Constrained Robots are Flat

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Abstract

In this paper we show that constrained robots are flat systems. Using this structural property of this class of control systems, a technique of simultaneous contact forces and position tracking along the constrained surface will be developed.

1 Introduction

The concept of flatness, introduced by Fliess *al.* [6], is strongly related to the problem of dynamic feedback linearizability (see for instance [17], [1]) and was shown to be useful to solve many important control problems (see [9] for several applications and [8] for stabilization around a reference trajectory).

The control of constrained robots has been an active area of research. The main problem concerning this subject is the simultaneous tracking of contact forces and the position along the constraint surface. (see for instance [2], [3], [13], [15] and the references therein).

The dynamic model of constrained robots consists of a set of implicit nonlinear equations and hence one can not apply to this class of control systems, the standard state space techniques. For instance, in order to tackle this problem with the so called geometric differential approach (see Isidori [11] or Nijmeijer and Van der Schaft [16]), one must adapt or generalize these results to the context of implicit equations (see for instance [12]).

We shall show that the notion of flatness is also a good tool to consider the control of constrained robots. We will adopt the approach of Fliess et al. [7], [8], that can be directly applied to implicit and generalized systems. The resulting control technique is closely related to the one of

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the section 3 of [12]. Our main contribution is to point out that this technique is indeed a direct consequence of the fact that the constrained robot is flat.

The paper is organized as follows. In section 2 we present a very brief exposition about Diffieties and Differential Flatness [8], [9]. In the section 3 we present a particular class of constrained systems, that are obtained from a flat system by adding some constraints on its flat output and we show that these constrained systems are flat, indeed. In the section 4 we show that the problem of simultaneous tracking of position and contact forces of a constrained robot can be considered using the ideas of section 3. Finally, in section 5 we will apply this method to an academic example.

2 Diffieties and differential flatness

In this section we will consider the concept of differential flatness given in [8], that is based in the differential geometry of infinite jets [18], [19]. It is not our aim here to give a complete and precise presentation of this subject and so we will restrict ourselves to the basic ideas. The first paragraphs of this sections are based in [5].

Let I be a countable set, finite or infinite with cardinal l. Denote by $\mathbb{R}^l$ the set of mappings $x : I \mapsto \mathbb{R}$. For each $i \in I$ we can associate the coordinate function $x_i : \mathbb{R}^l \mapsto \mathbb{R}$ such that $x_i(x) = x(i)$. Consider the set $\mathbb{R}^l$ with the Fréchet topology [19], *i.e.*, the usual topology of $\mathbb{R}^l$ when l is finite and the projective limit of the usual topology when l is infinite. For any open subset $\mathcal{O}$, write $C^\infty(\mathcal{O})$ the set of functions that depend only on a finite number of variables among the x_i and are C^∞.

The concept of infinite dimensional Fréchet C^∞ manifolds may be defined in this context by a topological space M that is equipped with a C^∞ atlas consisting on $\mathbb{R}^l$ valued charts [19]. The notion of C^∞ functions, fields and forms on an open subset of M are defined similarly as in the finite dimensional case. If $\{x_i | i \in I\}$ are local coordinates, we recall that a vector field f on M has in general an infinite expression $f = \sum_{i\in I} \zeta^i \frac{\partial}{\partial x_i}$, whereas one-forms have always finite expressions like $\sum_{finite} \omega_i dx_i$.

The notion of (local) C^∞ maps (or morphisms) between infinite dimensional Fréchet C^∞ manifolds is now obvious, as well as the notion of (local) diffeomorphisms (or isomorphisms). A (local) submersion (resp. immersion) is a C^∞ map such that there exist coordinates where its local expression is a canonical projection (resp. injection)[1].

A diffiety is a C^∞ Fréchet manifold equipped with involutive distribution Δ of finite dimension δ, called the *Cartan* dimension. Any (local) section f of Δ is called a (local) *Cartan field.* A diffiety is called *ordinary* if $\delta = 1$, otherwise the diffiety is said to be *partial.* In this work we restrict our attention to the ordinary case.

A *Lie-Backlünd* map $\phi : \tilde{M} \mapsto M$ between two diffieties $(\tilde{M}, \tilde{\Delta})$ and (M, Δ) is a C^∞ map such that $\phi_* \tilde{\Delta}\big|_x \subset \Delta|_{\phi(x)}$, where $\phi_* : T_x\tilde{M} \mapsto T_{f(x)}M$ is the tangent mapping.

[1] The implicit function theorem do not hold true for infinite dimensional Fréchet manifolds. As a consequence, the usual (local) characterization of submersions and immersions via the classification of tangent map is no longer valid

Intrinsic definitions of systems, their state variables representation and differential flatness are given in [5], [7], [8]. It is not our aim here to give a complete exposition of these concepts, but only to present the main ideas. The reader may refer to [5], [8], [7] for a more elegant and intrinsic presentation.

A *system* (M, f) is an ordinary diffiety M equipped with a privileged Cartan field f called the *total derivation* and a privileged function $t : M \mapsto \mathbb{R}$ called *time* such that $f(t) = \langle dt, f \rangle = 1$. Given a system (M, f), and a function $\phi : M \mapsto \mathbb{R}$ we will denote the (Lie) derivatives $L_f^k(\phi) = \phi^{(k)}$, corresponding to the k-fold derivatives of the function with respect to time.

A *Lie-Bäcklund morphism* (resp. immersion, submersion) between two systems $(\tilde{M}, \tilde{f})$ and (M, f) is a Lie Bäcklund morphism (resp. immersion, submersion) $\psi : \tilde{M} \mapsto M$ such that the Cartan fields $\tilde{f}$ and f are ψ-related, *i.e.*, $\psi_* \tilde{f} = f \circ \psi$.

A local state variables representation for a system (M, f) is a set of coordinate functions for M of the form $\left\{t, x_i, u_j^{(k)} | i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}, k \in \mathbb{N}\right\}$ in $C^\infty(\mathcal{O})$, where $\mathcal{O}$ is an open region of M. In other words the Cartan field f is locally given by[2] :

$$f = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{k \in \mathbb{N}} \sum_{j=1}^{m} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}$$

The set $(x_1, \ldots, x_n)$ is the state and $(u_1, \ldots, u_m)$ is the input of the state variables representation.

A system (M, f) is said to be locally differentially flat[3] if there exists a set of functions $\{y_1, \ldots, y_m\}$ in $C^\infty(\mathcal{O})$, where $\mathcal{O}$ is an open region of M, called *flat outputs*, such that the set $\left\{t, y_j^{(k)} | j \in \{1, \ldots, m\}, k \in \mathbb{N}\right\}$ is a set of local coordinate functions for M. In other words, the Cartan field f is locally given by

$$f = \frac{\partial}{\partial t} + \sum_{k \in \mathbb{N}} \sum_{j=1}^{m} y_j^{(k+1)} \frac{\partial}{\partial y_j^{(k)}}$$

Example 2.0-1 Consider the differential equations[4]

$$\ddot{q} = R(q, \dot{q}) + S(q)\lambda + T(q)\tau \tag{2.1}$$

where $T(q)$ is a square nonsingular matrix of smooth functions for all $q \in \mathbb{R}^n$.

Define a system consisting in the diffiety of global coordinates $\{q, q^{(1)}, (\lambda^{(i)}, \tau^{(i)} : i \in \mathbb{N})\}$ and the Cartan field

$$\begin{array}{c} f = \frac{\partial}{\partial t} + q^{(1)} \frac{\partial}{\partial q} + [R + S\lambda + T\tau] \frac{\partial}{\partial q^{(1)}} + \\ + \sum_{k \in \mathbb{N}} [\lambda^{(k+1)} \frac{\partial}{\partial \lambda^{(k)}} + \tau^{(k+1)} \frac{\partial}{\partial \tau^{(k)}}] \end{array}$$

[2]Note that the functions f_i may depend on $x, u, \ldots, u^{(\alpha)}$, corresponding to a "generalized" state variables representation.

[3]For simplicity, we do not distinguish here the notions of topological (orbital) and differential flatness [7], [5].

[4]We will see in the section 4 (equation (4.6)) that differential equations of this form may be useful in Robotics.

corresponding to a state representation of the system, with input (λ, τ), and state $(q, q^{(1)})$. Note that this system is flat and (λ, q) is a possible choice of a flat output. In fact, the regular static-state feedback $v = R + S\lambda + T\tau$, $w = \tau$ corresponds to a new coordinate system $\{q, q^{(1)}, (v^{(i)}, w^{(i)} : i \in \mathbb{N})\}$ for which the Cartan field is given by

$$f = \frac{\partial}{\partial t} + q^{(1)}\frac{\partial}{\partial q} + v\frac{\partial}{\partial q^{(1)}} + \sum_{k \in \mathbb{N}}[w^{(k+1)}\frac{\partial}{\partial w^{(k)}} + v^{(k+1)}\frac{\partial}{\partial v^{(k)}}]$$

□

3 On a class of constrained systems

Let (M, f) be a flat system with flat output $y = (y_1, \ldots, y_m)$. Consider the submanifold $\tilde{M}$ obtained from M by adding r constraints $\phi_i(y_1, \ldots, y_m) = 0 \quad (i = 1, \ldots r)$ to the system M. Assume that the Jacobian matrix $J\phi = \frac{\partial \phi}{\partial y}$ of the map $\phi = (\phi_1, \ldots, \phi_r)^T$ is of full row rank r for all y in the open domain of ϕ. We shall show that the "restriction" $\tilde{f}$ of the Cartan field f to the submanifold $\tilde{M}$ is a Cartan field for $\tilde{M}$. Furthermore, the system $(\tilde{M}, \tilde{f})$ is a flat system with flat output $(\tilde{y}_{r+1}, \ldots, \tilde{y}_m)$, where $\tilde{y}_i$ denotes the "restriction" of the function y_i to $\tilde{M}$, and the canonical insertion of $\tilde{M}$ in M is a Lie-Bäcklund immersion. In particular, since the (implicit) system $(\tilde{M}, \tilde{f})$ is flat, this system is well posed as a control system. So there exists no singularities that could appear in the context of implicit systems and one can parametrize all the solutions by the choice of the flat outputs as smooth functions of time.

For what follows in this section, note that the set of smooth functions

$$\Big\{t, (y_i^{(j)} : i \in \{1, \ldots, r\}, j \in \mathbb{N}),$$
$$(y_k^{(j)} : k \in \{r+1, \ldots, m\}, j \in \mathbb{N})\Big\}$$

denoted by $\left\{t, Y^1, Y^2\right\}$ is a local coordinate system for M and in this coordinates we have

$$f = \frac{\partial}{\partial t} + \sum_{j \in \mathbb{N}} \left\{ \sum_{i=1}^{r} y_i^{(j+1)} \frac{\partial}{\partial y_i^{(j)}} + \sum_{k=r+1}^{m} y_k^{(j+1)} \frac{\partial}{\partial y_k^{(j)}} \right\}$$

We begin by considering a more particular situation :

Proposition 3.0-2 *Let $\tilde{M} \subset M$ be the submanifold of M defined by the set of points of M such that $y_1^{(j)} = \ldots = y_r^{(j)} = 0$ for all $j \in \mathbb{N}$. Consider the canonical insertion mapping $i : \tilde{M} \mapsto M$. Then the restriction $\tilde{f}$ of f to $\tilde{M}$ is a Cartan field for $\tilde{M}$, the system $(\tilde{M}, \tilde{f})$ is flat with flat output $(\tilde{y}_{r+1}, \ldots, \tilde{y}_m)$, where $\tilde{y}_i$ denotes the restriction of y_i to $\tilde{M}$. Furthermore, $i : \tilde{M} \mapsto M$ is a Lie-Bäcklund immersion.*

Proof. Note first that the set $\left\{t, \tilde{Y}^2\right\}$ is a local coordinate chart for $\tilde{M}$, where $\tilde{Y}^2$ denotes the restriction of the set of functions $Y^2 = \left\{y_k^{(j)} : k \in \{r+1, \ldots, m\}, j \in \mathbb{N}\right\}$

to $\tilde{M}$ (by simplicity, the restriction of the function t to $\tilde{M}$ is also denoted by t).

Note that the restrictions of the functions defined on M to $\tilde{M}$ are obtained by $\tilde{y}_k^{(j)} = y_k^{(j)} \circ i$. By definition it is clear that in these coordinates, $i(t, \tilde{Y}^2) = (t, 0, \tilde{Y}^2)$ and $i_* \frac{\partial}{\partial t} = \frac{\partial}{\partial t}$, $i_* \frac{\partial}{\partial \tilde{y}_i^{(j)}} = \frac{\partial}{\partial y_i^{(j)}} \circ i$ for $\tilde{y}_i^{(j)} \in \tilde{Y}^2$. In particular the linear mapping $i_*(q) : T_q\tilde{M} \mapsto T_{i(q)}M$ is injective. So the equation

$$i_*\tilde{f} = f \circ i$$

defines a unique field $\tilde{f}$ on $\tilde{N}$ given by

$$\tilde{f} = \frac{\partial}{\partial t} + \sum_{j \in I\!N} \sum_{k=r+1}^{m} \tilde{y}_k^{(j+1)} \frac{\partial}{\partial \tilde{y}_k^{(j)}}$$

and the proposition is proved. Note that we can call $\tilde{f}$ by the "restriction" of f to M by using the usual identification $T_q\tilde{M} \cong i_*(T_q\tilde{M}) \subset T_{i(q)}M$, *i.e.*, $T_q\tilde{M}$ can be canonically identified with a subspace of $T_{i(q)}M$. □

Now return to the situation where that the constraints added to M are of the form $\phi_i(y_1, \ldots, y_m), i \in \{1, \ldots, r\}$ where $\phi = (\phi_1, \ldots, \phi_r)$ is such that $\frac{\partial \phi}{\partial(y_1, \cdots, y_m)}$ has rank r.

Corollary 3.0-3 *Let $\tilde{M} \subset M$ be the submanifold of M defined by the set of points p of M such that $\phi_1^{(j)}(p) = \ldots = \phi_r^{(j)}(p) = 0$ for all $j \in I\!N$. Then the restriction $\tilde{f}$ of f to $\tilde{M}$ is a Cartan field for $\tilde{M}$, the system $(\tilde{M}, \tilde{f})$ is (locally) flat and the canonical insertion map $i : \tilde{M} \mapsto M$ is a Lie-Bäcklund immersion.*

Furthermore, if we choose a set of functions $\{\psi_1, \ldots, \psi_{m-r}\}$ in a way that the Jacobian matrix $J \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} J\phi \\ J\psi \end{pmatrix}$ is locally nonsingular, then the restrictions $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_{m-r}\}$ of the functions $\{\psi_1, \ldots, \psi_{m-r}\}$ to $\tilde{M}$ form a (local) flat output for the system $(\tilde{M}, \tilde{f})$.

Proof. By the inverse function theorem, it is clear that the map $(y_1, \ldots, y_r, y_{r+1}, \ldots, y_m) \mapsto (\phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_{m-r})$ is a local diffeomorphism. In particular, we see that $(\phi_1, \ldots, \phi_r, \psi_1, \ldots, \psi_{m-r})$ is also a (local) flat output for the system (M, f). Hence the proposition 3.0-2 can be applied, showing the desired result. □

The following result is a generalization of the canonical form (23) and (26) of [12], at least whwn the "zero dynamics" is not present :

Corollary 3.0-4 *Consider the analytic system*

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y^1 & = & h^1(x) \\ y^2 & = & h^2(x) \end{array} \tag{3.2}$$

with input $u = (u_1, \ldots, u_m)$, state $x = (x_1, \ldots, x_n)$ and output $y = (y^1, y^2)$, where $y^1 = (y_1, \ldots y_k)$ and $y^2 = (y_{k+1}, \ldots, y_p)$. Consider that we add to the system the constraint

$$y^2 = 0 \tag{3.3}$$

Let $\{\sigma_1, \dots, \sigma_n\}$ the algebraic structure at infinity [4] of the nonconstrained system (3.2) *with input u, state x and output y. Assume that the system is right invertible (i.e., the rank of this system is $\sigma_n = p$) and that $\sum_{i=0}^{n} \sigma_i = (n+1)p$. Then the constrained system* (3.2)-(3.3) *is flat, with flat output y^1.*

Proof. Using the extension algorithm of [4] one verifies easily that y is a flat output for the nonconstrained system 3.2. The result follows from the proposition 3.0-2. □

4 Control of Constrained Robots

In this section we study the control of constrained robots. We restrict ourselves to the case of rigid robots, although it is not difficult to adapt these techniques to more complex models, taking into account elastic joints and/or actuator dynamics.

It is well known that rigid robots can be modeled by equations of the form

$$M(q)\ddot{q} + H(q, \dot{q}) = \tau \tag{4.4}$$

where $q = (q_1, \dots, q_n)^T$ is the vector of generalized displacements, $M(q)$ is the inertia matrix, which is symmetric and definite positive, $H(q, \dot{q})$ represents the Coriolis, centripetal and gravity forces, and τ the vector of generalized forces applied to each joint.

Constrained robots are robots for which their displacements are restricted by some physical contact surface. This is equivalent to add r holonomic constraints $\phi_i(q) = 0 \quad (i = 1, \dots, r)$ to its original equations. The following model can be obtained, by taking into account the contact forces [14], [12] :

$$\begin{aligned} M(q)\ddot{q} + H(q, \dot{q}) &= u && (4.5a)\\ u &= (J\phi)^T(q)\lambda + \tau && (4.5b)\\ \phi_i(q) &= 0 \;\; (i = 1, \dots, r) && (4.5c) \end{aligned}$$

where $J\phi(q) = \frac{\partial \phi}{\partial q}$, $\lambda = (\lambda_1, \dots, \lambda_r)^T$ is a vector of multipliers that parametrizes the effect of contact forces. We will assume that $\frac{\partial \phi}{\partial q_1, \dots, q_r}$ has rank r for all q in the operation region of the robot.

Now consider the system

$$\begin{aligned} M(q)\ddot{q} + H(q, \dot{q}) &= u \\ u &= (J\phi)^T(q)\lambda + \tau \end{aligned} \tag{4.6}$$

obtained from the system (4.5a)-(4.5b) by disregarding the constraints (4.5c). For this (nonconstrained) system (4.6), with state $(q, \dot{q})$ and input (λ, τ) it is easy to verify (see example 2.0-1) this system is flat and (λ, q) is a flat output for this system. Now choose a set of functions $\psi = (\psi_1, \dots, \psi_{n-r})$ in a way that the Jacobian matrix $\begin{pmatrix} J\phi \\ J\psi \end{pmatrix}$ is nonsingular. It is now clear that (λ, ϕ, ψ) is also a flat output for the noncontrained system (4.6). In particular, the corollary 3.0-3 may be applied and one concludes that $\zeta = (\lambda, \psi_1, \ldots, \psi_{n-r})$ is a local flat output for the constrained robot (4.5a)-(4.5b)-(4.5c) (by simplicity we do not distinguish ψ from $\tilde{\psi}$ in the notation of this section). Hence, one can fix a reference trajectory $\zeta_{ref}(t)$ and construct a control law in order to track $\zeta_{ref}(t)$ asymptotically.

It is convenient to re-write the nonconstrained robot equations considering the new flat output (λ, ϕ, ψ). For this note that $\dot{\phi} = (J\phi)\dot{q}$ and so $\ddot{\phi} = (J\phi)\ddot{q} + F(q, \dot{q})$, where $F(q, \dot{q})$ is a vector with r components given by $F_j = \dot{q}^T \mathcal{H}_j \dot{q}$ and $\mathcal{H}_j = \frac{\partial^2 \phi_j(q)}{\partial q, \partial q}$ is the Hessian matrix of ϕ_j, $j \in \{1, \ldots r\}$. Analogously, $\dot{\psi} = (J\psi)\dot{q}$ and $\ddot{\psi} = (J\psi)\ddot{q} + G(q, \dot{q})$. Hence

$$\begin{aligned} \ddot{\phi} &= A(q)\lambda + B(q)(-H(q,\dot{q}) + \tau) + F(q,\dot{q}) \\ \ddot{\psi} &= C(q)(-H(q,\dot{q}) + (J\phi)^T\lambda + \tau) + G(q,\dot{q}) \end{aligned} \tag{4.7}$$

where $A(q) = (J\phi)M^{-1}(J\phi)^T$, $B(q) = (J\phi)M^{-1}$ and $C(q) = (J\psi)M^{-1}$. Let

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} F(q,\dot{q}) \\ G(q,\dot{q}) \end{pmatrix} + \\ &+ \begin{pmatrix} J\phi \\ J\psi \end{pmatrix} M^{-1}(q) \left[-H(q,\dot{q}) + \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \right] \end{aligned} \tag{4.8}$$

It follows that

$$\begin{aligned} \ddot{\phi} &= A(q)\lambda + v_1 \\ \ddot{\psi} &= C(q)(J\phi)^T\lambda + v_2 \end{aligned} \tag{4.9}$$

Now note that, adding the constraint $\phi(q) = 0$ to the equations (4.7) implies that $\ddot{\phi} = 0$. In other words, one obtains :

$$\lambda = -A^{-1}(q)v_1$$

Note that the matrix $A(q) = (J\phi)M^{-1}(J\phi)^T$ is invertible because $M(q)$ is symmetric and positive definite and $J\phi(q)$ is full row rank. So the dynamics of the constrained robot is given by

$$\phi = 0 \tag{4.10a}$$
$$\lambda = -A^{-1}(q)v_1 \tag{4.10b}$$
$$\ddot{\psi} = -C(q)(J\phi)^T A^{-1}(q)v_1 + v_2 \tag{4.10c}$$

By the implicit function theorem, the solution of the equation (4.10a) is locally given by $q = \eta(\psi)$ for a convenient smooth map η. Hence, the equation (4.10c) for $q = \eta(\psi)$ is a (nonconstrained) state representation of the dynamics of the constrained robot, with state $(\psi, \dot{\psi})$ and input (v_1, v_2). In particular the equation (4.8) with $q = \eta(\psi)$ defines a regular state feedback for the constrained robot.

Now let $\zeta_{ref}(t) = (\lambda_{ref}(t), \psi_{ref})$ be a reference trajectory of the constrained system.

Let $e(t) = \psi_{ref} - \psi$. Then, letting

$$v_1 = -A(q)\lambda_{ref} \tag{4.11a}$$
$$v_2 = -C(q)(J\phi)^T\lambda_{ref} + \ddot{\psi}_{ref} + \tag{4.11b}$$
$$+ \Lambda_1(\dot{\psi}_{ref} - \dot{\psi}) + \Lambda_2(\psi_{ref} - \psi) \tag{4.11c}$$

where Λ_1 and Λ_2 are symmetric and definite positive matrices. Then it follows that the variables $\zeta(t) = (\lambda(t), \psi(t))$ will converge asymptotically to the reference trajectory. In fact :

$$\begin{aligned} \lambda(t) - \lambda_{ref} &= 0 \\ \ddot{e}(t) + \Lambda_1 \dot{e}(t) + \Lambda_2 e(t) &= 0 \end{aligned}$$

Remark. When the model of the robot or the real shape of the constraint surface are not precisely known, such a method may produce bad results. Note that small errors in the constraint map ϕ may produce big deviations in the expected value of $J\phi$. The choice of the functions ψ should be made in a way to have, locally, a nonsingular Jacobian matrix $\begin{pmatrix} J\phi \\ J\psi \end{pmatrix}$. In many cases one may choose more than one local flat output "ψ" and switch the corresponding control law according the operation region.

5 Example

In this section we will apply the technique of the last section to an academic example.

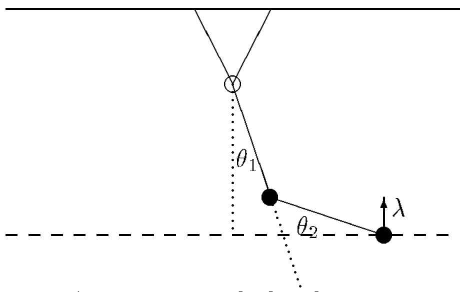

Figure 1: *Two link robot arm.*

Consider the equations of the two link robot arm of figure 1 [10] . The constraint surface is represented by the horizontal dashed line. The two dark disks represents unit masses and we can apply control torques to each degree of freedom corresponding to θ_1 and θ_2. The contact force is represented by the vector λ and the two arm lenghts are equal to one meter. The corresponding model of the nonconstrained robot is giving by :

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} \dot{\theta} \\ -M^{-1}(\theta)\left[C(\theta,\dot{\theta}) + K(\theta)\right] \end{pmatrix} + \\ &+ \begin{pmatrix} 0 \\ M^{-1}(\theta) \end{pmatrix} \tau \end{aligned} \tag{5.12}$$

where $\theta = (\theta_1, \theta_2)$ is the vector of angular displacements, $\tau = (\tau_1, \tau_2)$ is the vector of torques and

$$M(\theta) = \begin{pmatrix} 3 + 2\cos\theta_2 & 1 + \cos\theta_2 \\ 1 + cos\theta_2 & 1 \end{pmatrix}$$

$$C(\theta,\dot\theta) = \begin{pmatrix} -\dot\theta_2(2\dot\theta_1 + \dot\theta_2)\sin\theta_2 \\ \dot\theta_1^2 \sin\theta_2 \end{pmatrix}$$

$$K(\theta) = \begin{pmatrix} 2g\sin\theta_1 + g\sin(\theta_1+\theta_2) \\ g\sin(\theta_1+\theta_2) \end{pmatrix}$$

and g is the gravity constant. Consider constraint function

$$0 = \phi(\theta) = \cos\theta_1 + \cos(\theta_1+\theta_2) - L \tag{5.13}$$

corresponding to restrict the trajectory of the end of the second arm to the dashed straight line of figure 1. Note that L is the vertical distance of the dashed line from the first joint from the top to the bottom of figure 1. The corresponding constrained robot equations are :

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \theta \\ \dot\theta \end{pmatrix} &= \begin{pmatrix} \dot\theta \\ -M^{-1}(\theta)\left[C(\theta,\dot\theta) + K(\theta)\right] \end{pmatrix} + \\ &+ \quad (J\phi)^T\lambda + \begin{pmatrix} 0 \\ M^{-1}(\theta) \end{pmatrix}\tau \qquad (5.14) \\ 0 = \phi(\theta) &= \cos\theta_1 + \cos(\theta_1+\theta_2) - L \qquad (5.15) \end{aligned}$$

where $J\phi = \begin{pmatrix} -\sin\theta_1 - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \end{pmatrix}$ and λ is the contact force between the robot and the constraint surface. We can choose $\psi = \sin(\theta_1) + \sin(\theta_1+\theta_2)$, corresponding to the position along the constraint of the system. Note that the coordinates $(\psi, -\phi + L)$ corresponds to cartesian coordinates (x, y) for the figure 1. A control system using the previous development was constructed and some computer simulations are now presented. For all the plotted curves, the x-axis represents time in seconds.

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References

[1] B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control Optim.*, 29:38–57, 1991.

[2] X. Chen and M. A. Shayman. Dynamics and control of constrained nonlinear systems with application to robotics. In *Proc. Am. Control Conference*, pages 2962–2966, 1992.

[3] A. A. Cole. Control of robot manipulators with constrained motion. In *Proc. IEEE Conf. Decision and Control*, pages 1657–1658, 1989.

[4] M. D. Di Benedetto, J. W. Grizzle, and C. H. Moog. Rank invariants of nonlinear systems. *SIAM J. Control Optim.*, 27:658–672, 1989.

[5] M. Fliess, J. Lévine, P. Martin, F. Ollivier, and P. Rouchon. Flatness and dynamic feedback linearizability: two approaches. In *Proc. 3rd European Control Conference*, 1995.

[6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C. R. Acad. Sci. Paris Sér. I Math.*, 315:619–624, 1992.

[7] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund. *C. R. Acad. Sci. Paris Sér. I Math.*, 317:981–986, 1993.

[8] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Nonlinear control and Lie-Bäcklund transformations: Towards a new differential geometric standpoint. In *IEEE Conf. Decision and Contr.*, pages 339–344, 1994.

[9] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of non-linear systems: introductory theory and examples. *Internat. J. Control*, 61:1327–1361, 1995.

[10] L. C. J. M. Gras and H. Nijmeijer. Decoupling in nonlinear systems, from linearity to nonlinearity. *IEE Proceedings*, 136, Pt. D:53–62, 1989.

[11] A. Isidori. *Nonlinear Control Systems.* Springer–Verlag, 2nd edition, 1989.

[12] H. Krishnan and N. H. McClamroch. Tracking in nonlinear differential–algebraic control systems with applications to constrained robot systems. *Automatica J. IFAC*, 30:1885–1897, 1994.

[13] J. S. Liu. Hybrid position/force tracking for a class of constrained mechanical systems. *Systems Control Lett.*, 17:395–399, 1991.

[14] N. H. McClamroch and D. Wang. Feedback stabilization and tracking of constrained robots. *IEEE Trans. Automat. Control*, 33:419–426, 1988.

[15] J. K. Mills and A. A. Goldenberg. Force and position control of manipulators during constrained motion tasks. *IEEE J. Robotics and Automation*, 5:30–46, 1989.

[16] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer-Verlag, New York, 1990.

[17] W. F. Shadwick. Absolute equivalence and dynamic feedback linearization. *Systems Control Lett.*, 15:35–39, 1990.

[18] A. M. Vinogradov. Local symmetries and conservation laws. *Acta Appl. Math.*, 2:21–78, 1984.

[19] V. V. Zharinov. *Geometrical Aspects of Partial Differentials Equations.* World Scientific, Singapore, 1992.

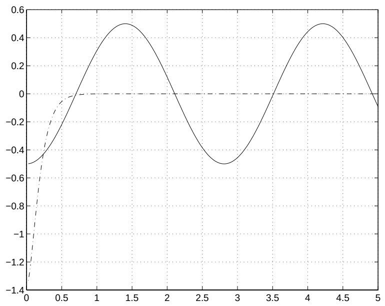

Figure 2: Reference position $\psi_{ref}(t)$ (continuous line) and trackink error e(t) (dashed line) in meters.

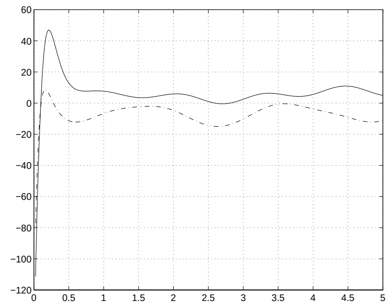

Figure 4: Control torques τ_1 and τ_2 in $N \times m$.

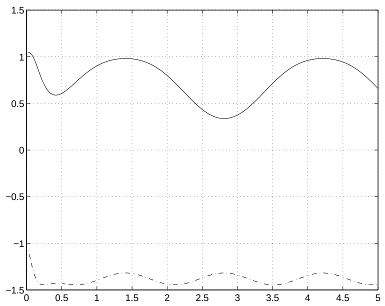

Figure 3: Angular positions θ_1 (continuous line) and θ_2 (dashed line) in radians.

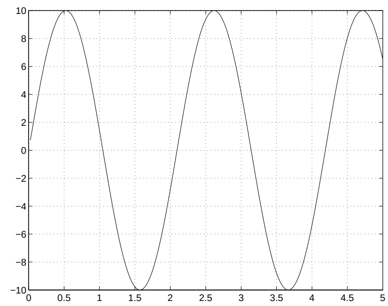

Figure 5: Contact force $\lambda(t)$ in N.