Flatness of nonlinear control systems : a Cartan-Kähler Approach

Paulo Sérgio Pereira da Silva[‡]
[‡]Escola Politécnica da USP - PTC
São Paulo - SP - 05508-900, Brazil
paulo@lac.usp.br

Keywords : Nonlinear systems, differential flatness, feedback linearization, exterior differential systems, Cartan-Kähler theorem.

Abstract

In this work we consider the concept of differential flatness defined by Fliess, Levine, Martin and Rouchon. The structural property of k-flatness (the case in which flat outputs depends up to the kth derivative of the inputs) is studied based on the properties of the dynamic extension algorithm and the Cartan-Kähler theorems on the existence of integral manifolds of exterior differential systems. In this context, it is shown that there may exist two kind of flat systems — the *nonsingular* and the *singular* cases. Since the *nonsingular* case contains properly the class of state-feedback linearizable systems, this induces a classification of nonlinear flat systems as *state-feedback linearizable*, *nonsingular* and *singular* systems, in an ascending order of complexity.

Necessary and sufficient conditions of k-flatness for the nonsingular case are developed. Necessary and sufficient conditions for the singular case are also presented, although they are much more involved and are not in closed form. Examples are presented to illustrate the results.

1 Introduction and motivation

Feedback linearization is an important structural problem in control systems theory. This problem was completely solved in the static-state feedback case [20, 18] but necessary and sufficient conditions for feedback linearizability by dynamic state feedback are not yet known (see [3, 29, 4, 16, 32, 34, 30, 1, 27, 33, 26, 17, 35] for several results on this subject).

Fliess et al introduced the notion of differential flatness for nonlinear control systems in [9, 11]. This structural concept is strongly related to the problem of feedback linearization, and corresponds to a complete and finite parametrization of all solutions of a control system by a differentially independent family of functions. Differential flatness was originally defined in a differential algebraic setting [9, 11] and restated in a infinite dimensional geometric setting recently introduced in control theory [10, 25, 12, 14, 13].

In this paper we consider analytic nonlinear affine control systems of the form

$$\dot{x}(t) = f(x(t)) + \sum_{j=0}^{m} g_j(x(t))u_j(t) \tag{1}$$

where $x(t)$ evolves on an analytic manifold X of dimension n, and the input $u(t)$ is a vector of dimension m. We will assume that this dynamics is well formed (see [28]) *i.e.*, the fields $\{g_j(x), j \in \lfloor m \rceil\}$ are linearly independent for every $x \in X$.

Roughly speaking, a system (1) is said to be *flat* if there exists a set of differentially independent functions $y = (y_1, \ldots, y_m)$ called *flat output* such that every variable of the system is a function of time and of y and its derivatives[1] (see [14]). The flat output is a set of functions that may depend on the state x and on the input u and its derivatives $u^{(s)}$ for $s = 0, \ldots, k$. In this case, the system is said to be k-flat. When the flat output depends only on x, the system is said to be 0-flat.

Instrumental for our purposes are the structure algorithms, like *Singh's algorithm* [31] and the *Dynamic Extension Algorithm (DEA)* [6, 21, 7, 5, 24]. Our main results combine the geometric properties of the DEA with *Cartan-Kähler theory* [2, Chap.3] to parametrize that the jets of the flat output candidates and to obtain necessary and sufficient conditions of k-flatness can be obtained. The results developed here indicate that two kind of flatness exists. For the first one, called *nonsingular*, we give a complete and effective characterization; For the second one, called *singular*, a characterization is also given, but the computations in this case may be related to more sophisticated results like *Cartan-Kähler theorem* and *Cartan-Kuranishi prolongation theorem* [2, Chaps.3, 6 and 8]. In the nonsingular case, one can check flatness of a given system using only standard geometric operations like Lie-derivations, exterior differentiations *etc*. Our results about the singular case are inspired by the ideas of Fliess *et al* [15] which show that checking flatness of a system is closely related to the problem of finding integral manifolds of exterior differential systems. In this paper we show how to compute this exterior differential system through some symbolic operations.

If a system is flat, by definition it is k-flat for k big enough. An important question that remains open is if there exists a

[1] The definition of flatness given here is not rigourous. The reader may refer the cited literature for a complete presentation.

bound on k depending on the number of the states and inputs [15].

The paper is organized as follows. In section 2 we present notation and some results about exterior differential systems. In section 3 we present the notion of structure codistributions, instrumental for parametrizing the jet of the flat-ouput candidate. In sections 4 and 5 we present the main results respectively about the singular and the nonsingular case. In section 6 some examples are discussed. Finally in section 7 we state some conclusions.

2 Preliminaries and notation

2.1 Notation

The field of real numbers will be denoted by $I\!R$. The subset of natural numbers $\{1, \ldots, k\}$ will be denoted by $\lfloor k \rceil$. A symmetric multiindex K of class s and length k is a set of elements of the form $(i_1, \ldots, i_k)$, where $i_j \in \lfloor s \rceil$ for $j \in \lfloor k \rceil$, and where all the permutations of $(i_1, \ldots, i_k)$ are identified with each other. The set of all symmetric multiindeces of class s is denoted by $\Sigma(s)$. The length of $K \in \Sigma(s)$ is denoted by $\|K\|$. Note that $K \in \Sigma(s)$, $\|K\| = k$ is identified to some $(i_1, \ldots, i_k)$ such that $i_1 \leq i_2 \leq \ldots \leq i_k \leq s$. Given $K \in \Sigma(s)$, $K = (i_1, \ldots, i_k)$ and $i \in \lfloor s \rceil$ then (Ki) stands for $K = (i_1, \ldots, i_k, i)$.

We will use the standard notations of differential geometry and exterior algebra [36, 2]. Let $\mathcal{P}$ be a smooth manifold of dimension p. Let $\mathcal{F}$ be a set of smooth functions defined on $\mathcal{P}$. Let $Z = \{z \in \mathcal{P} \mid f(z) = 0, \forall f \in \mathcal{F}\}$ be the set of common zeroes of all $f \in \mathcal{F}$. Then z is an *ordinary zero* if : (i) There exists a subset $F = \{f_1, \ldots, f_r\} \subset \mathcal{F}$ such that the set $df = (df_1, \ldots, df_r)$ is independent on z ; (ii) There exists an open neighborhood U of z such that the set of common zeros of F that are inside U coincides with $Z \cup U$. In particular, $Z \cup U$ is a submanifold of $\mathcal{P}$.

Given a field f and a 1-form ω on $\mathcal{P}$, we denote $\omega(f)$ by $\langle f, \omega \rangle$. The set of smooth k-forms on $\mathcal{P}$ will be denoted by $\Lambda_k(\mathcal{P})$ and $\Lambda(\mathcal{P}) = \cup_{k \in I\!N} \Lambda_k(\mathcal{P})$.

Given two forms η and ξ in $\Lambda(\mathcal{P})$, then $\eta \wedge \xi$ denotes their *wedge* multiplication. The *exterior derivative* of $\eta \in \Lambda(\mathcal{P})$ will be denoted by $d\eta$. Note that the graded algebra $\Lambda(\mathcal{P})$, as well as its homogeneous elements $\Lambda_k(\mathcal{P})$ of degree k, have a structure of $C^\infty(\mathcal{P})$-module (see [36] for details). Given a family $\nu = (\nu_1, \ldots, \nu_k)$ of a $C^\infty(\mathcal{P})$-module, then span $\{\nu_1, \ldots, \nu_k\}$ stands for the span over $C^\infty(\mathcal{P})$.

An *ideal* $\mathcal{I}$ is a $C^\infty(\mathcal{P})$-submodule of $\Lambda(\mathcal{P})$ such that, given two forms ω and θ in $\mathcal{I}$ then $\omega \wedge \theta \in \mathcal{I}$. Given a subset $\mathcal{S} \subset \Lambda(\mathcal{P})$ then $\{\mathcal{S}\}$ stands for the least ideal that contains $\mathcal{S}$.

A *differential ideal* $\mathcal{I}$ is an ideal that is closed by the exterior differentiation, *i.e.*, $d\mathcal{I} \subset \mathcal{I}$. A differential ideal is also called an *exterior differential system*.

2.2 Exterior Differential Systems with independence condition

We present some definitions and results about exterior differential systems. The reader may refer to the treatise [2] for details.

Let M be an analytic manifold with dimension m and let $\mathcal{I}$ be a differential ideal defined on M. Let $\Omega = \omega_1 \wedge \ldots \wedge \omega_n, \notin \mathcal{I}$ be an n-form on M. Then the pair $(\mathcal{I}, \Omega)$ is called *Exterior Differential System with independence condition*.

An integral element E of $\mathcal{I}$ on $x \in M$ is a subspace of $T_x M$ such that $\theta|_E = 0$ for all forms θ of $\mathcal{I}$. An integral element of $(\mathcal{I}, \Omega)$ on $x \in M$ is a subspace E of dimension n of $T_x M$ such that $\theta|_E = 0$ for all form θ of $\mathcal{I}$ and $\Omega|_E \neq 0$ (this last condition is called *independence condition*). An integral manifold of $(\mathcal{I}, \Omega)$ is an immersed manifold $i : N \to M$ dimension n of $T_x M$ such that, for every point $\xi \in N$, its tangent space $E = i_*(T_\xi N)$ is an integral element of $(\mathcal{I}, \Omega)$. For every immersed manifold $i : N \to M$ (not necessarily an integral manifold) we define the *restriction of* $(\mathcal{I}, \Omega)$ *to* N by $(\tilde{\mathcal{I}}, \tilde{\Omega}) = (i^*\mathcal{I}, i^*\Omega)$. We denote by $G_n(TM)$ the Grassmann bundle of all n-subspaces $E \subset T_x M$. The bundle of all integral elements E of $\mathcal{I}$ of dimension n is denoted by $V_n(\mathcal{I})$ and is a subbundle of $G_n(TM)$. Similarly, $G_n(TM, \Omega)$ denotes the bundle of all n-subspaces $E \subset T_x M$ obeying the independence condition $\Omega|_E \neq 0$. The bundle of all integral elements of $(\mathcal{I}, \Omega)$ is denoted by $V_n(\mathcal{I}, \Omega)$ and is a subbundle of $G_n(TM, \Omega)$. Let $E \in G_n(TM, \Omega)$. Denote a basis of E by $e = (e_1, \ldots, e_n)$. It can be shown that

$$\begin{aligned} V_n(\mathcal{I}) \; = \; & \{E \in G_n(TM) \mid \theta(e_1, \ldots, e_n) = 0, \\ & \forall \theta \in \mathcal{I}, \theta \text{ of degree } n\} \end{aligned}$$

If ϕ is an n-form on M, define a function ϕ_Ω on $V_n(\mathcal{I}, \Omega)$ by the formula $\phi|_E = \phi_\Omega(E)\, \Omega|_E$. Let

$$\mathcal{F}_\Omega(\mathcal{I}) = \{\phi_\Omega \mid \phi \in \mathcal{I}, \phi \text{ has degree n}\}. \tag{2}$$

Then

$$V_n(\mathcal{I}, \Omega) \text{ is the subset of common zeroes of } \mathcal{F}_\Omega(\mathcal{I}) \tag{3}$$

2.3 Prolongations

Let $(\mathcal{I}, \Omega)$ be an exterior differential system with independence condition. The first prolongation is a Pfaffian system $(I^{(1)}, \Omega^{(1)})$ defined on a manifold $M^{(1)}$. To define the prolongation we define first a Pfaffian system $(\mathcal{L}, \Phi)$ on $G_n(TM, \Omega)$ in the following way. Let $\pi : G_n(TM, \Omega) \to M$ be the canonical projection and for each $(x, E) \in G_n(TM, \Omega)$ we define

$$\begin{aligned} I^{(1)}(x, E) &= \pi^* E^\perp \\ J^{(1)}(x, E) &= \pi^* T_x^* M \end{aligned}$$

Then the filtration $I^{(1)} \subset J^{(1)} \subset T^* G_n(TM, \Omega)$ defines a Pfaffian system with independence condition [2, p.104]. In particular, we may take $\mathcal{L} = \{I^{(1)}, dI^{(1)}\}$ and $\Phi = \pi^*\Omega$. To see what this gives in coordinates, assume that

$(x^1, \ldots, x^n, y^1, \ldots, y^s)$ is a local coordinate system for M such that $\Omega = dx_1 \wedge \ldots \wedge dx_n$. Then, an n-plane E of $G_n(TM, \Omega)$ is generated by $e_1, \ldots, e_n$, where

$$e_i = \frac{\partial}{\partial x_i} + \sum_\sigma p_i^\sigma \frac{\partial}{\partial y^\sigma} \tag{4}$$

where $(x^i, y^\sigma, p_i^\sigma)$ is a local chart for $G_n(TM, \Omega)$. In other words, E is defined by the equations $\theta^\sigma(E) = 0, \sigma \in \lfloor s \rceil$ where

$$\theta^\sigma = dy^\sigma - \sum_i p_i^\sigma dx^i \tag{5}$$

It follows that $I^{(1)}$ is generated by the forms θ^σ and we may take $\Phi = \pi^*\Omega = dx_1 \wedge \ldots \wedge dx_n$.

Definition 2.1 *Let $M^{(1)} = V_n(TM, \Omega)$ and assume that $M^{(1)}$ is a submanifold of $G_n(TM)$. Let $\iota : M^{(1)} \to G_n(TM)$ be the canonical injection. Then the first prolongation is the (linear) Pfaffian system $(\iota^*\mathcal{L}, \iota^*\Phi)$.*

Note that one can always define the canonical map $\bar{\iota} : M^{(1)} \to M$ as the composition of the canonical immersion $\iota : M^{(1)} \to G_n(TM)$ with the canonical projection $\pi : G_n(TM) \to M$.

Remark 2.1 *Let $X = \operatorname{span}\{dx^1, \ldots, dx^n\} \subset T^*M$. Assume that the canonical map $\bar{\iota} : M^{(1)} \to M$ is a surjective submersion. Let $\tilde{X} = \bar{\iota}^* X$, $\tilde{I}^{(1)} = \iota^* I^{(1)}$ and $\tilde{J}^{(1)} = \iota^* J^{(1)}$ $= \bar{\iota}^* TM$.*

Note that $\tilde{J}^{(1)} = \tilde{I}^{(1)} \oplus \tilde{X}$. Let $\tilde{\pi} : \tilde{J}^{(1)} \to \tilde{X}$ be the canonical projection. We can define a correspondence $\omega \in T^*M \to \pi_1(\omega) \in \tilde{X} \subset T^*M^{(1)}$ by the rule $\pi_1(\omega) = \tilde{\pi}\,\bar{\iota}^*\omega$. In coordinates (x, y) for M, let $\omega = \sum_{\sigma=1}^s \alpha_\sigma dy^\sigma + \sum_{i=1}^n \beta_i dx^i$. Then $\pi_1(\omega) = \sum_{i=1}^n \{\beta_i + \sum_{\sigma=1}^s \alpha_\sigma p_i^\sigma\} dx^i$. Note that $\pi_1(\omega)$ is the unique 1-form such that $\pi_1(\omega) - \bar{\iota}^*\omega$ $\in \tilde{I}^{(1)}$. Note also that $\omega \in X$ implies that $\pi_1(\omega) \in \tilde{X}$.

The next Lemma says that an integral manifold induces canonically an integral manifold for the prolongations of any order.

Lemma 2.1 *Assume that the kth prolongation of a differential system with independence condition $(\mathcal{I}, \Omega)$ is well defined for all $k \in I\!N$. Assume that $i : X \to M$ is an integral manifold of $(\mathcal{I}, \Omega)$. Then there is a unique integral manifold $i^{(k)} : X \to M^{(k)}$ associated to $i : X \to M$ by a canonical lifting.*

Proof. Note that $i_* : TX \to TM$ induces a map $j : X \to V_n(\mathcal{I}, \Omega)$ such that $j(x) = i_*(T_x M)$. Note also that this map (canonical lifting) is an integral manifold for the first prolongation (see [2, pp.147–149]). The proof may be completed by induction on k. $\Box$

The next Lemma says that, under some natural assumptions, *the prolongation of the pull-back is the pull-back of the prolongation*[2].

Lemma 2.2 *Let Z, M, N be analytic manifolds and let Ω be a volume form on Z. Let $\mathcal{M} = Z \times M$ and $\mathcal{N} = Z \times N$. Let $\pi_1 : \mathcal{M} \to Z$, and $\pi_2 : \mathcal{N} \to Z$ be the canonical projections. Let $i : \mathcal{M} \to \mathcal{N}$ be an immersion such that $\pi_2 \circ i = \pi_1$. Let $(\mathcal{I}, \Omega)$ be an exterior differential system defined on $\mathcal{N}$ and let $(\mathcal{J}, \Omega)$ be the exterior differential system defined on $\mathcal{N}$ by $\mathcal{J} = i^* I$. Assume that both prolongations $(\mathcal{J}^{(1)}, \Omega)$ and $(\mathcal{I}^{(1)}, \Omega)$ are well defined respectively on $\mathcal{M}^{(1)}$ and $\mathcal{N}^{(1)}$. Let $j : M^{(1)} \to N^{(1)}$ be the map defined by $j(E_\xi) = i_* E_\xi$. Then $\mathcal{J}^{(1)} = j^* \mathcal{I}^{(1)}$.*

Proof. The proof is an easy application of the definitions of § 2.3. $\Box$

Remark 2.2 *It is easy to show that the integral manifolds i and $i^{(1)}$ of Lemma 2.1 and the map $\bar{\iota}$ of Def. 2.1 obey the relation $i = \bar{\iota} \circ i^{(1)}$.*

2.4 Prolongation and Symbolical Calculus

In this section we will state some definitions that allow to perform symbolical calculations with functions that are the solution of a partial differential equation[3]. The idea is to define symbolical operations in a way that, when specialized to an integral manifold, these symbolical operations becomes the usual manipulations of differential calculus. For instance, if one denotes symbolically the partial derivatives $\partial y^\sigma / \partial x^i$ of a function y^σ by p_i^σ, then one can perform all the differential operations (of first order) with this function in a symbolical fashion. Using the language of exterior differential systems and their prolongations, we will consider this idea in a much more abstract way.

Let $M = X \times N$ be a manifold and let $\Omega = dx_1 \wedge \ldots \wedge dx_n$ be a volume form on X. Let $(\mathcal{I}, \Omega)$ be an exterior differential system with independence condition defined on M. Let $(\mathcal{I}^{(k)}, \Omega)$ be its prolongation defined on $M^{(k)}$. Assume that the canonical maps $\bar{\iota}_k : M^{(k+1)} \to M^{(k)}$ are surjective submersions[4] for $k \in I\!N$ and that $M^{(k)} = X \times N^{(k)}$. Note that $\bar{\iota}_k(x, \eta_k) = (x, \eta_{k-1})$ and hence the map $(\iota_k)_*(\xi)$ restricted to $T_\xi X \subset T_\xi M^{(k)}$ is an isomorphism.

Denote $T_\zeta M = T_\zeta X \oplus T_\zeta N$. A *symbolic field* is a field $f : M \to TX \subset TM$ such that, for every point ξ on M, f is contained in $T_{\pi_x(\xi)} X$. Let $\bar{\iota}$ be the canonical map of Def. 2.1. Since $\bar{\iota}_*(\xi) | T_\xi X$ is an isomorphism onto $T_{\bar{\iota}(\xi)} X$, to every symbolic field we can associate a unique symbolic field $f^{(1)} : M^{(1)} \to TX \subset TM^{(1)}$ such that $\bar{\iota}_* f^{(1)} = f \circ \bar{\iota}$. We may define the symbolic field $f^{(k)}$ on $M^{(k)}$ inductively by the rule $(\bar{\iota}_{k-1})_* f^{(k)} = f^{(k-1)} \circ \bar{\iota}_{k-1}$.

Consider the map $\tilde{\pi} : \tilde{J}^{(1)} \to \tilde{X}$ defined in Rem. 2.1. One may define inductively the $C^\infty(M)$-linear map $\tilde{\pi}_k : \tilde{J}^{(k)} \subset T^*M^{(k)} \to X^{(k)} \subset T^*M^{(k)}$. This map may be extended in a natural way to a map from $\Lambda(M^{(k)})$ to $\{X^{(k)}\} \subset \Lambda(M^{(k)})$. It follows that the composition

$$\pi_k = \tilde{\pi}_k \bar{\iota}^*_{k-1}, \ k = 0, 1, 2, \ldots \tag{6}$$

[2]We abuse notation by using the same symbol Ω for the independence condition on all manifolds.

[3]Solutions of a partial differential equation are integral manifolds of exterior differential systems, as will be shown in § 2.6.

[4]This assumption is needed for considering the map π_1 of Rem. 2.1.

defines a map from $\Lambda(M^{(k-1)})$ to $\{X^{(k)}\} \subset \Lambda(M_k)$, $k \in \mathbb{N}$.

Definition 2.2 *Given a function $\phi : M \to \mathbb{R}$ and a symbolical field we define $\langle\!\langle d\phi, f\rangle\!\rangle = \langle \pi_1 d\phi, f^{(1)}\rangle$. Note that $\psi = \langle\!\langle d\phi, f\rangle\!\rangle$ is a function defined on $M^{(1)}$. If ω is a one form on M we define $\langle\!\langle \omega, f\rangle\!\rangle = \langle \pi_1\omega, f^{(1)}\rangle$. Given a form ω defined on M, denote $\bar\omega = \bar\iota^*\omega$. Analogously, given a codistribution Γ on M we define $\bar\Gamma = \bar\iota^*\Gamma$. Note that both objects $\bar\omega$ and $\bar\Gamma$ are defined on $M^{(1)}$. A symbolic 1-form ω is a form defined on M such that $\omega \in \operatorname{span}\left\{dx^1,\ldots,dx^n\right\}$. A codistribution Γ on M is a symbolic codistribution if $\Gamma \subset \operatorname{span}\left\{dx^1,\ldots,dx^n\right\}$. For a symbolical codistribution Γ, define the flag $\Gamma^{(1)} = \{\bar\omega \in \bar\Gamma \mid d\bar\omega \in \{I^{(1)} + \bar\Gamma\}\}$. We say that Γ is* symbolicaly *involutive if $\Gamma^{(1)} = \bar\Gamma$.*

Define $\Gamma^{(0)} = \Gamma$. As in the "nonsymbolical" case one may compute the derived flag $\Gamma^{(k)} = \left\{\Gamma^{(k-1)}\right\}^{(1)}$ defined on $M^{(k)}$ for $k \in \mathbb{N}$. Let $n = \dim X$ and let $\zeta_n \in M^{(n)}$. Denote by $\zeta_{k-1} = \bar\iota_k(\zeta_k)$. We say that ζ_n is a regular point of the derived flag $\{\Gamma^{(k)}, k \in \{0,1,\ldots,n\}$, if ζ_k is a regular point of $\Gamma^{(k)}$ for $k \in \{0,1,\ldots,n\}$.

Definition 2.3 *Let ϕ be a function and f be a symbolic field, both defined on M. Then $\mathcal{L}_f\phi$ stands for $\langle\!\langle d\phi, f\rangle\!\rangle$. Define $\mathcal{L}_f^k\phi$ inductively by $\langle \pi_k d\mathcal{L}_f^{k-1}\phi, f^{(k)}\rangle$. Note that $\mathcal{L}_f^k\phi$ is a function defined on $M^{(k)}$. Let ω be a symbolical 1-form on M. Define $\mathcal{L}_f\omega = \tilde\pi\left\{\imath(f^{(1)})(\pi_1 d\omega) + d(\langle \bar\iota^*\omega, f^{(1)}\rangle)\right\}$, where $\imath(\cdot)(\cdot)$ denotes the interior product (see [36, 2.25 (d)]) and π_1 is defined by* (6). *If θ is a symbolical form defined on $M^{(k-1)}$ define $\mathcal{L}_{f^{(k)}}\theta = \tilde\pi_k\left\{\imath(f^{(k)})(\pi_k d\theta) + d(\langle \bar\iota_{k-1}^*\theta, f^{(k)}\rangle)\right\}$. Then define inductively $\mathcal{L}_f^k\omega = \mathcal{L}_{f^{(k)}}(\mathcal{L}_f^{k-1}\omega)$.*

Note that, in coordinates we have $\mathcal{L}_f\left(\sum_{i=1}^n \alpha_i dx^i\right) = \sum_{i=1}^n\left\{\mathcal{L}_f(\alpha_i)\, dx^i + \tilde\pi\left(\alpha_i d\langle dx^i, f^{(1)}\rangle\right)\right\}$.

The next proposition shows that, when specialized to integral manifolds the symbolical calculations defined above coincides to the "standard" differential calculus.

Proposition 2.1 *Let $i : X \to M$ be a local integral manifold of $\mathcal{I}$ and let $i^{(k)} :\to M^{(k)}$ be the the local integral manifold of $\mathcal{I}^{(k)}$ induced by $i^{(k)}$ (see Lemma 2.1). Note that there exists a unique field f defined on X such that $\langle i^*dx^i, \tilde f\rangle = \langle dx^i, f\circ i\rangle$. Let $\phi : M \to \mathbb{R}$ be a function, let f be a symbolic field, let ω be a symbolic 1-form, and let Γ be a symbolic distribution. Then*

(i) $\langle\!\langle d\phi, f\rangle\!\rangle \circ i^{(1)} = \langle d(\phi\circ i), \tilde f\rangle$

(ii) $\mathcal{L}_f^k\phi \circ i^{(k)} = L_{\tilde f)}^k \phi\circ i$

(iii) $(i^{(k)})^*\mathcal{L}_f^k\omega = L_{\tilde f}^k(i^*\omega)$.

(iv) *Assume that the symbolic distribution Γ is symbolically involutive. Then $i^*\Gamma$ is involutive as a codistribution defined on X.*

(v) *Let ζ_n be a regular point of the derived flag obtained from Γ. Assume that ζ_n is contained in the image of $i^{(n)}$. Then there exist some $q \le \dim\Gamma \le n$ such that $\Gamma^{(q-1)}$ is involutive. In particular, $(i^{(q-1)})^*\Gamma^{(q-1)}$ is integrable as a codistribution defined on X.*

2.5 Jet-spaces, contact-forms and prolongations

We denote by $\mathcal{J}^r(Z,Y)$ the set of r-jets of all smooth maps $y : Z \to Y$ between smooth manifolds Z and Y. Then $\mathcal{J}^r(Z,Y)$ has a structure of smooth manifold. For instance, consider the manifold $\mathcal{J}_y^r(Z,Y)$, where Z has dimension t and $Y = \mathbb{R}^s$. If Z has local coordinates $z = (z^1,\ldots,z^t)$ then $\mathcal{J}^r(Z,Y)$ has local coordinates $(z^i, y^\sigma, y_K^\sigma : i \in \lfloor t\rceil$, $\sigma \in \lfloor s\rceil$ $K \in \Sigma(t)$, $\|K\| \le r)$, where $y^\sigma \in \mathbb{R}$ represents the function evaluation $y^\sigma(z)$ and y_K^σ represents, for a symmetric multiindex $K = (i_1\ldots i_s)$, the partial derivative $\left.\frac{\partial y^\sigma}{\partial z^{i_1}\ldots\partial z^{i_s}}\right|_z$. On the manifold $\mathcal{J}^r(Z,Y)$, we may define the *contact forms*

$$\begin{array}{c} dy^\sigma - \sum_{j=1}^n y_j^\sigma dz^j,\ \sigma \in \lfloor s\rceil \\ dy_k^\sigma - \sum_{j=1}^n y_{kj}^\sigma dz^j,\ \sigma \in \lfloor s\rceil, k \in \lfloor t\rceil \\ \cdots \\ dy_K^\sigma - \sum_{j=1}^n y_{Kj}^\sigma dz^j,\ \sigma \in \lfloor s\rceil, K \in \Sigma(t),\ \|K\| \le r-1 \end{array} \tag{7}$$

Then the integral manifolds of the Pfaffian system $\mathcal{I}^r$ generated by the contact-forms above with independence condition $\Omega = dz_1 \wedge \ldots \wedge dz_t$ are the jets of functions $y^\sigma : Z \to \mathbb{R}$, $\sigma \in \lfloor s\rceil$ (see [2, Theo. 3.2, p.26]). Furthermore it is easy to see that the prolongation of $\mathcal{I}^r$ is $\mathcal{I}^{r+1}$. It can be shown also that every integral element of $\mathcal{I}^r$ at $(z, y^\sigma, y_K^\sigma : \sigma \in \lfloor s\rceil$, $K \in \Sigma(t)$, $\|K\| \le r)$ is generated by the set $\{e_1,\ldots,e_t\}$ given by

$$e_i = \frac{\partial}{\partial z^i} + \sum_{\sigma=1}^s \sum_{\substack{J\in\Sigma(n)\\ \|J\|\le r}} y_{Ji}^\sigma \frac{\partial}{\partial y_J} \tag{8}$$

where $y_{J_1 i_1}^\sigma = y_{J_2 i_2}^\sigma$ if the symmetric multiindeces $(J_1 i_1)$ and $(J_2 i_2)$ are identified by a convenient permutation.

Using Cartan's test [2, Thm. 1.11, pp. 74] one may show that every integral element of $\mathcal{I}^1$ is involutive and, by the Cartan-Kähler Theorem[2, Cor. 2.3, p.86] it admits local integral manifolds around every point. Using [2, Theo. 2.1, p.248], that shows that the prolongation of an involutive system is also involutive, one may show that $\mathcal{I}^r$ is involutive for all r.

2.6 Involutive codistributions and restricted Jet-spaces

In the sequel we abuse notation and $\mathcal{J}^r(Z,Y_1)\times\mathcal{J}^r(Z,Y_2)$ stands for $\mathcal{J}^r(Z, Y_1\times Y_2)$. Denote by (z,η_i) the points of $\mathcal{J}^{r_i}(Z,Y_i)$, i = 1, 2. In a similar vein, $\mathcal{J}^{r_1}(Z,Y_1)\times\mathcal{J}^{r_2}(Z,Y_2)$ will stands for the set of points (z,η_1,η_2) (with a common z).

Given a nonsingular involutive codistribution $\Gamma = \operatorname{span}\left\{\omega^1,\ldots,\omega^s\right\}$ defined on Z, with $\dim\Gamma = s$, then one may define a particular-class of contact forms on a manifold $R = Z\times\mathbb{R}^{s_0}\times\mathbb{R}^s$ with local coordinates $(z^i, y^\sigma, \alpha_j : i \in$

$\lfloor n \rceil, \sigma \in \lfloor s_0 \rceil, j \in \lfloor s \rceil)$ given by :

$$dy^\sigma - \sum_{j=1}^{s} \alpha_j^\sigma \omega^j, \ \sigma \in \lfloor s_0 \rceil \tag{9}$$

Denote the Pfaffian system generated by these forms by $\mathcal{I}$. Since Γ is involutive, by the Frobenius theorem we may assume without loss of generality that the local coordinate system $(z_1, \ldots, z_s, z_{s+1}, \ldots, z_t)$ is such that $\Gamma =$ span $\{dz_1, \ldots, dz_s\}$. Hence, in a new local coordinate system $(z^i, y^\sigma, \beta_j^\sigma : i \in \lfloor t \rceil, \sigma \in \lfloor s_0 \rceil, j \in \lfloor s \rceil)$ for R, these forms can be written as

$$\theta^\sigma = dy^\sigma - \sum_{j=1}^{s} \beta_j^\sigma dz^j, \sigma \in \lfloor s_0 \rceil \tag{10}$$

Consider the independence condition $\Omega = dz_1 \wedge \ldots \wedge dz_n$. One can show that every n-integral element of R and its prolongations of any order admit corresponding integral manifolds (they are all involutive with respect to the independence condition $\Omega = dz_1 \wedge \ldots \wedge dz_n$). In particular, one can show the following result

Proposition 2.2 *Let $R = Z \times I\!R^{s_0} \times I\!R^{s_0 s}$ be a manifold with local coordinates $(z^i, y^\sigma, \alpha_j^\sigma : i \in \lfloor t \rceil, j \in \lfloor s \rceil, \sigma \in \lfloor s_0 \rceil)$. Let $\mathcal{I}$ be the Pfaffian system generated by the forms* (9). *There exist integral manifolds of $(\mathcal{I}^{(k)}, \Omega)$ at every point of the prolongation $R^{(k)}$ for every $k \in I\!N$.*

Note that the construction above may be considered as a *restricted jet space* in the sense that the partial-derivatives are subject to some relations. The following definition generalizes this situation.

Definition 2.4 *Let Z be an analytic manifold and let Ω be a volume form on Z. Let $N = \mathcal{J}^{r_1}(Z, Y_1) \times \ldots \times \mathcal{J}^{r_p}(Z, Y_p)$, where $Y_i = I\!R, i \in \lfloor p \rceil$. Let $\mathcal{I}$ be the Pfaffian system generated on N by the contact forms associated to all the jet spaces $\mathcal{J}^{r_1}(Z, Y_1), \ldots, \mathcal{J}^{r_p}(Z, Y_p)$. Let $\mathcal{M}$ be an analytical manifold of the form $Z \times M$. A* restricted jet space *is an analytic immersion $i : \mathcal{M} \to N$ such that $\pi \circ i = \pi_1$ where $\pi_1 : \mathcal{M} \to Z$ and $\pi : N \to Z$ are the canonical projections. The pull-back $\mathcal{J} = i^*(\mathcal{I})$ defines the* restricted exterior differential system *generated by the* restricted contact forms.

Note that solving any partial differential equations with p-unknown functions $(y^1, \ldots, y^p)$ (depending on the variables $z_1, \ldots, z_t$, with order r_σ on y^σ for $\sigma \in \lfloor p \rceil$), is equivalent to finding integral manifolds of a convenient restricted exterior differential system.

The next proposition implies that the prolongation of a restricted jet-space is a restricted jet-space of greater order.

Proposition 2.3 *Consider the same notation stated in the last definition. Assume that the prolongation $\mathcal{J}^{(1)}$ is well defined on $\mathcal{M}^{(1)}$. Let $j : \mathcal{M}^{(1)} \to N^{(1)}$ be the map defined by $j(E_\xi) = i_* E_\xi$. Then $\mathcal{J}^{(1)} = j^* \mathcal{I}^{(1)}$.*

Proof. Note that the prolongation $M^{(k)}$ of M is always well defined and is given by

$$\mathcal{J}^{r_1+k}(Z, Y_1) \times \ldots \times \mathcal{J}^{r_p+k}(Z, Y_p).$$

Hence, the proof is a straightforward application of Lemma 2.2. $\square$

2.7 Extension of Restricted Jets-Spaces

Let $M = Z \times N$ be a restricted jet space. Let I be the codistribution generated on N by the restricted-contact forms and let $\mathcal{I} = \{I, dI\}$. Let $\Omega = dz^1 \wedge \ldots \wedge dz^t$ be a volume form on Z. Assume that the prolongations $\mathcal{I}^{(k)}$ are well defined on $M^{(k)}$ and are involutive for all $k \in I\!N$. We will define an extension of this space by adjoining the restricted 1-jet of ν new functions. For this, let $\Gamma \subset$ span $\{dz^1, \ldots, dz^t\}$ be a nonsingular symbolical codistribution on M generated by a basis $\{\omega^1, \ldots, \omega^s\}$. Assume that Γ is symbolically involutive, *i.e.*, $d\Gamma$ mod $\{I + \Gamma\} \equiv 0$[5]. Let $Y = Y_1 \times \ldots \times Y_\nu$ and let $A = A_1 \times \ldots \times A_\nu$, where $Y_i = I\!R, i \in \lfloor s_0 \rceil$, and $A_j = I\!R^s$, $j \in \lfloor s \rceil$. Consider the manifold $Y \times A$ with global coordinates $(y^b, \alpha_j^b : b \in \lfloor \nu \rceil, j \in \lfloor s \rceil)$, and let $M_1 = M \times Y \times A$. Define the forms on M_1 given by

$$\theta_0^b = dh^b - \sum_{j=1}^{s} \alpha_j^b \omega^j, b \in \lfloor \nu \rceil \tag{11}$$

Define the codistribution $I_1 =$ span $\{\theta_0^b : b \in \lfloor \nu \rceil\} + I$ and let $\mathcal{I}_1 = \{I_1, dI_1\}$. Abusing notation, let Ω stands for its pull-back from Z to M_1. We have the following result :

Proposition 2.4 *Consider the construction of M_1 and $\mathcal{I}_1$ above. Assume that the canonical maps $\bar{\iota}_k : M^{(k)} \to M^{(k-1)}$ are surjective submersions. Then*

(i) *The exterior differential system with independence condition (I_1, Ω) defined on M_1 is is involutive.*

(ii) *The prolongations $(\mathcal{I}_1^{(k)}, \Omega)$ are well defined on the restricted jet-spaces $M_1^{(k)}$ and are involutive for all $k \in I\!N$.*

(iii) *The canonical maps $\bar{\jmath}_k : M_1^{(k+1)} \to M_1^{(k)}$ are surjective submersions.*

3 Structure codistributions

In this section we discuss some structural properties of a nonlinear control system of the form

$$\dot{x}(t) = f(x(t)) + \sum_{j=0}^{m} g_j(x(t)) u_j(t) \tag{12a}$$

$$y = h(x(t)) \tag{12b}$$

3.1 Dynamic extension algorithm (DEA)

Let us recall the main aspects of the dynamic extension algorithm (in the version of [7]). Given an analytic system (12a)–(12b), the dynamic extension algorithm is a sequence of applications of regular static-state feedbacks and extensions of the state by integrators.

It is well known (see [7]) that the dynamic extension algorithm has an intrinsic interpretation based on the algebraic structure at infinity $\{\sigma_1, \ldots, \sigma_n\}$. The integer $\rho = \sigma_n$ is

[5]Note that, by Prop. 2.1 part (iv), this implies that Γ restricted to an integral manifold of $\mathcal{I}$ is involutive.

called *output rank* [8, 7]. The following result is well known[6] but is restated here in the differential geometric setting of [14] :

Theorem 3.1 *[7, 5, 24] Assume that system* (12a) *is analytic. Let S be the system associated to* (12a) *in the sense of [14]. Consider the filtrations (defined on S)*

$$\begin{aligned} \mathcal{Y}_k &= \text{span}\left\{dx, dy, \ldots, dy^{(k)}\right\}, k \in I\!N \\ Y_k &= \text{span}\{dy, \ldots, dy^{(k)}\}, k \in I\!N \end{aligned}$$

Let $S_k \subset S$ be the open and dense set of regular points of the codistributions Y_i and $\mathcal{Y}_i$ for $i = 0, \ldots, k$. In the kth step of the dynamic extension algorithm, one may construct around $\xi \in S_{k-1}$ a partition (and a reordering) of the output $y = (\bar{y}_k, \widehat{y}_k)$ with card $\bar{y}_k = \sigma_k$*, a new local state representation (x_k, u_k) of the system S with state $x_k = (x, \bar{y}_1^{(1)}, \ldots, \bar{y}_k^{(k)})$ and input $u_k = (\dot{\bar{y}}_k^{(k)}, \hat{u}_k)$ such that*

(i) span $\{dx_k\} =$ span $\left\{dx, dy, \ldots, dy^{(k)}\right\}$.

(ii) span $\{dx_k, du_k\} =$ span $\left\{dx, dy, \ldots, dy^{(k+1)}, du\right\}$.

(iii) $\dot{\bar{y}}_k^{(k)} \subset \bar{y}_{k+1}^{(k+1)}$

(iv) *Let $\mathcal{D}(\mathcal{C})$ denote the generic dimension of a codistribution $\mathcal{C}$ generated by the differentials of a finite set of analytic functions. The sequence $\sigma_k = \mathcal{D}(\mathcal{Y}_k) - \mathcal{D}(\mathcal{Y}_{k-1})$ is nondecreasing, the sequence $\rho_k = \mathcal{D}(Y_k) - \mathcal{D}(Y_{k-1})$ is nonincreasing, and both sequences converge to the same integer ρ, called the* output rank, *for some $k^* \leq n = \dim x$. In particular, we have $\rho_k \geq \rho \geq \sigma_n$ for $k = 0, \ldots, n$ and the the sequence $\nu_k = \mathcal{D}(Y_k \cap$ span $\{dx\})$ converges for $k = k^* - 1$.*

Proof. The reader may refer to [24] for a complete proof which an adaptation of known results to the geometric setting of [14]. $\square$

3.2 Structure codistributions

Consider the system (12a)–(12b), defined in the sense of [14]. Define the codistributions Ω_k calculated from $\mathcal{Y}_k =$ span $\left\{dx, dy, \ldots, dy^{(k)}\right\}$ in the following way :

Definition 3.1

$$\begin{aligned} \Omega_0 &= \text{span}\{dx\} \\ \Omega_k &= \text{span}\left\{\omega \in \Omega_{k-1} \mid \omega^{(k)} \in \mathcal{Y}_k\right\}, k \in \lfloor n \rceil \end{aligned} \tag{13}$$

The involutive closure $\Gamma_k = \bar{\Omega}_k$ for $k \in \lfloor n \rceil$ are system invariants called structure codistributions *of* (1).

This codistributions may be computed based on the same ideas presented in [1].

Proposition 3.1 *Consider system* (12a) *with output y. Assume that the structure at infinity of this system is $\{\sigma_1, \ldots, \sigma_n\}$. Denote by $y = (\bar{y}_k, \widehat{y}_k)$ the partition of the output corresponding to the application k step of the DEA (see the notation of § 3.1), where $\bar{y}_k$ are, up to a convenient reordering, the first σ_k components of y. Then* span $\{d\widehat{y}_k\} \subset \Gamma_k$.

Proof. By Theo. 3.1 it follows that $\{dx, d\bar{y}_1^{(1)}, \ldots, d\bar{y}_k^{(k)}\}$ is a basis of $\mathcal{Y}_k$ and we may take $\widehat{y}_k \subset \widehat{y}_{k-1}$. In particular, $d\widehat{y}_k^{(j)} \in \mathcal{Y}_k$ for $j = 0, \ldots, k$. The desired result follows from Definition 3.1. $\square$

The following proposition characterizes 0-flatness.

Proposition 3.2 *Consider an analytic system S defined by* (1) *in the sense of [14] and assume that the system is well formed. i.e.,* span $\{du\} \subset$ span $\{dx, d\dot{x}\}$*. Consider the output $y_i = h_i(x)$, $i \in \lfloor m \rceil$. Let $\xi \in S$ be a regular point of Y_k and $\mathcal{Y}_k$ for $k = 0, 1, \ldots, n$. Then S is (locally) 0-flat around ξ with (local) flat output y if and only if there exist $k^* \leq n$ such that one of the following equivalent conditions are satisfied :*

(i) *The algebraic structure at infinity obeys the following condition*

$$n + \sum_{i=1}^{k^*-1} \sigma_i = mk^* \tag{14a}$$

$$\sigma_{k^*} = m \tag{14b}$$

(ii) span $\{dx\} \subset Y_{k^*-1}$.

4 The nonsingular case

In order to develop sufficient conditions of k-flatness we will define a *symbolic* version of the dynamic extension algorithm (SDEA). Choosing a nondecreasing integer solution $(\sigma_1, \ldots, \sigma_n)$ of equation (14), this algorithm constructs a *restricted* jet-space that parametrizes the flat-output candidates $y_1 = h_1(x), \ldots, y_m = h_m(x)$. The corresponding exterior differential system $(\mathcal{J}, \Omega)$ (generated by the restricted contact forms, see Definition 2.4) is constructed in a way that these *restrictions* correspond to the properties stated in Proposition 3.1. In each step of the algorithm, the restricted jet space is extended by the procedure of § 2.7. So by Prop. 2.4, the exterior differential system constructed in step k is involutive. If some further dimensional properties are fulfilled, the existence of an integral manifold of $(\mathcal{J}, \Omega)$ will imply, by Proposition 3.2, the existence of a 0-flat output of the system.

Remark 4.1 *It is important to point out the meaning of the word* "nonsingular" *employed here. This means that we will consider that we are working in a neighborhood points where our (symbolic) objects are nonsingular, i.e., we consider only generic points*[7].

4.1 SDEA

Consider the analytic system (1) defined on an open set X with global coordinates $x = (x^1, \ldots, x^n)$. Denote $(f_0, g_0) = (f, g)$. Let $s_0 = n$, $\Gamma_0 =$

[6] The approaches of [7, 5] are algebraic. Some other properties of the DEA considered in the geometric approach of [19] can be found in [22].

[7] This remark refers to step (S3) of SDEA, described in § 4.3.

span $\{\omega_0^i : i = 1, \ldots, n\}$, where $\omega_0^i = dx_i$. Define $\mathcal{I}_0 = \{0\}$. The SDEA can be summarized as follows.

(S0) Let $\sigma_0 = 0$. For every nondecreasing integer solution $(\sigma_1, \ldots, \sigma_{k^*})$ of equation (14), consider the manifold $Z = X \times I\!R^{\sigma_1} \times \ldots \times I\!R^{\sigma_{k^*}}$, with local coordinates $(x, \bar{v}_1, \ldots, \bar{v}_{k^*})$. Let $M_0 = Z$.
Then, for $k = 1$ to k^*, execute:

(S1) (a) Extend the restricted jet-space as described in § 2.7 by adding $\sigma_k - \sigma_{k-1}$ more functions to the the restricted jet-space M_{k-1} constructed in the previous step. Note that M_{k-1} is immersed in $\mathcal{J}^{r_1}(Z, H_1) \times \ldots \times \mathcal{J}^{r_k}(Z, H_k)$, where $H_j = I\!R^{\sigma_j}, j \in \lfloor k \rceil$. Define a restricted jet space $T_k = M_{k-1} \times L_k$, where $L_k = I\!R^{\mu_k}$, $\mu_k = s_{k-1}(\sigma_k - \sigma_{k-1})$ and L_k has canonical coordinates $(h^i, \alpha_j^i : i \in \{\sigma_{k-1}+1, \ldots, \sigma_k\}, j \in \{1, \ldots, s_{k-1}\}$.

(S1) (b) Let $\hat{\mathcal{J}}_k$ be a Pfaffian system defined on T_k generated by $\mathcal{J}_{k-1}$ and the forms :

$$\theta^i = dh^i - \textstyle\sum_{j \in s_{k-1}} \alpha_j^i \omega_{k-1}^j,\ i \in \{\sigma_{k-1}+1, \ldots, \sigma_k\}$$

where the symbolic distribution $\Gamma_{k-1} = \text{span}\{\omega_{k-1}^1, \ldots, \omega_{k-1}^{s_{k-1}}\} \subset \text{span}\{dx\}$ was computed in the previous step.

(S2) Compute (symbolically) the kth step of the Dynamic Extension Algorithm. This will produce the (symbolical) system (f_k, g_k).

(S3) Compute the (symbolical) structure codistribution

$$\Gamma_k \quad = \quad \text{span}\left\{\omega_k^1, \ldots, \omega_k^{s_k}\right\} \subset \text{span}\left\{dx\right\}.$$

The steps (S2) and (S3) will be described in detail in the sequel[8].

4.2 Step (S2)

This step corresponds to the symbolical computation of the kth step of the dynamical extension algorithm. In the $(k-1)$th iteration we have computed a symbolical system (f_{k-1}, g_{k-1}), with state $x_{k-1} = (x^T, \bar{v}_1^T, \ldots, \bar{v}_{k-1}^T)$, input u_{k-1} and output y *i.e.*, f_{k-1} and g_{k-1} are symbolical fields defined on M_{k-1} (see § 2.4). Let $h_k = (h^1, \ldots, h^{\sigma_k})^T$. Compute

$$\begin{array}{rcl} \bar{y}_k^{(k)} & = & \mathcal{L}_{f_{k-1}}^k h_k + (\mathcal{L}_{g_{k-1}} \mathcal{L}_{f_{k-1}}^{k-1} h_k) u_{k-1} \\ & = & \bar{a}_k + \bar{b}_k u_{k-1} \end{array} \tag{15}$$

Note that $\bar{a}_k$ and $\bar{b}_k$ are matrices of functions defined on $T_k^{(k)}$. Assume that the generical rank of b_k on $T_k^{(k)}$ is σ_k. If this is not true, no solution is possible with the structure at infinity chosen. Otherwise, up to a reordering of the input components, assume that $\bar{b}_k = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \end{pmatrix}$ where $\bar{b}_{11}$ is generically nonsingular (with rank σ_k). Define

$$\beta_k = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ 0 & I \end{pmatrix}^{-1} \quad ; \ \alpha_k = \beta_k \begin{pmatrix} -\bar{a}_k(x) \\ 0 \end{pmatrix}$$

and let

$$u_{k-1} = \alpha_k + \beta_k \, v_k$$

be an analytic regular static state feedback, where $v_k = \begin{pmatrix} \bar{v}_k^T & \widehat{v}_k^T \end{pmatrix}^T$ Note that α and β are respectively matrices $m \times 1$ and $m \times m$ of functions defined on $T_k^{(k)}$. Add a dynamic extension :

$$\begin{array}{l} \bar{u}_k = \dot{\bar{v}}_k \\ \widehat{u}_k = \widehat{v}_k \end{array}$$

and let $u_k = \left(\bar{u}_k^T, \widehat{u}_k^T\right)^T$.

4.3 Step (S3)

This step corresponds to the computation of the symbolical structure codistribution Γ_k (see [1] for similar computations).

In iteration $k-1$ we have computed $\Gamma_{k-1} = \text{span}\left\{\omega_{k-1}^1, \ldots, \omega_{k-1}^{s_{k-1}}\right\}$ and their derivatives $\frac{d}{dt}^{k-1} \omega_{k-1}^j \in \text{span}\left\{dx, d\bar{v}_1, \ldots, d\bar{v}_{k-1}\right\}$. In this step we have to find functions γ_j such that

$$\bar{\omega} = \sum_j \gamma_j \frac{d}{dt}^k \omega_{k-1}^j \in Z_k = \text{span}\left\{dx, d\bar{v}_1, \ldots, d\bar{v}_k\right\}. \tag{16}$$

where $\frac{d}{dt}^k \omega_{k-1}^j = \mathcal{L}_{f_k + g_k u_k}\left(\frac{d}{dt}^{k-1} \omega_{k-1}^j\right)$. If γ_j are so, then Ω_k is spanned by all symbolic forms $\bar{\omega}$ that obey condition (16).

Let $\theta^j = \frac{d}{dt}^{k-1} \omega_{k-1}^j$ (computed in the iteration $k-1$). Compute $\dot{\theta}^j = \mathcal{L}_{f_k + g_k u_k} \theta^j$. ¿From (16) one may find a basis $\{\mu_k^1, \ldots, \mu_k^{r_k}\}$ of Ω_k by solving the linear equations.

$$\sum_j \gamma^j \langle\!\langle \theta^j, g_{k_l} \rangle\!\rangle = 0, \ l \in \lfloor m \rceil$$

We stress the standard assumption that we are working around (generic) nonsingular points, *i.e.*, points where the rank of the matrices of analytical functions, codistributions, etc. are maximal (see Rem 4.1).

Using the symbolic derived flag (see § 2.4), we may obtain the greater involutive symbolic codistribution $\Gamma_k = \text{span}\left\{\omega_k^1, \ldots, \omega_k^{s_k}\right\}$ contained in Ω_k. Then compute $\frac{d}{dt}^k \omega_k^j = \mathcal{L}_{f_k + g_k u_k}\left(\frac{d}{dt}^{k-1} \omega_k^j\right)$(note that one may use the expressions of $\frac{d}{dt}^r \omega_r^j$ for $r = 1, \ldots, k-1$ computed in the previous iterations).

Denote by $M_k = T_k^{(p_k)}$ and $\mathcal{J}_k = \hat{\mathcal{J}}_k^{(p_k)}$, where p_k is big enough for performing all these symbolic calculations.

[8] The step (S2) includes a finite number of choices represented by a reordering of the inputs.

4.4 A sufficient condition for k-flatness

Theorem 4.1 *Let $\Sigma = (\sigma_1, \ldots, \sigma_n)$ be a candidate for the structure at infinity obeying condition* (14)*. Assume that the Symbolic Dynamic Extension Algorithm for system* (12a)*, associated to Σ, constructs in the kth-iteration the symbolic structure codistributions Γ_k and the symbolic system (f_k, g_k) such that rank $\bar{b}_k$ is generically equal to σ_k. Then the system is 0-flat*

Proof. By construction, in the k-th iteration of the DEA we have a manifold M_{k-1} and a differential system $\mathcal{J}_{k-1}$ and we contruct a manifold T_k, and a differential system $\hat{\mathcal{J}}_k$ as described in § 2.7. Note also that $(\mathcal{J}_k, \Omega)$ is a p_k-prolongation of $(\hat{\mathcal{J}}_k, \Omega)$ for a convenient p_k. By proposition 2.4 it follows that this exterior differential system is involutive. In particular it admits local integral manifolds passing through every point. These integral manifolds induce functions $(h_1(x), \ldots, h_m(x))$ defined on an open neighborhood of X. By Prop. 3.1, Prop. 3.2, Prop. 2.1 and from the fact the SDEA is a symbolic version of the DEA it follows that $(h_1(x), \ldots, h_m(x))$ is a local flat output of system (1). □

Definition 4.1 *A 0-flat system is called nonsingular if the conditions of Theo. 4.1 holds.*

Proposition 4.1 *A system* (12a) *that is static-feedback linearizable is nonsingular in the sense of Def. 4.1.*

5 Singular Case

In this section we give necessary and sufficient conditions for 0-flatness of a system (12a) in the general case. We construct a restricted jet-space N and an exterior differential system $(\mathcal{I}, \Omega)$, generated by restricted contact-forms, such that (local) 0-flatness of system (12a) is equivalent to the existence of (local) integral manifolds of $(\mathcal{I}, \Omega)$.

Let $X = I\!R^n$ be the state space of system (12a). Let $U^{(k)} = I\!R^m, k \in \{0, \ldots, n\}$. Consider the manifold $Z = X \times U^{(0)} \times \ldots \times U^{(n)}$ with global coordinates $(x_i, u_j^{(k)} : i \in \lfloor n \rceil,\ j \in \lfloor m \rceil,\ k \in \{0, \ldots, n\})$. Consider the field $\hat{f} = \sum_{i=1}^n (f_i(x) + \sum_{j=1}^m g_i(x) u_j^{(0)}) \frac{\partial}{\partial x_i} + \sum_{j=1}^m \sum_{k=0}^{n-1} u_j^{(k+1)} \frac{\partial}{\partial u_j(k)}$. Let $y = h(x)$ be an output for system (12a). Then it is easy to show that $y^{(k)} = L_{\hat{f}}^k h(x)$, $k \in \{0, \ldots, n\}$ (see [22]).

Now let $Y = I\!R^m$ with global coordinates $(y^1, \ldots, y^m)$. Consider the jet-space $\mathcal{J}^{n+1}(X, Y)$ and let I^{n+1} be the codistribution generated by the corresponding contact-forms (the contact forms are given by (7) replacing Z by X, z^j by x^j, t by n, r by n and s by m).

Now let $M = U^{(0)} \times \ldots \times U^{(n)} \times \mathcal{J}^{n+1}(X, Y)$ with global coordinates $(x_i, u_j^{(k)}, y^\sigma, y_K^\sigma : i \in \lfloor n \rceil,\ j \in \lfloor m \rceil,\ k \in \{0, \ldots, n\},\ \sigma \in \lfloor m \rceil,\ K \in \Sigma(n), \|K\| \le n+1)$. Denote by I the pull-back of I^{n+1} from $\mathcal{J}^{n+1}(X, Y)$ to M. Let $\tilde{\mathcal{I}} = \{I, dI\}$ and let

$$\tilde{\Omega} = dx^1 \wedge \ldots \wedge dx^n \wedge du_1^{(0)} \wedge \ldots \wedge du_m^{(0)} \wedge \ldots \wedge du_1^{(n)} \wedge \ldots \wedge du_m^{(n)}.$$

Then $(\tilde{\mathcal{I}}, \tilde{\Omega})$ is an exterior differential system with independence condition defined on M.

Now let $\{e_1, \ldots, e_n\}$ be defined by (8) replacing Z by X, z^j by x^j, t by n, r by n and s by m. Define the field $\tilde{f}$ on M by $\tilde{f} = \sum_{i=1}^n (f_i(x) + \sum_{j=1}^m g_i(x) u_j^{(0)}) e_i + \sum_{j=1}^m \sum_{k=0}^{n-1} u_j^{(k+1)} \frac{\partial}{\partial u_j(k)}$. Let $k^* \le n$ be an integer. Consider the following restriction :

$$\begin{aligned} &\text{span}\left\{dx^i : i \in \lfloor n \rceil\right\} \bmod I \subset \\ &\text{span}\left\{dL_{\tilde{f}}^{k^*} y^\sigma : \sigma \in \lfloor m \rceil\right\} \bmod I \end{aligned} \tag{17}$$

It can be shown that (17) is equivalent to $\{\alpha_i = 0, i \in \lfloor l \rceil\}$ where α_i are analytic functions defined on M. So assume that this restriction defines an immersion $\jmath : N \to M$ (in the general case the analytical manifold can be decomposed in a *strata*). Then let $\mathcal{I} = \jmath^* \tilde{\mathcal{I}}$ and $\Omega = \jmath * \tilde{\Omega}$. Let θ be a 1-form on M and let $\pi\theta$ be the unique 1-form in $T^*Z \subset T^*M$ such that $\pi\theta - \theta \bmod I \equiv 0$. We have the following result :

Theorem 5.1 *The system* (12a) *is 0 flat around a point of $\zeta \in Z$ if and only if :*

(i) $(\mathcal{I}, \Omega)$ *admits a local integral manifold that passes through $\xi \in N$ such that $i(\zeta) = \xi$*

(ii) *The codistribution*

$$\text{span}\left\{\pi dL_{\tilde{f}}^r y^\sigma : r = 0, 1, \ldots, k^*, \sigma \in \lfloor m \rceil\right\}$$

is nonsingular in $\jmath \circ i(\zeta)$.

For the proof of a version of Theo. 5.1 and the details about the constructions above see [23].

Remark 5.1 *Note that necessary and sufficient conditions of k-flatnes can be obtained by putting k integrators in series with the inputs of the system (see [23]).*

6 Examples

We begin with a very simple academic example :

Example 1 *Consider the system with input $u = (u_1, u_2)$ evolving on $X = I\!R^5$, given by*

$$\begin{aligned} \dot{x}^1 &= x^4 x^3 \\ \dot{x}^2 &= x^4 \\ \dot{x}^3 &= x^5 \\ \dot{x}^4 &= u_1 \\ \dot{x}^5 &= u_2 \end{aligned}$$

It is easy to verify that this system is not linearizable by static-state feedback because the conditions of [20, 18] are not fulfilled.

A possible solution of condition (14) *is $k^* = 3$, $\sigma_1 = 0$, $\sigma_2 = 1$ and $\sigma_3 = 2$. We will execute the steps of SDEA for this choice of structure at infinity.*

S0. *Let $\sigma_0 = 0$, $Z = X \times I\!R^{\sigma_2} \times I\!R^{\sigma_3}$ with global coordinates $(x, \bar{v}_2, \bar{v}_3)$. Let $\Gamma_0 = \text{span}\left\{dx^1, dx^2, dx^3, dx^4, dx^5\right\}$ and*

Set $f_0 = f$, $g_0 = g$, $M_0 = Z$ and let $\mathcal{I}_0 = \{0\}$.

Iteration 1.
(S1) Since $\sigma_1 - \sigma_0 = 0$, we have $T_1 = M_0$ and $\hat{\mathcal{J}}_1 = \{0\}$.
(S2) Since $\sigma_1 = 0$, we have $f_1 = f_0$, $g_1 = g_0$, $x_1 = x$, $u_1 = u$.
(S3) Note that $\Omega_1 = \{\omega \in \Gamma_0 \mid L_{f_1+g_1u_1}\omega \in \text{span}\{dx_1\}\}$. Simple computations give $\Omega_1 = \text{span}\left\{\omega_1^1, \omega_1^2, \omega_1^3\right\}$, where $\omega_i = dx^i : i \in \lfloor 3\rceil$ Compute $\dot{\omega}_1^j = \mathcal{L}_{f_1+g_1u_1}\omega_i^j$ for $i \in \lfloor 3\rceil$, obtaining $\dot{\omega}_1^1 = x^3du^1 + u^1dx^3$, $\dot{\omega}_1^2 = dx^4$, $\dot{\omega}_1^3 = dx^5$. Note that $\hat{\mathcal{J}}_1 = \{0\}$ implies that symbolical involutivity reduces to the involutivity of Ω_1. In particular, $\Gamma_1 = \Omega_1$, $M_1 = M_0$ and $\mathcal{J}_1 = \{0\}$.

Iteration 2.
(S1) We have $\sigma_2 - \sigma_1 = 1$. Hence $T_2 = X \times L_2$ where $L_2 = I\!R \times I\!R^3$ with coordinates $(h^1, \alpha_j^1 : j \in \lfloor 2\rceil)$. Let $\mathcal{I}_2 = \{dh^1 - \sum_{j=1}^3 \alpha_j^1 dx^j\}$.
(S2) Let $h_2 = (h^1 \dots h^{\sigma_2}) = h^1$ Note that

$$\begin{array}{rcl} \bar{y}_2^{(2)} & = & \mathcal{L}_{f_1}^2 h_2 + (\mathcal{L}_{g_1}\mathcal{L}_{f_1}h_2)u_1 \\ & = & \bar{a}_2 + \bar{b}_2 u_1 \end{array}$$

where

$$\bar{b}_2 = \left(\begin{array}{cc} \alpha_1^1 x_3 + \alpha_2^1 & \alpha_3^1 \end{array} \right).$$

Hence we may take

$$\beta_2 = \left(\begin{array}{cc} \frac{1}{\alpha_1^1 x_3 + \alpha_2^1} & -\frac{\alpha_3^1}{\alpha_1^1 x_3 + \alpha_2^1} \\ 0 & 1 \end{array} \right) \quad \alpha_2 = \beta_2 \left(\begin{array}{c} -\bar{a}_2 \\ 0 \end{array} \right)$$

Denoting by $\beta_2 = \left(\bar{\beta}_2\ \hat{\beta}_2\right)$ where $\bar{\beta}_2$ is the first column and $\hat{\beta}_2$ is the second column of β_2, then (f_2, g_2) is a system with state $x_1 = (x, \bar{v}_2)$ and input $u_2 = (\bar{u}_2, \hat{u}_2)$ given by :

$$\begin{array}{rcl} \dot{x} & = & f_1 + g_1\alpha_2 + g_1\bar{\beta}_2\bar{v}_2 + g_2\hat{\beta}_2\hat{u}_2 \\ \dot{\bar{v}}_2 & = & \bar{u}_2 \end{array}$$

(S3) To compute Γ_2, let

$$\Omega_2 = \text{span}\left\{\omega \in \Gamma_1 \,|\mathcal{L}_{f_2+g_2u_2}\dot{\omega} \in \text{span}\{dx, d\bar{v}_2\}\right\}$$

Let $\omega = \sum_{i=1}^3 \gamma_i\omega_1^i$ Then $\dot{\omega} = \sum_{i=1}^3\{\gamma_i\dot{\omega}_1^i + \dot{\gamma}_i\omega_1^i\}$. It is easy to show that $\mathcal{L}_{f_2+g_2u_2}\dot{\omega} \in \text{span}\{dx, d\bar{v}_2\}$ if and only if $\langle\sum_{i=1}^3 d\dot{\omega}_1^i, g\hat{\beta}\rangle = 0$. Hence

$$\gamma_1 x^3 \frac{-\alpha_3^1}{\alpha_1^1 x^3 + \alpha_2^1} + \gamma_2 \frac{-\alpha_3^1}{\alpha_1^1 x^3 + \alpha_2^1} + \gamma_3 = 0.$$

It follows that $\Omega_2 = \text{span}\left\{\omega_1^2, \omega_2^2\right\}$, where

$$\omega_2^1 = dx^1 - x^3dx_2, \quad \omega_2^2 = (\alpha_1^1x^3 + \alpha_2^1)dx^2 + \alpha_3^1dx^3$$

Further computations show that Ω_2 is symbolically involutive and so $\Gamma_2 = \Omega_2$ and that :

$$\begin{array}{rcl} \frac{d^2}{dt}\omega_2^1 & = & \theta_1 - x^5dx^4 - x^4dx_5 \\ \frac{d^2}{dt}\omega_2^2 & = & \theta_2 + d\bar{v}_2 \end{array}$$

where $\theta_1 \in \text{span}\left\{dx^3, dx^2\right\}$ and $\theta_2 \in \text{span}\left\{dx^1, dx^2, dx^3, dx^4, dx^5\right\}$.

Iteration 3. *(S1) We have $\sigma_3 - \sigma_2 = 1$. Hence $T_3 = M_3 \times L_3$ where $L_2 = I\!R \times I\!R^3$ with coordinates $(h^2, \alpha_j^2 : j \in \lfloor 2\rceil)$. Let $\mathcal{I}_2 = \{dh^1 - \sum_{j=1}^3 \alpha_j^1\omega_j^2\}$.*
(S2) Let $h_3 = (h^1 \dots h^{\sigma_3})$. After tedious computations it follows that

$$\begin{array}{rcl} \bar{y}_3^{(3)} & = & \mathcal{L}_{f_1}^3 h_3 + (\mathcal{L}_{g_1}\mathcal{L}_{f_1}^2 h_3)u_1 \\ & = & \bar{a}_3 + \bar{b}_3 u_1 \end{array}$$

where

$$\bar{b}_3 = \left(\begin{array}{cc} 1 & 0 \\ \psi(x_2) & \alpha_1^2(x^5\alpha_3^1 + x^4) + \alpha_2^2 \end{array} \right).$$

It follows that $\bar{b}_3$ is generically nonsingular. By Theorem 4.1 it follows that the system is 0-flat. The flat output is (y^1, y^2) where $dy^1 \in \text{span}\left\{dx^1, dx^2, dx^3\right\}$ and $dy^2 \in \Gamma_2$ is such that $\bar{b}_3$ is nonsingular.

The next example shows that there exist flat systems that are singular with respect to Def. 4.1 :

Example 2 *Consider the system with input $u = (u_1, u_2)$ evolving on $X = I\!R^5$, given by*

$$\begin{array}{rcl} \dot{x}^1 & = & x^4 \\ \dot{x}^2 & = & x^5 \\ \dot{x}^3 & = & x^4x^5 \\ \dot{x}^4 & = & u_1 \\ \dot{x}^5 & = & u_2 \end{array}$$

Note that this example corresponds to the one of [15] with $\alpha_1 = \alpha_2 = 1$ where one has extended the state by integrators in series to the inputs. There exists two solutions of equation (14) *given by : $\{k^* = 4,\ \sigma_1 = \sigma_2 = \sigma_3 = 1,\ \sigma_4 = 2\}$ and $\{k^* = 3,\ \sigma_1 = 0, \sigma_2 = 1, \sigma_3 = 2\}$. After some tedious computations one can show that the condition of Theo. 4.1 is not satisfied by either of these structures at infinity because* rank $\bar{b}_{k^*} < 2$ *in both cases. It is shown in [15] that this system is 0-flat (the original system is 1-flat[9]). Hence this example shows that there exist 0-flat systems that are singular with respect to Def. 4.1.*

7 Conclusions

In this paper it is shown that there may exist two kind of 0-flat systems — the *nonsingular* and the *singular* cases. The *nonsingular* case contains properly the class of state-feedback linearizable systems. Hence 0-flat systems may be classified as *state-feedback linearizable*, *nonsingular* and *singular* systems, in an ascending order of complexity.

Necessary and sufficient conditions of k-flatness for the nonsingular case are developed. For the singular case, necessary and sufficient conditions of k-flatness relies on a convenient application of Cartan-Kähler and Cartan-Kuranishi theorems.

[9] In [23] it is shown that the example of [15] is not 0-flat.

Acknowledgments

This work was partially supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPq) and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), under grants 300492/95-2 and 97/04668-1.

References

[1] E. Aranda-Bricaire, C. H. Moog, and J. B. Pomet. A linear algebraic framework for dynamic feedback linearization. *IEEE Trans. Automat. Control*, 40:127–132, 1995.

[2] R. Briant, S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths. *Exterior Differential Systems*. Springer Verlag, 1991.

[3] B. Charlet, J. Lévine, and R. Marino. On dynamic feedback linearization. *Systems Control Lett.*, 13:143–151, 1989.

[4] B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control Optim.*, 29:38–57, 1991.

[5] E. Delaleau and P. S. Pereira da Silva. Filtrations in feedback synthesis: Part I — systems and feedbacks. *Forum Math.*, 10(2):147–174, 1998.

[6] J. Descusse and C. H. Moog. Decoupling with dynamic compensation for strong invertible affine nonlinear systems. *Internat. J. Control*, 42:1387–1398, 1985.

[7] M. D. Di Benedetto, J. W. Grizzle, and C. H. Moog. Rank invariants of nonlinear systems. *SIAM J. Control Optim.*, 27:658–672, 1989.

[8] M. Fliess. Automatique et corps différentiels. *Forum Math.*, 1:227–238, 1989.

[9] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C. R. Acad. Sci. Paris Sér. I Math.*, 315:619–624, 1992.

[10] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund. *C. R. Acad. Sci. Paris Sér. I Math.*, 317:981–986, 1993.

[11] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of non-linear systems: introductory theory and examples. *Internat. J. Control*, 61:1327–1361, 1995.

[12] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Deux applications de la géométrie locale des diffiétés. *Ann. Inst. H. Poincaré Phys. Théor.*, 66:275–292, 1997.

[13] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Nonlinear control and diffieties, with an application to physics. In J. Krasil'shchik M. Henneaux and A. Vinogradov, editors, *Secondary Calculus and Cohomological Physics*, volume 219 of *Contemporary Math.*, pages 81–92, 1998.

[14] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Control*, 44(5):922–937, 1999.

[15] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Some open question related to flat nonlinear systems. In V.D. Blondel, E. Sontag, M. Vidyasagar, and J.C. Willems, editors, *Open Problems in Mathematical Systems and Control Theory*, pages 99–103, London, 1999. Springer Verlag.

[16] R. B. Gardner and W. F. Shadwick. The GS algorithm for exact linearization to Brunovsky normal form. *IEEE Trans. Automat. Control*, 37:224–230, 1992.

[17] M. Guay, P. J. McLellan, and D. W. Bacon. A condition for dynamic feedback linearization of control-affine nonlinear systems. *Internat. J. Control*, 68(1):87–106, 1997.

[18] L. R. Hunt, R. Su, and G. Meyer. Design for multi-input nonlinear systems. In R. Brocket, R. Millmann, and H. J. Sussmann, editors, *Differential Geometric Methods in Nonlinear Control Theory*, pages 268–298, 1983.

[19] A. Isidori. *Nonlinear Control Systems*. Springer–Verlag, 3nd edition, 1995.

[20] B. Jakubczyk and W. Respondek. On linearization of control systems. *Bull. Acad. Pol. Sc., Ser. Sci. Math.*, 28:517–522, 1980.

[21] H. Nijmeijer and W. Respondek. Dynamic input-output decoupling of nonlinear control systems. *IEEE Trans. Automat. Control*, 33:1065–1070, 1988.

[22] P. S. Pereira da Silva. On the nonlinear dynamic disturbance decoupling problem. *J. Math. Systems Estim. Control*, 6:1–26, 1996.

[23] P. S. Pereira da Silva. Flatness of nonlinear control systems and exterior differential systems. In *Second Nonlinear Control Network (NCN) Workshop*, Paris, June, 5–8, 2000. Submitted.

[24] P. S. Pereira da Silva. *Geometric Properties of the Dynamic Extension Algorithm*. Internal Report BT/PTC/00xx, Escola Politécnica da Universidade de São Paulo http://www.lac.usp.br/∼paulo/down.html, 2000.

[25] J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In B. Jackubczyk, W. respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, pages 319–339, Warsaw, 1995. Banach Center Publications.

[26] J.-B. Pomet. On dynamic feedback linearization of four-dimensional affine control systems with two inputs. *ESAIM Control Optim. Calc. Var.*, 2:151–230 (electronic), 1997.

[27] P. Rouchon. Necessary condition and genericity of dynamic feedback linearization. *J. Math. Systems Estim. Control*, 5(3):345–358, 1995.

[28] J. Rudolph. Well-formed dynamics under quasi-static state feedback. In B. Jackubczyk, W. Respondek, and T. Rzezuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, pages 349–360, Warsaw, 1995. Banach Center Publications.

[29] W. F. Shadwick. Absolute equivalence and dynamic feedback linearization. *Systems Control Lett.*, 15:35–39, 1990.

[30] William F. Shadwick and Willem M. Sluis. Dynamic feedback for classical geometries. In *Differential geometry and mathematical physics (Vancouver, BC, 1993)*, volume 170 of *Contemp. Math.*, pages 207–213. Amer. Math. Soc., Providence, RI, 1994.

[31] S. N. Singh. A modified algorithm for invertibility in nonlinear systems. *IEEE Trans. Automat. Control*, AC–26:595–598, 1981.

[32] W. M. Sluis. A necessary condition for dynamic feedback linearization. *Systems Control Lett.*, 21:277–283, 1993.

[33] W. M. Sluis and D. M. Tilbury. A bound on the number of integrators needed to linearize a control system. *Systems Control Lett.*, 29(1):43–50, 1996.

[34] D. Tilbury, R. M. Murray, and S. R. Sastry. Trajectory generation for the n-trailer problem using Goursat normal form. *IEEE Trans. Automat. Control*, 40:802–819, 1995.

[35] M. van Nieuwstadt, M. Rathinam, and R. M. Murray. Differential flatness and absolute equivalence of nonlinear control systems. *SIAM J. Control Optim.*, 36(4):1225–1239 (electronic), 1998.

[36] F. W. Warner. *Foundations of differentiable manifolds and Lie Groups*. Scott, Foresman and Company, Glenview, Illinois, 1971.