar{y}, \widehat{u}, \widehat{u}_{k*-1}^{(1)}, \ldots, \widehat{u}_0^{(k^*)}, Z\} \tag{32}$$

is also a local coordinate system which is unbounded in $W = \{\widehat{u}, \widehat{u}_{k*-1}^{(1)}, \ldots, \widehat{u}_0^{(k^*)}, Z\}$. Furthermore $\{dt, d\bar{y}, d\widehat{u}, d\widehat{u}_{k*-1}^{(1)}, \ldots, d\widehat{u}_0^{(k^*)}\}$ is a basis of $\operatorname{span}\ \{dt, du, \ldots, du^{(k^*)}\}$.

Proof. First note that, by parts 1 and 2 of lemma 3 (see also (S1) and (S2) in appendix A) it is easy to see that the choice of $\widehat{u}_k$ described above is a possible choice of such subsets of the inputs in the dynamic extension algorithm (DEA). So denote (x_k, u_k) the state representation obtained in the step k of DEA. The proof proceeds by induction. In step 0 of the DEA, write the global coordinate system induced by the state representation (x, u) as $\{t, x, u, \ldots, u^{(k^*)}, Z_0\}$, with $Z_0 = \{u^{(k^*+k+1)} : k \in I\!N\}$. By lemma 2 and the description of the DEA it follows that $\{t, x_0, u_0, \ldots, u_0^{(k^*-1)}, \widehat{u}_0^{(k^*)}, Z_0\}$ is a local coordinate system that is unbounded in $Z_1 = (\widehat{u}_0^{(k^*)}, Z_0)$. Continuing in this way, in the step k of DEA, we shall have constructed a chart $\{t, x_k, u_k, \ldots, u_k^{(k^*-k-1)}, \widehat{u}_k^{(k^*-k)}, Z_k\}$, where $Z_k = \{\widehat{u}_{k-1}^{(k^*-k+1)}, \ldots, \widehat{u}_0^{(k^*)}, Z_0\}$, that is unbounded in $\{\widehat{u}_k^{(k^*-k)}, Z_k\}$. Note that, in each step, lemma 1 also ensures that $\{dt, dx_k, du_k, \ldots, du_k^{(k^*-k-1)}, \widehat{u}_k^{(k^*-k)}, \widehat{u}_{k-1}^{(k^*-k+1)}, \ldots, \widehat{u}_0^{(k^*)}\}$ is a basis of $\operatorname{span}\{dt, dx, du, \ldots, du^{(k^*)}\}$. Proceeding in this way, the first part of result is easily proved by induction. The fact that (32) is a local chart with the claimed properties is an easy consequence of lemma 1. □

6.2. **Establishing the main result.** For simplicity, in this section we may consider the following assumptions for the explicit system (5):

(33) The set $L = \{dt, dy^{(0)}, \ldots, dy^{(k^*)}\}$ is independent for every $\xi \in S$.

(34) There exists a fixed choice of the functions of (32) in a way that $\{dt, d\widehat{x}, d\bar{y}, d\widehat{u}, d\widehat{u}^{(1)}_{k*-1}, \ldots, d\widehat{u}^{(k^*)}_0\}$ is globally a basis of $\text{span}\,\{dt, dx, du, \ldots, du^{(k^*)}\}$.

(35) The input of the implicit system is $\widehat{u} = \widehat{u}^{(0)}_{k^*}$ (which is a subset of u).

We shall see that condition (33) will assure the global convergence of our main result. If the set L is dependent for some points outside Γ then our result will hold only locally. The condition (34) will guarantee that one does not need to seek for different choices of the functions (32) during the process of integration[16]. The condition (35) implies that the input of the implicit system is a canonical input. In particular, the behavior of the system is not impulsive.

Now we will construct the system (1). Let $M = I\!R \times \mathcal{X} \times (I\!R^m)^{k^*+1}$ with global coordinates $(t, x, u^{(0)}, \ldots, u^{(k^*)})$, where $x \in I\!R^n$ and $u^{(k)} \in I\!R^m, k = 0, \ldots, k^*$. Consider the partition[17] $u^{(0)} = (\bar{u}, \widehat{u})$ where $\dim \bar{u} = \sigma_{k^*}$ and $\dim \widehat{u} = m - \sigma_{k^*}$. Let $N = I\!R \times \mathcal{X} \times I\!R^{m-\sigma_{k^*}} \times (I\!R^m)^{k^*}$, with global coordinates $(t, x, \bar{u}, u^{(1)}, \ldots, u^{(k^*)}) = (t, x, \nu)$. Note that, after a reordering, the coordinates of M are $(t, x, \nu, \widehat{u})$, with $\nu = (\bar{u}, u^{(1)}, \ldots, u^{(k^*)})$.

Let S be the system (in the sense of [20]), with global coordinates $\{t, x, (u^{(k)} : k \in I\!N)\}$ defined by (5). Let $\frac{d}{dt}$ be the Cartan field associated to S (given by (2)). As before, let $y = a(t,x) + b(t,x)u^{(0)}$ be the output of S and denote $y^{(0)} = y$ and $y^{(k)} = L_{\frac{d}{dt}} y^{(k-1)}$. Let $\pi_M : S \to M$ be the canonical projection. Abusing notation, we consider that $\{y^{(0)}, \ldots, y^{(k^*)}\}$ are functions defined on M. Let $\xi \in S$, and $\pi_M(\xi) = \mu = (t, x, \nu, \widehat{u})$. With the same abuse of notation, we may consider that $\mathbb{B} = \{dt, d\widehat{x}, dy^{(0)}, \ldots, dy^{(k^*)}, d\widehat{u}, d\widehat{u}^{(1)}_{k^*-1}, \ldots, d\widehat{u}^{(k^*)}_0\}$ is the local basis of $T^*_\nu M$ induced by Prop. 6. Define $\tau_{\widehat{x}} = \frac{d}{dt}\widehat{x} = \widehat{f}(x) + \widehat{g}(x)u$ and let $\widehat{y} = (y^{(0)}, \ldots, y^{(k^*)})$ and $\widehat{v} = (\widehat{u}^{(1)}_{k^*-1}, \ldots, \widehat{u}^{(k^*)}_0)$. Note that, up to some reordering of coordinates, we have $M \cong N \times \widehat{U}$ where $\widehat{U} = I\!R^{m-\sigma_{k^*}}$. So let $\tau : M \times \widehat{U} \to TM$, be the parameterized field defined by:

$$dt(\tau) = \tau_t = 1 \tag{36a}$$

$$d\widehat{x}(\tau) = \tau_{\widehat{x}} = \widehat{f}(x) + \widehat{g}(x)u \tag{36b}$$

$$d\widehat{y}(\tau) = \tau_{\widehat{y}} = -\gamma\widehat{y} \tag{36c}$$

$$d\widehat{v}(\tau) = \tau_{\widehat{v}} = -\beta\widehat{v} \tag{36d}$$

$$d\widehat{u}(\tau) = \tau_{\widehat{u}} = \widehat{u}^{(0)} \tag{36e}$$

In order to help the reader, we summarize our notations of this section in the tabular arrangement of figure 1.

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[16] It may be desirable, at least from the numerical point of view, to choose these functions pointwise in order to improve the conditioning of matrix T of (37).

[17] With a possible reordering of the components of $u^{(0)}$.

Coordinates	Notation	Comments
$u^{(0)} = u$	$(\bar{u}, \widehat{u})$	$\widehat{u}$ is the input of the DAE
$(\bar{u}, u^{(1)}, \ldots, u^{(k^*)})$	ν	Derivatives of u, apart $\widehat{u}$
(t, x, ν)	ζ	state of $\bar{\tau}$ (canonical coordinates of N)
$(t, x, \nu, \widehat{u})$	$(\zeta, \widehat{u})$	Canonical coordinates of M)
$(\widehat{u}^{(1)}_{k*-1}, \ldots, \widehat{u}^{(k^*)}_0)$	$\widehat{v}$	Part of the coordinate system of (34)
$(y^{(0)}, \ldots, y^{(k^*)})$	$\widehat{y}$	Output derivatives
$(t, \widehat{x}, \widehat{y}, \widehat{v}, \widehat{u})$		New coordinates for M (see (34))

FIGURE 1. Table of notations of this section.

Let T be the matrix formed by the differentials $dt, d\widehat{x}, d\widehat{y}, d\widehat{v}, d\widehat{u}$ when written in the coordinates $\{t, x, u^{(0)}, \ldots, u^{(k^*)}\}$ as row vectors. Let

$$T = \begin{pmatrix} dt \\ d\widehat{x} \\ d\widehat{y} \\ d\widehat{v} \\ d\widehat{u} \end{pmatrix} \tag{37}$$

The equation (36) is equivalent to[18]

$$T\tau = \begin{pmatrix} \tau_t \\ \tau_{\widehat{x}} \\ \tau_{\widehat{y}} \\ \tau_{\widehat{v}} \\ \tau_{\widehat{u}} \end{pmatrix} = \widehat{\tau} \tag{38}$$

Let $\Pi : M \to N$ be the canonical projection. Note that, in the coordinates $(t, x, \nu, \widehat{u})$ for M and (t, x, ν) for N, $\Pi(t, x, \nu, \widehat{u}) = (t, x, \nu)$. Let $\bar{\tau} : M \to TN$ be the parameterized field defined by:

$$\bar{\tau} = \Pi_* \tau \tag{39}$$

Equation (39) means that the parameterized field $\bar{\tau}$ in cordinates (t, x, ν) is obtained from the field τ written in coordinates by $(t, x, \nu, \widehat{u})$ by eliminating its components in the directions $\frac{\partial}{\partial \widehat{u}}$. Since $\bar{\tau}$ is parameterized by $\widehat{u}, \widehat{u}^{(0)}$, then $\bar{\tau}$ defines a (nonclassical) control system with state (x, ν) and input $\widehat{u}$.

The part (ii) of the next theorem means that every solution of (4) corresponds to a solution of (40) with initial condition on an invariant manifold Υ of (40). Furthermore, the parts (iii) and (iv) shows that every solution of (40) converges to Υ and every solution of (40) that is close Υ is also close to a solution of (4).

Theorem 2. *Assume that the conditions* (33)–(34)–(35) *hold. Denote a point* $(t, x, \nu, \widehat{u}) \in M = N \times \widehat{U}$ *by* $(\zeta, \widehat{u})$, *where* $\zeta = (t, x, \nu)$. *Let* π_x *be the canonical projection* $\pi_x : N \to \mathcal{X}$ *defined by* $\pi_x(\zeta) = x$ *and* $\pi_u : M \to I\!R^m$ *defined by* $\pi_u(\zeta, \widehat{u}) = u^{(0)}$. *Choose a smooth input* $\widehat{u} : [t_0, t_1] \to I\!R^{m-\sigma_{k*}}$. *Consider the control*

[18]From a numerical point of view, it is better to solve the linear equation $T\tau = \widehat{\tau}$ rather than compute $\tau = T^{-1}\widehat{\tau}$. In the same vein, when integrating numerically the explicit equation there is no need to include the equation $\dot{t} = 1$.

system defined on N with input $\widehat{u}(t) \in \widehat{U}$ given by:

$$\dot{\zeta}(t) = \bar{\tau}(\zeta(t), \widehat{u}(t), \widehat{u}^{(1)}(t)) \tag{40a}$$

$$\zeta(t_0) = \zeta_0 \tag{40b}$$

Where $\bar{\tau}$ is defined by (38)–(39). Consider $y^{(k)} = \frac{d}{dt}^k(y), k = 0, \ldots, k^*$ as a function defined on M and let $\widehat{y} = (y^{(0)}, \ldots, y^{(k^*)})$. Define the set $\Upsilon = \{(\zeta, \widehat{u}) \in M \mid \widehat{y}(\zeta, \widehat{u}) = 0\}$. Then the following properties hold:

- (i) *If $(\zeta, \widehat{u}) = (t, x, \nu, \widehat{u}) \in \Upsilon$, then $\dot{x}_i = \langle dx_i, \tau\rangle|_{(\zeta,\widehat{u})} = \frac{d}{dt}(x_i)|_{(\zeta,\widehat{u})} = f_i(x, u^{(0)})|_{(\zeta,\widehat{u})}$, $i = 1, \ldots, n$.*
- (ii) *Choose an input $\widehat{u}(\cdot)$ and let $\zeta(t)$ be a solution of (40) with $(\zeta(t_0), \widehat{u}(t_0)) \in \Upsilon$. Then $x(t) = \pi_x(\zeta(t))$ is a solution of (4) with input $u(t) = \pi_u(\zeta(t), \widehat{(t)})$. Conversely, if $x(t)$ is a smooth solution of (4), then $x(t) = \pi_x(\zeta(t))$ for some solution of (40) with $(\zeta(t_0), \widehat{u}(t_0)) \in \Upsilon$ and $\widehat{v}(\zeta(t), \widehat{u}(t)) \equiv 0$.*
- (iii) *Let $\zeta(t)$ be a solution of (40) with initial condition ζ_0 and input $\widehat{u}(t)$. Assume that $\zeta(t)$ is well defined for $t \in ([t_0, t_1]$, then $\|\widehat{y}(t)\| \leq e^{-\gamma t}\|\widehat{y}(t_0)\|$ for all $t \in [t_0, t_1]$.*
- (iv) *Let $L \subset M$ be a compact set. Let $\epsilon_1 > 0$ and $L_1 = \{\mu \in M | dist(\mu, L) < \epsilon_1\}$.*

  Assume that there exists $\alpha > 0$ such that, if $\|\widehat{u}(t)\| \leq \alpha$ for $t \in [t_0, t_1]$, then every solution $\zeta(t)$ of (40) with initial condition $(\zeta(t_0), \widehat{u}(t_0) \in L_1$ is such that $(\zeta(t), \widehat{u}(t))$ is well defined and is inside a compact set $R \subset M$ for every $t \in [t_0, t_1]$.

  Then there exists $\epsilon > 0$ such that, if $\zeta(t), t \in I = [t_0, t_f]$ is a solution of (40) with initial condition inside L, and $\|\widehat{y}(\zeta(t_0), \widehat{u}(t_0))\| < \epsilon$, then there exist $\kappa_1, \kappa_2 > 0$ and a solution $x(t)$ of (4) such that $\|\pi_x(\zeta(t)) - x(t)\| \leq \kappa_1\|\widehat{y}(t_0)\|e^{\kappa_2(t-t_0)}$ for all $t \in [t_0, t_f]$.

Proof. **(i).** Let $\pi_M : S \to M$ be the canonical projection. Let $\xi \in S$ and ϕ be a function in the set $\Psi = \{\Psi_1, \ldots, \Psi_\gamma\} = \{t, \widehat{x}, y^{(0)}, \ldots, y^{(k^*-1)}\}$. In this part of the proof we will distinguish the function x_i defined on S from the function $\widetilde{x}_i$ defined on M such that $x_i = \widetilde{x}_i \circ \pi_M$. So we may write $\phi = \widetilde{\phi} \circ \pi_M$, etc. The notation $\widetilde{\Psi}$ is then clear from the context.

Note that, by (36) we have $d\widetilde{\phi}(\tau)|_{\pi_M(\xi)} = d\phi(\frac{d}{dt})|_\xi$ for any $\xi \in S$ such that $\pi_M(\xi) \in \Upsilon$. Since $\{d\Psi\}$ is a basis of $\mathcal{Y}_{k^*-1}$, by part 7 of lemma 3 it follows that

$d\widetilde{x} \in \text{span}\left\{d\widetilde{\Psi}\right\}$. In particular we must have $d\widetilde{x}_i = \sum_{j=1}^{\gamma} \alpha_{ij} d\widetilde{\Psi}_j$. Note now that

$$f_i(t,x,u) = \langle dx_i, \frac{d}{dt}\rangle \tag{41a}$$
$$= \langle d(\widetilde{x}_i \circ \pi_M), \frac{d}{dt}\rangle \tag{41b}$$
$$= \langle \pi_M^* d\widetilde{x}_i, \frac{d}{dt}\rangle \tag{41c}$$
$$= \langle d\widetilde{x}_i, (\pi_M)_* \frac{d}{dt}\rangle \tag{41d}$$
$$= \langle \sum_{j=1}^{\gamma} \alpha_{ij} d\widetilde{\Psi}_j, (\pi_M)_* \frac{d}{dt}\rangle \tag{41e}$$
$$= \sum_{j=1}^{\gamma} \alpha_{ij} \langle d\widetilde{\Psi}_j, (\pi_M)_* \frac{d}{dt}\rangle \tag{41f}$$

It is easy to show that, for every $(\zeta, \widehat{u}) \in \Upsilon$, every ξ such that $(\zeta, \widehat{u}) = \pi_M(\xi)$ and every function $\widetilde{\Psi}_j \in \widetilde{\Psi}$ for $j = 1, \ldots \gamma$ we have

$$\langle d\widetilde{\Psi}_j, \tau\rangle|_{(\zeta,\widehat{u})} = \langle d\widetilde{\Psi}_j, (\pi_M)_* \frac{d}{dt}|_\xi\rangle = \langle d\Psi_j, \frac{d}{dt}\rangle|_\xi \tag{42}$$

(this is an easy consequence of (36) and the fact that $0 = \langle dy^{(j)}, \tau\rangle|_{(\zeta,\widehat{u})}$ for every $(\zeta, \widehat{u}) \in \Upsilon$). From (41a), (41f) and (42) it follows that:

$$\begin{aligned} f_i(t, \pi_x(\zeta), \pi_u(\zeta, \widehat{u})) &= \sum_{j=1}^{\gamma} \alpha_{ij} \langle d\widetilde{\Psi}_j, (\pi_M)_* \frac{d}{dt}(\xi)\rangle_{(\zeta,\widehat{u})} \\ &= \sum_{j=1}^{\gamma} \alpha_{ij} \langle d\widetilde{\Psi}_j, \tau\rangle|_{(\zeta,\widehat{u})} \\ &= \sum_{j=1}^{\gamma} \langle \alpha_{ij} d\widetilde{\Psi}_j, \tau\rangle|_{(\zeta,\widehat{u})} \\ &= \langle d\widetilde{x}_i, \tau\rangle_{(\zeta,\widehat{u})} \end{aligned}$$

for every $(\zeta, \widehat{u}) \in \Upsilon$, showing (i).

(ii). Construct the system evolving on M with input $\widehat{u}^{(1)}$ and state $\mu = (\zeta, \widehat{u})$ given by

$$\dot{\mu} = \tau(\mu, \widehat{u}^{(1)}) \tag{43}$$

It is clear that (43) is the prolongation by integrators of (40). Hence the smooth solutions of (40) corresponds to the solutions of (43) for convenient initial conditions. Furthermore,

$$\begin{pmatrix} \bar{\tau}(\zeta, \widehat{u}, \widehat{u}^{(1)}) \\ \widehat{u}^{(1)} \end{pmatrix} = \tau(\zeta, \widehat{u}, \widehat{u}^{(1)})$$

From (36c) it is clear that Υ is an invariant set for system (43). In particular, by (i) we have $dx(\tau) = \dot{x} = f(x,u)$ along $\mu \in \Upsilon$, and so the first part of (ii) holds.

Now let $x(t)$ be a solution of (4). By Proposition 1, there exists a corresponding solution $\xi_1(t)$ of S with $y(t) \equiv 0$ and ${\xi_1}_x(t) = x(t)$, ${\xi_1}_u(t) = u(t)$ obeys the differential equation (4a). It follows that $\dot{\widehat{x}} = \tau_{\widehat{x}} = \widehat{f} + \widehat{g}u$ and so $\widehat{x}(t)$ obeys (36b).

Note also that proposition 6 assure that the local chart $\{t, \widehat{x}, \widehat{y}, \widehat{u}, \widehat{v}, (u^{(k^*+k+1)} : k \in I\!N)\}$, where $\widehat{v} = (\widehat{u}_1^{(k^*-1)}, \ldots, \widehat{u}_{k^*}^{(0)})$, is unbounded in $Z = \{\widehat{v}, (u^{(k^*+k+1)} : k \in I\!N)\}$. Since $y(\xi_1(t)) \equiv 0$, in these coordinates, we have $\xi_1(t) = (t, \widehat{x}(t), 0, \widehat{u}(t), Z(t))$. As this local chart is unbounded in $Z(t)$, we may define $\xi_2(t) = (t, \widehat{x}(t), 0, \widehat{u}(t), 0)$ by taking $Z(t) \equiv 0$. By construction, span $\{dx, du\} \subset$ span $\{d\widehat{x}, d\widehat{y}, d\widehat{u}\}$. Hence, its is clear that ${\xi_1}_x(t) = {\xi_2}_x(t)$ and ${\xi_1}_u(t)) = {\xi_2}_u(t)$. Now take $\mu_2(t) = (t, x(t), \nu(t), \widehat{u}(t)) = (\zeta_2(t), \widehat{u}(t)) = \pi_M(\xi_2(t))$. So, $\mu_2(t)$ obeys (43) and hence $\zeta_2(t)$ is a solution of (40) with initial condition $\zeta_0 = \zeta_2(t_0)$ and input $\widehat{u}(t)$. As $Z(\xi_2(t)) \equiv 0$, we have $\widehat{v}(\mu_2(t)) \equiv 0$.

(iii). Straightforward from (36c).

(iv). We know that every smooth function is globally Lipchitz inside a compact R. Hence,there exist some $k_1 > 0$ such that $\|\tau(\zeta_1, \widehat{u}_1) - \tau(\zeta_2, \widehat{u}_2)\| \leq k_1 \|(\zeta_1, \widehat{u}_1) - (\zeta_2, \widehat{u}_2)\|$. Taking $\widehat{u} = \widehat{u}_1 = \widehat{u}_2$ it follows that $\|\tau(\zeta_1, \widehat{u}) - \tau(\zeta_2, \widehat{u})\| \leq k_1 \|\zeta_1 - \zeta_2\|$ for every $(\zeta_1, \widehat{u})$, $(\zeta_1, \widehat{u}) \in R$.

Since two solutions $\zeta_1(t)$ and $\zeta_2(t)$ of (40) with bounded input $\widehat{u}(\cdot)$ and with initial conditions respectively $(\zeta_1(t_0), \widehat{u}(t_0)), (\zeta_2(t_0), \widehat{u}(t_0))$, both in L_1 are well defined in $[t_0, t_1]$ and are such that $(\zeta_1(t), \widehat{u}(t)) \in R$ and $(\zeta_2(t), \widehat{u}(t)) \in R$ for $t \in [t_0, t_1]$, from the same idea of the proof of the classical result of continuous dependence of the solutions of Lipchitz continuous differential equations[19], we have

$$\|\zeta_1(t) - \zeta_2(t)\| \leq K_1 e^{K_2(t-t_0)} \|\zeta_1(t_0) - \zeta_2(t_0)\|, t \in [t_0, t_1]. \tag{44}$$

for convenient positive real numbers K_1, K_2.

Now around every point $\mu \in L$ we may construct open sets V_μ, U_μ and a local chart $\phi_\mu : U_\mu \to V_\mu$ of M, such that $(t, x, \nu, \widehat{u}) \mapsto (t, \widehat{x}, \widehat{y}, \widehat{v}, \widehat{u})$. This construction may be done in a way that V_μ is a rectangular open set containing $\phi_\mu(\mu)$ and the closure of U_μ is compact. Since Υ is closed, for every $\mu \notin \Upsilon$ we may choose V_μ, U_μ in a way that $\bar{U}_\mu \cap \Upsilon = \emptyset$, where $\bar{U}_\mu$ denotes the closure of U_μ. In this case we, denote $\widehat{Y}_\mu = \min_{a \in \bar{U}_\mu} \|\widehat{y}(a)\|$. Note that $\widehat{Y}_\mu = 0$, when $\mu \in \Upsilon$, and $\widehat{Y}_\mu > 0$, when $\mu \notin \Upsilon$

Furthermore, for every pair $\mu_i \in U_\mu, i = 1, 2$, the mean-value inequality applied to ϕ_μ^{-1} gives

$$\begin{aligned} \|\mu_1 - \mu_2\| \leq \; & K_\mu \left(\|t(\mu_1) - t(\mu_2)\| + \|\widehat{x}(\mu_1) - \widehat{x}(\mu_2)\| + \right. \\ & \left. \|\widehat{y}(\mu_1) - \widehat{y}(\mu_2)\| + \|\widehat{v}(\mu_1) - \widehat{v}(\mu_2)\| + \|\widehat{u}(\mu_1) - \widehat{u}(\mu_2)\|\right) \end{aligned} \tag{45}$$

Note now that the family $\mathcal{C} = \{U_\mu : \mu \in L\}$ is an open covering of the compact set L. So we may take a finite subcovering $\{U_{\mu_i} : i \in \Lambda\}$ and let $K = \max_{i \in \Lambda} K_{\mu_i}$, where K_{μ_i} is defined in (45). Note that this class is divided in two subclasses $\mathcal{C}_1 = \{U_\mu : \mu \in \Lambda, U_\mu \cap \Upsilon = \emptyset\}$ and $\mathcal{C}_2 = \{U_\mu : \mu \in \Lambda, U_\mu \cap \Upsilon \neq \emptyset\}$. Let $\widehat{Y} = \min\{\widehat{Y}_\mu : \mu \in L \mid U_\mu \in \mathcal{C}_1\}$. By construction, if we take $\epsilon_2 = \widehat{Y}$, then

$$a \in L \text{ and } \|\widehat{y}(a)\| \leq \epsilon_2 \text{ implies that } a \text{ is inside some } U_\mu \in \mathcal{C}_2. \tag{46}$$

If $U_\mu \in \mathcal{C}_2$, we may define $\pi_\mu : U_\mu \to \Upsilon$ defined by $\pi_\mu(t, \widehat{x}, \widehat{y}, \widehat{v}, \widehat{u}) \to (t, \widehat{x}, 0, \widehat{v}, \widehat{u})$ (defined in the local coordinates ϕ_μ). By (45), it follows that

$$\|a - \pi_\mu(a)\| \leq K \|\widehat{y}(a)\|, a \in U_\mu, U_\mu \in \mathcal{C}_2 \tag{47}$$

[19]Which is a consequence of Bellman-Gronwall lemma.

Now let $\epsilon = \min\{\epsilon_2, \epsilon_1/K\}$. Then, for every $a \in L$, with $\|\widehat{y}(a)\| < \epsilon$, we may take the solution with initial condition $\mu_0 = (\zeta(t_0), \widehat{u}(t_0)) = \pi_\mu(a)$ with input $\widehat{u}(\cdot)$. From part (ii), as $\mu_0 \in \Upsilon$, it follows that $\pi_x(\zeta(t)$ is a solution of (4). From (47) it is clear that $\pi_\mu(a) \in L_1$.From (46), (47), and (44), the desired result follows easily. □

7. Example

We shall illustrate the main results of this paper with an academic example. Consider the system

$$
\begin{aligned}
\dot{t} &= 1\\
\dot{x}_1 &= 10x_3 + x_3u1\\
\dot{x}_2 &= u1\\
\dot{x}_3 &= u2\\
y1 &= x1 + ae^{x1} - 2 - 2a + x_2 - b\cos(t) = 0\\
y2 &= x2 + b\cos(t) = 0
\end{aligned}
$$

The symbolic derivatives of the output were computed using Matlab® symbolic package (Mapple®). The numerical dimension of $\mathcal{Y}_k$ is the numerical rank of the Jacobian matrix $J_k = \frac{\partial(x,y,\dot{y},y^{(k)})}{\partial(t,x,u,\ldots,u^{(k)})}$. In order to determine the generical rank of J_k, their ranks have been computed for random values of $t, x, u, \ldots, u^{(k)}$, giving $\sigma_0 = 0$, $\sigma_1 = 1$, $\sigma_2 = 2$, indicating that, if the system is regular, the index is given $k^* = 2$ (during the numerical integration of (40), one may test pointwise the condition number of the matrix T and compute numerically the rank of J_k in order to test if our assumptions of regularity do not hold). By the same method, one may verify that the rank of $D_{k^*} = \frac{\partial(t,y,\dot{y},y^{(k^*)})}{\partial(t,x,u,\ldots,u^{(k^*)})}$ for random values of $(t, x, u, \ldots, u^{(k^*)})$ is equal to $1 + 2 \times (k^* + 1) = 7$, showing that the outputs are generically differentially independent. In this case the implicit system is completely determined. One may show that y is generically a flat output for S, which is verified by showing that the rank of D_{k^*} is the same of J_{k^*}, or equivalently, span $\{dx\} \subset$ span $\{dt, dy, \ldots, dy^{(k^*)}\}$ [31, 37, 34]. Then, it is clear that $\widehat{x} = \emptyset$. Testing numerically other random points and combinations, we have chosen $\widehat{v} = u_2^{(1)}$. In this way the coordinate system (32) for this system is $\{t, \widehat{x}, \widehat{y}, \widehat{v}, \widehat{u}\}$ with $\widehat{x} = \widehat{u} = \emptyset$, $\widehat{y} = (y, \dot{y}, \ddot{y})$, $\widehat{v} = \dot{u}_2$. It is important to mention that, in this example, $y^{(k^*)}$ does not depend on $u^{(k^*)}$. Hence it is easy to see that one may eliminate (as done in the example) the set $u^{(k^*)}$ of coordinates from M, reducing the dimension of the state space of system (40). The symbolical computations in Matlab/Mapple give

```
T = [ 1,      0,       0,       0,       0,       0,       0,      0;
   b*sin(t), 1+a*exp(x1),      1,       0,       0,       0,       0, 0;
     -b*sin(t),       0,       1,       0,       0,       0,       0,       0;
     b*cos(t),    a*exp(x1)*(10*x3+x3*u1),       0,
     (1+a*exp(x1))*(10+u1),  (1+a*exp(x1))*x3+1,       0,       0,       0;
     -b*cos(t),       0,       0,       0,       1,       0,       0,       0;
     -b*sin(t), a*exp(x1)*(10*x3+x3*u1)^2+a*exp(x1)*(10+u1)*u2+
     a*exp(x1)*x3*u1p,0,2*a*exp(x1)*(10*x3+x3*u1)*(10+u1)+
     (1+a*exp(x1))*u1p,2*a*exp(x1)*(10*x3+x3*u1)*x3+(1+a*exp(x1))*u2,
```

```
 (1+a*exp(x1))*(10+u1),  (1+a*exp(x1))*x3+1,     0;
b*sin(t),      0,      0,      0,      0,      0,      1,      0;
0,      0,      0,      0,      0,      0,      0,      1];
```

where [t, x1, x2, x3, u1, u2, u1p, u2p] $= [t, x_1, x_2, x_3 u_1, u_2, \dot{u}_1, \dot{u}_2]$. The solution of system (40) with initial conditions in Υ gives the results[20] of figures 2 and 3. The figure 4 shows that the distance between the point $\bar{\zeta}(t) = (\zeta(t), \widehat{u}(t))$ and Υ does not grow in time. In order to verify the numerical errors in the deriv-

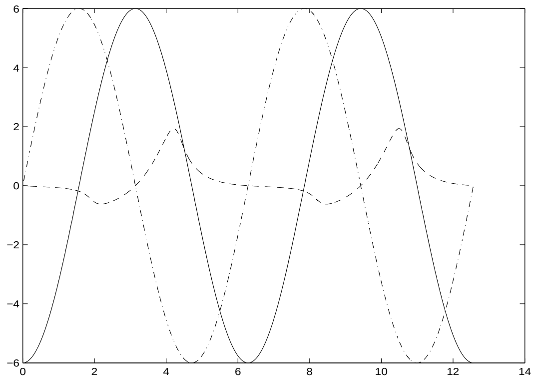

FIGURE 2. Curves of $\pi_x(\zeta(t))$ versus time. The curve $x_1(t)$ is continuous, $x_2(t)$ is dashed and $x_3(t)$ dashed-dotted

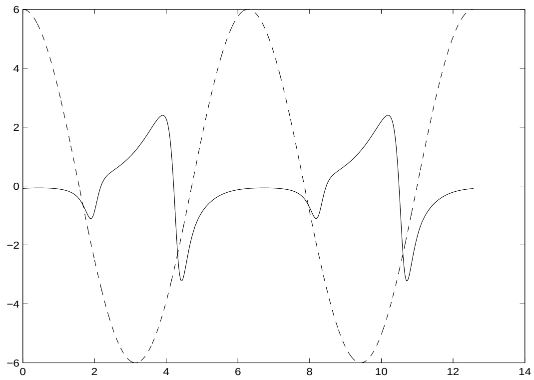

FIGURE 3. Curves of $\pi_u(\bar{\zeta}(t))$ versus time. The curve $u_1(t)$ is continuous, $u_2(t)$ is dashed.

ative of $x(t)$ of our method, we have computed the error $e(t) = (\langle dx, \tau(\bar{\zeta}(t))\rangle - f(t, \pi_x(\zeta(t)), \pi_u(\bar{\zeta}(t))) - g(\pi_x(\zeta(t))\pi_u(\bar{\zeta}(t))$. The ideal result would be zero, but small numerical errors are shown in figure (5).

Another test was performed by applying the input $\pi_u(\bar{\zeta}(t))$ of figure 3 to the system (5) with the same compatible initial conditions $\pi_x(\zeta(t_0))$. The ideal result of $y(t)$ obtained in this way would be zero, but small deviations have been detected

[20]Simulations have been made in Matlab/Simulink®.

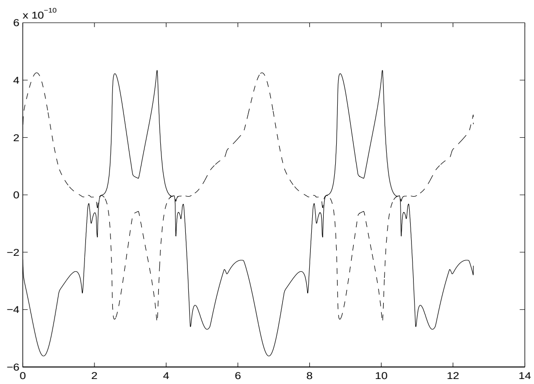

FIGURE 4. Curves of $y(\pi_x(\bar{\zeta}(t))$ versus time. The curve $y_1(t)$ is continuous, $y_2(t)$ is dashed.

in figure 6, due to numerical errors of our method and the errors in the numerical integration of the test itself.

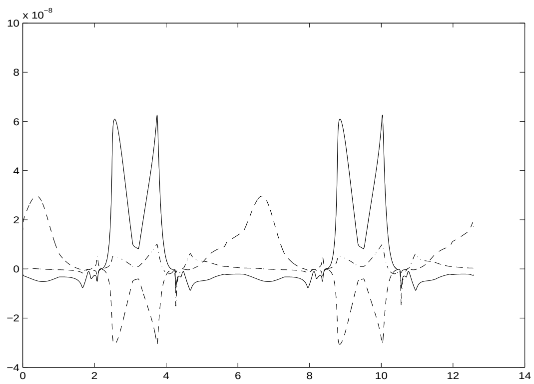

FIGURE 5. Error $e(t)$ in the derivative of $\pi_x(\zeta(t))$. The curve $e_1(t)$ is continuous, $e_2(t)$ is dashed and $e_3(t)$ is dashed-dotted.

Note that all the solutions of (40) will converge to Υ and hence a set of compatible initial conditions can be found simply by integrating the explicit system (40). We have chosen $a = 6, b = 0.2, \gamma = \beta = 70$ and initial conditions (Matlab long e format)[21]:

$$\begin{aligned} \zeta(0) &= [t, x_1, x_2, x_3, u_1, u_2, \dot{u}_1, \dot{u}_2](0) \\ &= [0 \;\; 2.595584190645906\text{e+}000 \;\; \text{-}5.999999999754128\text{e+}000 \\ &\quad 4.546949839215331\text{e-}012 \;\; 4.207790366139719\text{e-}012 \\ &\quad \text{-}8.330778672817486\text{e-}002 \;\; 5.999999999754128\text{e+}000 \;\; 0] \end{aligned}$$

Remark 4. *The "standard" differential index of this system is $\nu_d = 3$ ($k^* = 1$). Including the first order derivatives of the constraints into the set of constraints,*

[21]The files for Matlab 6 used in this test may be retrieved in `http://www.lac.usp.br/~paulo/implicit`.

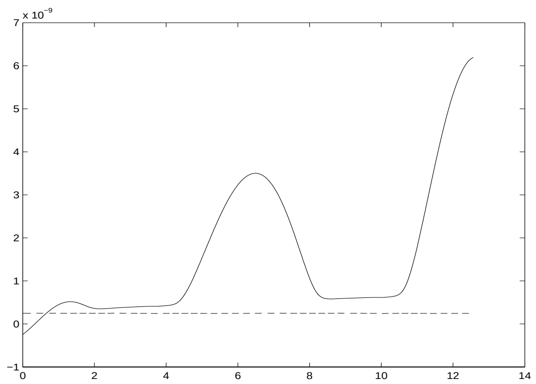

FIGURE 6. Curves $y_1(t)$ and $y_2(t)$ with $\pi_u(\zeta(t))$ as the input of system (5). The curve y_1 is continuous, y_2 is dashed.

one may reduce the index of this system to $\nu_d = 2$ *(*$k^* = 1$*) using the techniques of* [6, 4]*. However, the second derivatives of the constraints depend on the derivatives of the algebraic variables* u_1 *and* u_2*. Hence in order to reduce the index further one must use symbolic transformations, for instance the techniques of theorem 1. This simple example has the property that the explicit system obtained by adding an integrator in the first input is decoupable by static-state feedback, and so one could use the ideas of the beginning of section 5 in order to obtain another index-zero equivalent system. However, for more complex examples the symbolic reduction of the index by any means may be a hard task, whereas the techniques of theorem 2 could be applied.*

8. Conclusions and further research

Further research combining symbolic and numerical algorithms can be useful for improving numerical integration schemes for higher-index DAEs (see [11, 44]). In this paper we have established two potential methods of integration of DAEs. The first one, based in the dynamical extension algorithm (DEA), can be applied in particular cases, but it can produce very complex symbolic manipulations associated to the DEA. The second method is based on the geometric properties of DAEs and seems to be promising for the numerical integration of higher-index DAEs. This second method is based on the computation of the symbolic derivatives of the constraints and on the numeric solution of the (pointwise) linear equation $M\tau = \widehat{\tau}$ (see equation (38)). If the system is sparse, this property will reflect on the derivatives of the constraints, and then it will assure that the matrix T is also sparse. Hence, our second method is compatible with the application of linear-algebra packages for sparse matrices. The assumptions (34)–(35) of theorem 2 may be weakened, and the corresponding functions may be chosen pointwise using numerical methods, but this is the subject of future research.

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Appendix A. Proof of Lemma 3

Proof. Let $(x_{-1}, u_{-1}) = (x, u)$ be the original state representation of system S with output $y^{(0)}$ defined by (5). In step $k-1$ of this algorithm $(k = 0, 1, 2, \ldots)$ one has constructed a classical (local) state representation (x_{k-1}, u_{k-1}) with output $y^{(k)}$ defined on an open neighborhood U_{k-1} of $\xi \in S$. Assume that span $\left\{dt, dx_{k-1}, dy^{(k)}\right\}$ is nonsingular around ξ[22]. Note that we can give the following geometric description of DEA

- (S1) Choose $\bar{y}_k$ (possibly among the components of y) by completing $\{dx_{k-1}\}$ into a basis $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}\}$ for span $\left\{dt, dx_{k-1}, dy^{(k)}\right\}$. Now choose $\widehat{u}_k$ (possibly among the components of u_{k-1}) by completing $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}\}$ into a basis $\{dt, dx_{k-1}, d\bar{y}_k^{(k)}, d\widehat{u}_k\}$ of span $\{dt, dx_{k-1}, du_{k-1}\}$. According to the last paragraph of section 2.1, this defines a local state feedback. By construction, this state feedback have the properties (13).

[22]It is easy to show that this is equivalent to the fact that the matrix $b_k(x_{k-1})$ of (11) has constant rank around ξ.

- (S2) Define $x_k = (x_{k-1}, \bar{y}_k^{(k)})$, and $u_k = (\dot{\bar{y}}_k^{(k)}, \widehat{u}_k)$. This is an extension of the state of the form (14).

Note that, (see the end of section 2.1), we have that (S1) and (S2) produces a new local state representation (x_k, u_k) of system S defined in an open neighborhood $U_k \subset U_{k-1}$ of ξ. Note that the steps $(S1)$ and $(S2)$ describes the procedure of the Dynamic Extension Algorithm that could be performed, at least theoretically, for nonaffine systems[23]. In particular our geometric interpretation of Lemma 3 holds for nonaffine systems.

(i and ii). We show first that the state representation (x_k, u_k) is classical, *i. e.*, span $\{d\dot{x}_k\} \subset$ span $\{dt, dx_k, du_k\}$. This property holds for (x, u). By induction, assume that it holds for (x_k, u_k). Then from (S1) and (S2) we have span $\{d\dot{x}_{k+1}\} \subset$ span $\left\{dt, dx_k, d\dot{x}_k, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\right\} \subset$ span $\{dt, dx_{k+1}, du_{k+1}\}$.

In step $k=0$, we choose a partition $y^{(0)} = (\bar{y}_0^{(0)}, \widehat{y}_0^{(0)})$ and construct $\widehat{u}_0$ satisfying (S1) for $k=0$. Then $d\widehat{y}_0^{(0)} \in \text{span}\{dt, dx, d\bar{y}_0^{(0)}\}$. Thus, $d\dot{\widehat{y}}_0^{(0)} \in \text{span}\{dt, dx, d\dot{x}, d\bar{y}_0^{(0)}, d\dot{\bar{y}}_0^{(0)}\} \subset \text{span}\{dt, dx, du, d\bar{y}_0^{(0)}, d\dot{\bar{y}}_0^{(0)}\}$. So, $d\dot{y} \in \text{span}\{dt, dx_0, du_0\}$. Then it is easy to see that 1 and 2 are satisfied for $k=0$. Now assume that, in the step $k-1$ we have a local state representation (x_{k-1}, u_{k-1}) satisfying i and ii. Choose a partition $y^{(k)} = (\bar{y}_k^{(k)}, \widehat{y}_k^{(k)})$ in a way that (S1) is satisfied and construct $\widehat{u}_k$ satisfying (S2). By i for $k-1$ and (S1) it follows that, $\text{span}\{dt, dx_k\} = \text{span}\{dt, dx, dy, \ldots, dy^{(k)}\}$. By construction, notice that $d\widehat{y}_k^{(k+1)} \in \text{span}\{dt, dx_{k-1}, d\dot{x}_{k-1}, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\} \subset \text{span}\{dt, dx_{k-1}$ $du_{k-1}, d\bar{y}_k^{(k)}, d\dot{\bar{y}}_k^{(k)}\}$. By (S1) it follows that $dy^{(k+1)} \in \text{span}\{dt, dx_k, du_k\}$. We show now that if ii holds for $k-1$, then $\text{span}\{dt, dx_k, du_k\} = \text{span}\{dt, dx, dy, \ldots, dy^{(k+1)}, du\}$, completing the induction. In fact, note that $\text{span}\{dt, dx_k, du_k\} = \text{span}\{dt, dx_{k-1}, d\bar{y}_k^{(k)}, d\widehat{u}_k\} +$ span $\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. By (S1) and the induction hypothesis it follows that $\text{span}\{dt, dx_k, du_k\} = \text{span}\{dt, dx, du, dy, \ldots, dy^{(k)}\} +$ span $\left\{d\dot{\bar{y}}_k^{(k)}\right\}$. Since $dy^{(k+1)} \in \text{span}\{dt, dx_k, du_k\}$, then ii holds for k. This shows i and ii.

(iii, v, vi, vii). We show first that

$$\dim Y_k(\nu) - \dim Y_{k-1}(\nu) \geq \dim Y_{k+1}(\nu) - \dim Y_k(\nu) \text{ for every } \nu \in S_k \tag{48}$$

For this note that, if the 1-forms $\{\eta_1, \ldots, \eta_s\} \subset Y_k$ are linearly dependent mod Y_{k-1}, *i. e.*, if $\alpha_0 dt + \sum_{i=1}^{s} \alpha_i \eta_i + \sum_{i=1}^{p} \sum_{j=0}^{k-1} \beta_{ij} dy_i^{(j)} = 0$ then, differentiation in time gives $\dot{\alpha}_0 dt + \sum_{i=1}^{s} (\dot{\alpha}_i \eta_i + \alpha_i \dot{\eta}_i) + \sum_{i=1}^{p} \sum_{j=0}^{k-1} (\dot{\beta}_{ij} dy_i^{(j)} + \beta_{ij} dy_i^{(j+1)}) = 0$. In other words, $\dot{\eta}_1, \ldots, \dot{\eta}_s$ are linearly dependent mod Y_{k+1}. Let $\xi \in S_k$. From the nonsingularity of $Y_j, \mathcal{Y}_j, j = 0, \ldots, k$ in S_k, if $\dim Y_k - \dim Y_{k-1} = l$ in $\xi \in S_k$, then we may choose a partition $y = (\bar{y}^T, \widehat{y}^T)$ such that $\bar{y}$ has l components and we locally have $Y_k = \text{span}\left\{d\bar{y}^{(k)}\right\} + Y_{k-1}$. Let $\widehat{y}_j$ be any component of $\widehat{y}$ for $j \in \lfloor p-l \rceil$. By construction we have that $\{d\widehat{y}_j^{(k)}, d\bar{y}^{(k)}\}$ is linearly dependent mod Y_{k-1} for every $j \in \lfloor p-l \rceil$. From the remark above it follows that the set $\{d\widehat{y}_j^{(k+1)}, d\bar{y}^{(k+1)}\}$ is

[23] In this case the computations are much more difficult since one may apply the inverse function theorem to compute the feedback $u_{k-1} = \gamma(x_{k-1}, v_k)$ in each step of the algorithm. A description of a version of the DEA for nonaffine systems can be found in [29].

(locally) dependent mod Y_k for every $j \in \lfloor p-r \rceil$, showing (48). In particular the sequence ρ_k is nonincreasing.

We show now that

$$\dim \mathcal{Y}_k(\nu) - \dim \mathcal{Y}_{k-1}(\nu) \leq \dim \mathcal{Y}_{k+1}(\nu) - \dim \mathcal{Y}_k(\nu) \text{ for every } \nu \in S_k \tag{49}$$

Assume that (x_k, u_k) is a state representation constructed around a neighborhood U_k of a point $\xi \in S_k$ and satisfying (S1), (S2), i and ii. Since $d\dot{\bar{y}}_k^{(k)} \subset u_k$, it follows that the components of $d\dot{\bar{y}}_k^{(k)}$ are independent mod $\mathcal{Y}_k$ since they are also components of the input u_k and furthermore, span $\{dt, dx_k\} = \mathcal{Y}_k$. Hence $\bar{y}_{k+1}^{(k+1)}$ may be chosen satisfying iii, showing (49). In particular, $\sigma_{k+1} \geq \sigma_k$.

To show the convergence of sequences ρ_k and σ_k for some $k^* \leq n$, assume that $\nu \in S_k$. Denote span$\{dx\}$ by X. Then $\mathcal{Y}_k = X + Y_k$ and thus

$$\dim \mathcal{Y}_k(\nu) = \dim X(\nu) + \dim Y_k(\nu) - \dim(Y_k(\nu) \cap X(\nu)).$$

Denote for $k \in I\!N$:

$$\begin{aligned} s_k(\nu) &= \dim \mathcal{Y}_k(\nu) - \dim \mathcal{Y}_{k-1}(\nu) \\ p_k(\nu) &= \dim Y_k(\nu) - \dim Y_{k-1}(\nu) \end{aligned}$$

Note that $\rho_k = p_k(\nu)$ and $\sigma_k = s_k(\nu)$ are constant for every $\nu \in S_k$. We also have

$$s_k(\nu) = p_k(\nu) - \dim(Y_k(\nu) \cap X(\nu)) + \dim(Y_{k-1}(\nu) \cap X(\nu)). \tag{50}$$

We show now that

$$\begin{array}{c} \text{if there exists } k^* \text{ and some } \nu \in S_{k^*} \text{ such that } s_{k^*}(\nu) = p_{k^*}(\nu) = \rho, \\ \text{then } s_{k^*+1}(\xi) = p_{k^*+1}(\xi) = \rho \text{ for every } \xi \in S_{k^*}. \end{array} \tag{51}$$

Note that, from (51), a simple induction shows that $s_k(\xi) = p_k(\xi) = \rho$ for every $k \geq k^*$ and $\xi \in S_{k^*}$. Furthermore, this last affirmation implies that $S_k = S_{k^*}$ for $k \geq k^*$.

To show (51), assume that $p_{k^*}(\nu) = s_{k^*}(\nu) = \rho$ for some $\nu \in S_{k^*}$. From (50), it follows that

$$-\dim(Y_{k^*}(\nu) \cap X(\nu)) + \dim(Y_{k^*-1}(\nu) \cap X(\nu)) = 0.$$

Since the dimensions of $Y_{k^*} \cap X$ and of $Y_{k^*-1} \cap X$ are constant in S_{k^*}, it follows that, for every $\xi \in S_{k^*}$, we have

$$p_{k^*}(\xi) = s_{k^*}(\xi) = \rho \tag{52}$$

and

$$-\dim(Y_{k^*}(\xi) \cap X(\xi)) + \dim(Y_{k^*-1}(\xi) \cap X(\xi)) = 0.$$

Note from (50) that

$$s_{k^*+1}(\xi) - p_{k^*+1}(\xi) = -\dim(Y_{k^*+1}(\xi) \cap X(\xi)) + \dim(Y_{k^*}(\xi) \cap X(\nu)) \tag{53}$$

for every $\xi \in S_{k^*}$. By (48), (49) and (52), it follows that

$$s_{k^*+1}(\xi) - p_{k^*+1}(\xi) \geq 0.$$

Since

$$-\dim(Y_{k^*+1}(\xi) \cap X(\xi)) + \dim(Y_{k^*}(\xi) \cap X(\xi)) \leq 0,$$

the only possibility is to have both sides of (53) equal to zero for every $\xi \in S_{k^*}$. Using (48) and (49) again, then (51) follows. Note that a simple induction shows that (51) implies vii.

To complete the proof of v, vi and vii it suffices to show the existence of k^* such that (51) holds. For this note that $\dim(Y_k(\nu) \cap X(\nu))$ is nondecreasing for $k = 0, \ldots, n$ and it is least than or equal to $n = \dim X$. In particular, there exists some $k^* \leq n$ such that $\dim(Y_{k^*}(\nu) \cap X(\nu)) = \dim(Y_{k^*-1}(\nu) \cap X(\nu))$.

(iv). Easy consequence of i, ii and the way that $\widehat{u}_k$ is chosen in (S1).

(viii). The first part of viii follows easily from iii, from the fact that card $\bar{y}_k = \sigma_k$ and from v. The second part of viii follows easily from the equality card $\bar{y}_k = \sigma_k$, from the fact that the components of $d\bar{y}_k^{(k+1)}$ are independent $\mod Y_k$ and from the fact that $\sigma_k = \rho_k = \rho$ for $k \geq k^*$.

(9). Easy consequence of lemma 2. □

Appendix B. Computation of (α_k, β_k) in step k of the DEA

Let

$$\begin{aligned} \bar{y}_k^{(k)} &= \bar{a}(t, x_{k-1}) + \bar{b}(t, x_{k-1}) u_{k-1} \\ \widehat{y}_k^{(k)} &= \widehat{a}(t, x_{k-1}) + \widehat{b}(t, x_{k-1}) u_{k-1} \end{aligned}$$

where rank $\bar{b} = \sigma_k$ around $\bar{x}_{k-1}$. Up to some reordering of inputs, let $\bar{b} = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \end{pmatrix}$ where $\bar{b}_{11}$ is locally nonsingular. Then define locally around $(t, \bar{x}_{k-1})$:

$$\beta_k(t, x_{k-1}) = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} \bar{b}_{11}^{-1} & -\bar{b}_{11}^{-1} \bar{b}_{12} \\ 0 & I \end{pmatrix}$$

$$\alpha_k(x) = \beta_0(x) \begin{pmatrix} -\bar{a} \\ 0 \end{pmatrix}$$

It is easy to verify that such (α_k, β_k) is a possible choice that has the convenient properties.

Appendix C. Proof of Proposition 4

Proof. By the nonsingularity of the codistributions $Y_k, k \in I\!N$, and using the same idea of the proof of part 5 of lemma 3 (in particular the proof that the sequence ρ_k is nonincreasing), it is not difficult to show that there exist a local basis of Y_{k^*} of the form $\{dt, dy_1, \ldots, dy_1^{(\rho_1)}, \ldots, dy_r, \ldots, dy_r^{(\rho_r)}\}$. Around any $\xi \in \Gamma$, the part (i) of Lemma 5 (see the proof of part (ii) of this lemma) implies that $\mathbb{B} = \{dy_1^{(0)}, \ldots, dy_1^{(\rho_1)}, \ldots, dy_r^{(0)}, \ldots, dy_r^{(\rho_r)}\}$ is a basis of $\mathbb{Y}_{k*}$.

Let $\Delta \subset I\!R \times \mathcal{X}$ be the subset such that $y^{(k)}(t, x) = 0, k \in I\!N$. Let

$$\Delta' = \left\{ x \in \mathcal{X} \,|\, y_j^{(i_j)}(t, x) = 0, j \in \lfloor r \rceil, i_j \in \{0, 1, \ldots, \rho_j + 1\} \right\}.$$

We show now that, around any $(t, x) = \pi_{t,x}(\xi)$ with $\xi \in \Gamma$, we have $\Delta = \Delta'$. In fact, let $(t, x) \in \Delta'$. It is clear that $\Delta \subset \Delta'$. By part (ii) of Definition 5, we must have $dy^{(\rho_j + k)} \in Y_{k^*}$ for $k \in I\!N$. By part (i) of lemma 5 it follows that $dy^{(\rho_j + k)} \in \mathbb{Y}_{k^*}$ (otherwise one can construct a 1-form $\eta \neq 0$ such that $\eta \in \mathbb{Y}_{k^*} \cap \operatorname{span}\{dt\}$). In particular, $\mathbb{Y}_{k^*+k} = \mathbb{Y}_{k^*}$ for $k \in I\!N$. Hence, $dy^{(\rho_j + k)} = \sum_{j=1}^{r} \sum_{i=1}^{\rho_j} \alpha_{ij} dy_j^{(i)}$ for convenient functions $\alpha_{ij}(t, x)$ and for all $k \in I\!N$. Then $y_j^{(\rho_j + k + 1)}(x) = \langle dy_j^{(\rho_j + k)}; f + gu \rangle = \sum_{j=1}^{r} \sum_{i=0}^{\rho_j} \alpha_{ij} y_j^{(i+1)} = 0$ showing for every $x \in \Delta'$, showing that $\Delta' \subset \Delta$. By the nonsingularity of the codistribution $\mathbb{Y}_{k^*} = \mathbb{Y}_{k^*+1}$, it follows that Δ is an immersed submanifold of $I\!R \times \mathcal{X}$. Then it is clear that (i) is true. To show that (ii) is true, it suffices to show that Δ is an invariant manifold. But this is a straitghforward consequence of the fact that one may complete the set $\bar{y} =$

$\{y_1, \ldots, y_1^{(\rho_1)}, \ldots, y_r, \ldots, y_r^{(\rho_r)}\}$ into a local coordinate system $(t, \bar{y}, \phi)$ of $I\!R \times \mathcal{X}$. Since $d\bar{y}^{(1)} \in \mathbb{Y}_{k^*}$, we locally have $\bar{y}^{(1)} = \Psi(\bar{y}) = 0$. If $(t_0, x_0) \in \Delta$ then $\bar{y}^{(1)}\big|_{t_0} = 0$. It follows that $\bar{y} = 0$ is an equilibrium point differential equation

$$\bar{y}^{(1)} = \Psi(\bar{y})$$

and so, it follows that $\bar{y}(t, x(t)) \equiv 0$. Since $dy^{(k)} \in \text{span}\,\{\mathbb{B}\}$, one has $y^{(k)} = y^{(k)}(\bar{y})$ Hence, $y^{(k)}(\bar{y}(t)) \equiv y^{(k)}(\bar{y}(t_0)) \equiv 0$ and so $(t, x(t)) \in \Delta$. □

Appendix D. Proof of Theorem 1

Proof. Using the same arguments of the proof of lemma 4, one may construct an affine state representation of the same form of (18) after the step k^* of the dynamic extension algorithm (instead of step $k^* - 1$ in the proof of that lemma). In this case $\widetilde{x} = (x, \bar{y}_0^{(0)}, \ldots, \bar{y}_{k^*}^{(k^*)})$ and $\widetilde{u} = (\omega, \mu)$, where $\omega = \bar{y}_{k^*}^{(k^*+1)}$ and $\mu = \widehat{u}_{k^*}$. By parts 1 and 8 of lemma 3 we see that $dy^{(k)} \in \text{span}\,\{d\widetilde{x}, (d\omega^{(j)} : j = 0, \ldots, k - k^* - 1)\}$ for all $k \in I\!N$.

Making $\omega = 0$ (a nonregular feedback) and adding the constraint $y = h(x) = 0$ we will show that we obtain a pseudo-explicit system. In fact, making $\omega = 0$ defines a Lie-Bäcklund imersion $\iota : T \to S$, where the system T is defined by:

$$\dot{\tilde{x}} = \tilde{f}(t, \tilde{x}) + \widehat{g}(t, \tilde{x})\mu \tag{54}$$

Note that the local coordinates of S induced by the state representation $(\widetilde{x}, \widetilde{u})$ are $\{t, \widetilde{x}, (\mu^{(k)} : k \in I\!N)\}$ and the local coordinates of S are $\{t, \widetilde{x}, \Omega, \mathcal{M}\}$ where $\Omega = (\omega^{(k)} : k \in I\!N)$, and $\mathcal{M} = (\mu^{(k)} : k \in I\!N)\}$. In these coordinates we have $\iota(t, \widetilde{x}, \mathcal{M}) = (t, \widetilde{x}, 0, \mathcal{M})$. Let ϕ be a function defined on S. We abuse notation, denoting $\phi \circ \iota$ simply by ϕ. As ι is a Lie-Bäcklund immersion, it follows that $\phi^{(k)} \circ \iota = (\phi \circ \iota)^{(k)}$. By construction it is clear that

$$\begin{aligned} \iota^* dt &= dt \\ \iota^* d\widetilde{x} &= d\widetilde{x} \\ \iota^* d\mu^{(k)} &= d\mu^{(k)}, \text{ for all } k \in I\!N \\ \iota^* d\omega^{(k)} &= 0, \text{ for all } k \in I\!N. \end{aligned}$$

To show that T has the properties of definition 2, it suffices to consider the same properties of system S and observe that the pull-back ι^* will preserve these properties. In fact, as $\omega = \bar{y}_{k^*}^{(k^*+1)}$, noting that $y^{(k)} = y^{(k)}(\widetilde{x})$ for $k \leq k^*$ and $y^{(k)} = y^{(k)}(\widetilde{x}, \omega, \ldots, \omega^{(k-k^*-1)})$, it is easy to show that the sets $\Gamma_S = \{\xi \in S : y^{(k)}(\xi) = 0\}$ and $\Gamma_T = \{\nu \in T : \widetilde{y}^{(k)}(\nu) = 0\}$ are such that $\Gamma_S = \iota(\Gamma_T)$. Using part 1 of Lemma 3, and (55) it is easy to show that the dimensions of $\iota^* Y_k$ are preserved for $k = 0, \ldots, k^*$ and property 8 of Lemma 3 implies that $\iota^* Y_k = \iota^* Y_{k^*}$ for $k \geq k^*$. By similar arguments, it follows that the dimensions of $\iota^* \mathcal{Y}_k$ are preserved for $k = 0, \ldots, k^*$ and $\iota^* \mathcal{Y}_k = \iota^* \mathcal{Y}_{k^*}$ for $k \geq k^*$. The regularity of $\iota^* \mathbb{Y}_k$ around the points Γ_T can be easily deduced from the part (iii) of lemma 5 and the regularity of the other codistributions. The properties (i) and (ii) of definition 5 are consequence of the fact that[24] $\iota^* \mathcal{Y}_{k^*} = \iota^* \text{span}\,\{dt, d\widetilde{x}\} = \text{span}\,\{dt, d\widetilde{x}\}$ and that $\iota^* \mathcal{Y}_k = \iota^* \mathcal{Y}_{k^*}$ for $k \geq k^*$.

The result follows from the application of Prop. 5 to system T defined by (54). □

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[24] Abusing notation, we denote $t \circ \iota = t$ and $\widetilde{x} \circ \iota = \widetilde{x}$.

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