ON FLATNESS OF NONLINEAR IMPLICIT SYSTEMS

PAULO SERGIO PEREIRA DA SILVA*, SIMONE BATISTA†

*Escola Politécnica da USP
Av. Luciano Gualberto trav. 03, 158
05508-900 Cidade Universitária São Paulo – SP – BRAZIL

†Escola de Ciência e Tecnologia da UFRN
Avenida Hermes da Fonseca, nº 1111
59015-001 Tirol – Natal-RN — BRAZIL

Emails: paulo@lac.usp.br, ssimonebatista@ig.com.br

Abstract— This work considers a semi-implicit system Δ, that is, a pair (S, y), where S is a explicit system described by a state representation $\dot{x}(t) = f(t, x(t), u(t))$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, which is subject to a set of algebraic constraints $y(t) = h(t, x(t), u(t)) = 0$, where $y(t) \in \mathbb{R}^l$. An input candidate is a set of functions $v = (v_1, \dots, v_s)$, which may depend on time t, on x, and on u and its derivatives up to a finite order. Forgetting some technical assumptions, one says that this input is proper, if there exists some state z such that the implicit system admits a proper state representation $\dot{z} = g(t, z, v)$. Given an implicit system and an input candidate v, the problem studied in this paper is the question of verifying if v is proper input and if the corresponding state representation is linearizable by regular static-state feedback (considering the state z and the input v). Under some mild assumptions, the main result of this work shows necessary and sufficient conditions for the solution of this problem. These solvability conditions may be computed without the knowledge of z, and they rely on an integrability test that regards the explicit system S. The approach of this paper is the infinite-dimensional differential geometric setting of (Fliess et al., 1999).

Keywords— Nonlinear systems, flatness, exact linearization, state representations, implicit systems, realization theory, DAE's, differential geometric approach, diffieties

1 Introduction

The problem that is considered in this paper is defined in this section. The statement of this problem is a little bit imprecise for the moment, since this section concerns only the main ideas.

Let an implicit system Δ be given by

$$\Delta : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)) \\ y(t) &= h(t, x(t), u(t)) = 0 \end{cases} \tag{1}$$

where[1] $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$.

Let $\lfloor s \rceil$ stands for the set $\{1, 2, \dots, s\}$. Given a set of functions $v = (v_1, \dots, v_s)$, called input candidate, where $v_i = \phi_i(t, x(t), u(t), \dots, u^{(\alpha_i)})$, $\alpha_i \in \mathbb{N}, i \in \lfloor s \rceil$, one says that v is a proper input, if there exists a set $z = (z_1, \dots, z_q)$ of functions $z = \vartheta(t, x(t), u(t), \dots, u^{(\beta)})$ such that the system Δ admits a state representation

$$\dot{z}(t) = g(t, z(t), v(t)) \tag{2}$$

The problem studied in this paper regards the following aspects

(a) To check if v is a proper input.

(b) If (a) is true, then one may try to check if (2) is locally linearizable by regular static-state feedback.

[1]Note that $x(t)$ and $u(t)$ are not necessarily the state and the input of the implicit system since the algebraic constraints and their derivatives may induce (differential) relations linking their components.

Note that static-state feedback of this problem considers the state z and is applied to the input v, that is, is of the form $v(t) = \mathcal{V}(t, z(t), \eta(t))$, where $\eta(t)$ is the new input. The closed loop system is given by the linear controllable time-invariant system

$$\dot{\zeta}(t) = A\zeta(t) + B\eta(t)$$

where ζ is a new state. Note that v is not necessarily a physical input of the implicit system, but it may be a virtual input, as in the context of backstepping (Krstic et al., 1995).

Under some mild assumptions, the main result of this work shows necessary and sufficient conditions for the solution of this problem. These conditions rely on an integrability test that is computed from the explicit system S that is obtained by disregarding the constraints $y \equiv 0$. Furthermore, these conditions may be computed *without the knowledge of the state* z of (2) .

The problem of verifying if v is a proper input of the implicit system has been solved in (Pereira da Silva and Batista, 2010). The results of (Pereira da Silva, 2008) are combined with the ones of (Pereira da Silva and Batista, 2010) in order to solve this problem.

The approach of this paper is the infinite-dimensional differential geometric setting of (Fliess et al., 1999). The survey (Pereira da Silva et al., 2008) presents the results about this approach that are considered here. For completeness, a very brief summary of this approach is presented in appendix A, which introduces some standard vocabulary and notation of this approach.

The field of real numbers will be denoted by $\mathbb{R}$. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x = (x_1, \ldots, x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$. Let S be a system with Cartan field $\frac{d}{dt}$ (see appendix A). The Lie derivative $L_{\frac{d}{dt}}\eta$ of a function (or a form) η will be denoted by $\dot{\eta}$ (or $\eta^{(1)}$) and the k-fold Lie derivative $L^k_{\frac{d}{dt}}\eta$ of η will be denoted by $\eta^{(k)}$. If $\eta = (\eta_1, \ldots, \eta_m)$ is a set of functions (or forms), then $\eta^{(k)}$ stands for $\eta^{(k)} = (\eta_1^{(k)}, \ldots, \eta_m^{(k)})$. A codistribution $\Omega = \text{span}\,\{\omega_i, i \in A\}$ is said to be *integrable* if the exterior derivatives of each ω_i can be expressed as $d\omega_i = \sum_{j\in F} \eta_j \wedge \omega_j$ for convenient one forms $\eta_j, j \in F$, and F is a finite set[2]. Cartan's version of the Frobenius theorem can be used in the context of diffieties for finite dimensional integrable distributions Γ, when Γ is nonsingular (see (Pomet, 1995; Pereira da Silva et al., 2008)).

2 Some facts about implicit systems

Consider an implicit system Δ of the form (1), and suppose that also that all the functions defining (1) are smooth. One will call $x(t) \in \mathbb{R}^n$ the "pseudo-state" and $u(t) \in \mathbb{R}^m$ will be called the "pseudo-input"[3]. Recall from (3), that S is obtained from Δ by disregarding the constraints $y \equiv 0$. Furthermore, the functions $y = h(t, x, u)$ are considered to be outputs of S. Throughout this paper, the system Δ is the implicit system defined by (1), and S is the explicit system given by

$$S : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)) \\ y(t) &= h(t, x(t), u(t)) \end{cases} \quad (3)$$

System S can be viewed as a diffiety with Cartan field $\frac{d}{dt}$ and output[4] $y = h(t, x, u)$, in the framework of (Fliess et al., 1999) (see appendix A). Then $y^{(k)}$ stands for the function $L^k_{\frac{d}{dt}} y = \frac{d^k}{dt^k} y$ defined on S, which may depend on $x, u^{(0)}, u^{(1)}, \ldots$.

The following codistribution, defined on the system (diffiety) S given by (3), will be used in the sequel

$$\mathcal{Y} = \text{span}\left\{dt, (dy^{(k)} : k \in \mathbb{N})\right\}. \quad (4)$$

Definition 1 A *local output subsystem*[5] Y for the explicit system S with output y defined by (3), is a diffiety Y and a Lie-Bäcklund submersion $\pi : U \subset S \to Y$, where $U \subset S$ is an open subset, such that $\pi^*(T_\xi^* Y) = \mathcal{Y}|_\xi$ for all $\xi \in U$. A local state representation $((x_a, x_b), (u_a, u_b))$ of S is said to be strongly adapted[6] to the output subsystem Y if

(A) The Lie-Bäcklund submersion π is locally given by $\pi(t, x_a, x_b, (u_a^{(j)}, u_b^{(j)} : j \in \mathbb{N})) = (t, x_a, (u_a^{(j)}, j \in \mathbb{N}))$.

(B) The local state equations of S are of the form

$$\dot{x}_a = f_a(t, x_a, u_a) \quad (5a)$$
$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \quad (5b)$$

where (5a) are the local state equations[7] for Y.

(C) $\text{span}\left\{dx_a, (du_a^{(k)} : k \in \mathbb{N})\right\} = \text{span}\ \{dy^{(k)} : k \in \mathbb{N}\}$.

(D) The set of functions $\{x_a, u_a^{(k)} : k \in \mathbb{N})\}$ is contained in the set $\{y^{(k)} : k \in \mathbb{N}\}$. ♣

Remark 1 *It is important to point out that the components of the input (u_a, u_b) are redefined, that is, they are not necessarily a reordering of the original input u of S. The same remark applies to the components of (x_a, x_b), with respect to the original state x.*

Let $\tilde{\Delta}$ be the subset of the explicit system (diffiety) S defined by the points of S for which all the Lie derivatives $y^{(k)} = \frac{d}{dt}^k y$ vanish:

$$\tilde{\Delta} = \{\xi \in S \mid y^{(k)}(\xi) = 0, k \in \mathbb{N}\} \quad (6)$$

Definition 2 An implicit system Δ (defined by (1)) is regular if

- $\tilde{\Delta} \neq \emptyset$.
- There exists a *local output subsystem* Y for system S with output y around all $\xi \in \tilde{\Delta}$.
- Around all $\xi \in \tilde{\Delta}$, system S admits local state equations that are *strongly adapted* to Y.

♣

Sufficient conditions for showing that a given implicit system is regular are given in (Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva et al., 2008). They are essentially linked to the notion of zero dynamics that appears in the decoupling theory. For completeness, these results are re-stated in Appendix B, which shows how

[2]This means that the differential ideal generated by Ω is differentially closed).

[3]This terminology is justified in (Pereira da Silva and Batista, 2010; Pereira da Silva et al., 2008)

[4]It must be pointed out again that $y = h(t, x, u)$ is regarded an output rather than a constraint.

[5]See (Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva et al., 2008).

[6]The weaker definition of *adapted* state equations considered in Theorem 4.3 of (Pereira da Silva and Corrêa Filho, 2001) is obtained if one replaces the assumptions (C) and (D) by the only assumption that $\mathcal{Y} = \text{span}\left\{dt, dx_a, (du_a^{(k)} : k \in \mathbb{N})\right\}$. This last theorem also shows that the output subsystem is locally unique up to local Lie-Bäcklund isomorphisms.

[7]Using (C) and (D), one may show that f_a does not depend on t.

to compute the strongly adapted state representation.

The following definition regards an implicit system as an immersed submanifold.

Definition 3 Consider an implicit system Δ defined by (1) and the explicit system S defined by (3). A diffiety Γ is said to be equivalent to the implicit system Δ if:

- There exists a Lie-Bäcklund immersion $\iota : \Gamma \to S$.
- For every solution $\sigma(t)$ of Δ, there exists a solution $\nu(t)$ of Γ such that $\sigma(t) = \iota \circ \nu(t)$.

The equivalent system Γ is said to be canonical if $\Gamma \subset S$, the topology of Γ is the subset topology, ι is the insertion map, and Γ is a control system[8]. ♣

The concept defined above is compatible with the notion of equivalence by endogenous feedback. In fact, it is show in (Pereira da Silva et al., 2008) that, if an implicit system Δ is equivalent to Γ_1 and Δ is also equivalent to Γ_2, then Γ_1 and Γ_2 are equivalent by endogenous feedback. It can be shown that a regular implicit system defined by (1) is equivalent to an immersed system in the explicit system S defined by (3). This result is the Proposition 1 below.

Proposition 1 *(Pereira da Silva and Corrêa Filho, 2001; Pereira da Silva et al., 2008) Let Δ be a regular implicit system defined by* (1) *and let S be the explicit system associated to* (3)*. Then the subset $\tilde{\Delta} \subset S$ defined by* (6) *has a canonical structure of immersed (embedded) submanifold of S such that the canonical insertion $\iota : \tilde{\Delta} \to S$ is a Lie-Bäcklund immersion*[9]*. Furthermore $\tilde{\Delta}$ admits a local classical state representation around every point $\xi \in \tilde{\Delta}$. In particular, $\tilde{\Delta}$ is canonically equivalent to Δ.*

The idea of the proof of Proposition 1 is to consider the local state representation (5) that is strongly adapted to the output subsystem. It is shown that $\{t, x_a, x_b, U_a, U_b\}$ and $\{t, x_b, U_b\}$ are respectively local coordinates[10] for S and $\tilde{\Delta}$, where $U_a = \{u_a^{(j)} : j \in \mathbb{N}\}$ and $U_b = \{u_b^{(j)} : j \in \mathbb{N}\}$. In these coordinates $\iota(t, x_b, U_b) = (t, 0, x_b, 0, U_b)$.

It must be pointed out that the proof of proposition 1 shows also that the local state equations for $\tilde{\Delta}$ are given by $\dot{x}_b = f_b(t, 0, x_b, 0, u_b)$. In particular, the implicit system is equivalent to a kind of "zero dynamics", as pointed out in (Byrnes and Isidori, 1991; Krishnan and McClamroch, 1994). Note that these notions of equivalence allow one to define a state representation of an implicit system Δ as being a state representation of an equivalent system $\tilde{\Delta}$.

[8] By definition, Γ is a control system if it locally admits a state representation around every point $\gamma \in \Gamma$.

[9] Since ι_* injective, it can be shown that Cartan field $\partial_{\tilde{\Delta}}$ of $\tilde{\Delta}$ may be canonically defined by $\iota_* \partial_{\tilde{\Delta}} = \frac{d}{dt} \circ \iota$, where $\frac{d}{dt}$ is the Cartan field of S.

[10] Using the same name of x_b as a set of local coordinate functions of $\tilde{\Delta}$ and S is an abuse of notation. One could write for instance $\tilde{x}_b$ and consider that $\tilde{x}_b = x_b \circ \iota$.

3 Adapted projections

Given any state representation $(\tilde{x}, \tilde{u})$ defined on $U \subset S$, with $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$ of S that is adapted to the output subsystem Y, its clear that the $C^\infty(U)$-module $\mathcal{U} = \text{span}\left\{dt, dx_a, dx_b, (du_a^{(k)}, du_b^{(k)} : k \in \mathbb{N})\right\}$ is locally decomposed as $\mathcal{U} = \mathcal{B} \oplus \mathcal{Y}$, where $\mathcal{Y} = \text{span}\left\{dt, dx_a, (du_a^{(j)} : j \in \mathbb{N})\right\}$ and $\mathcal{B} = \text{span}\left\{dx_b, (du_b^{(j)} : j \in \mathbb{N})\right\}$. Here one may regard $\mathcal{U} = \mathcal{B} \oplus \mathcal{Y}$, $\mathcal{B}$, and $\mathcal{Y}$ as modules over $C^\infty(U)$. Another possible point of view is to work pointwise at some $\xi \in U$. Then, $\mathcal{U}|_\xi$, $\mathcal{B}|_\xi$, $\mathcal{Y}|_\xi$ become vector spaces over $\mathbb{R}$. Recall that a one-form defined on U may be written as

$$\begin{aligned} \omega &= \alpha_0 dt + \sum_{i=1}^{n_a} \alpha_i dx_{a_i} + \sum_{i=1}^{n_b} \beta_i dx_{b_i} \\ &+ \sum_{j=0}^{m_a} \sum_{k=0}^{\infty} \gamma_{jk} du_{a_j}^{(k)} + \sum_{j=0}^{m_b} \sum_{k=0}^{\infty} \epsilon_{jk} du_{b_j}^{(k)} \end{aligned} \qquad (7)$$

for adequate smooth functions α_i, β_i, γ_{jk}, ϵ_{jk}. Around any $\nu \in U$, a state representation defines a local coordinate system, so there exists some open neighborhood V_ν of ν, such that only a finite subset of those functions could be nonzero on V_ν. However, for the sake of defining our projection θ, one may consider the previous infinite sum without any problem.

One may define locally the projection $\theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B}$, called *adapted projection*, which associates a one-form ω to its projection

$$\theta(\omega) = \sum_{i=1}^{n_b} \beta_i dx_{b_i} + \sum_{k=0}^{\infty} \sum_{j=1}^{m_b} \epsilon_{jk} du_{b_j}^{(k)}. \qquad (8)$$

This projection is clearly a module morphism (or a linear map between vector spaces, when one works pointwise).

4 Proper Inputs

The notion of a proper input v is now stated in a precise manner.

Definition 4 (Proper input) Assume that a implicit system (1) is regular. Consider the explicit system S with output y defined by (3). Let $v = (v_1, \ldots, v_s)$ be a set of functions defined around a point $\nu \in \tilde{\Delta} \subset S$, where $\tilde{\Delta}$ is defined by (6)[11]. The set v is the *input candidate* of the implicit system. The input candidate v is said to be a *proper input* around some point $\nu \in \tilde{\Delta}$ if there exists a local proper state representation $((z_a, z), (v_a, v))$ for the explicit system S, defined around ν, that is strongly adapted to subsystem Y. ♣

Remark 2 *Roughly speaking, in the problem of testing if v is proper, note that the input candidate v is the data, and the question is to verify the existence of such z.*

The following theorem gives necessary and sufficient conditions for the solution a input candidate v to be a proper input.

Theorem 1 *(Pereira da Silva and Batista, 2010) Let $((x_a, x_b), (u_a, u_b))$ be a local proper state representation of system S defined by* (3) *that is adapted to Y. Let θ be the associated adapted projection (see section 3). Let $\mathcal{Y}$ be the codistribution, defined on S by* (4)*. Let γ be the least non-negative integer*[12] *such that one may locally write that span $\{dv\} \subset$ span $\left\{dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma)}\right\} \oplus \mathcal{Y}$. Then v is a proper input around some $\nu \in S$ if and only if there exists a non-negative integer δ such that, for the codistribution Γ_0 defined by*

$$\begin{aligned}\Gamma_0 \quad = \quad & \mathit{span}\,\Big\{dx_b, du_b^{(0)}, \ldots, du_b^{(\gamma)}, \\ & \theta\left(dv^{(0)}\right), \ldots, \theta\left(dv^{(\delta)}\right)\Big\}\end{aligned} \tag{9}$$

and for the codistributions Γ_k defined by

$$\Gamma_k = \mathit{span}\,\{\omega \in \Gamma_{k-1} \mid \dot{\omega} \in \Gamma_{k-1} + \mathcal{Y}\} \tag{10}$$

one has:
(i) Γ_k is finite dimensional and nonsingular and $\dim \Gamma_{k-1} - \dim \Gamma_k = \dim v$ *for $k = 1, \ldots, \delta + 1$.*
(ii) $\Gamma_{\delta+1} \oplus \mathcal{Y}$ is integrable.
(iii) $\Gamma_0 = \Gamma_1 \oplus$ span $\left\{\theta\left(dv^{(\delta)}\right)\right\}$.
(iv) The set $\mathbb{V}^{(k)} = \{\theta(dv^{(k)})\}$ is locally linearly independent for $k = 0, \ldots, \delta$.

An intrinsic version of the last theorem[13] can be found in (Pereira da Silva and Batista, 2010). However, the presented version is much more suitable for computations.

5 A sufficient condition of feedback-linearizability

This section presents the main result of this paper. One will combine the results of (Pereira da Silva, 2008) with the ones of (Pereira da Silva and Batista, 2010) in order to obtain a sufficient condition of flatness of the implicit system (1).

Theorem 2 *Let v be an input candidate that is a local* proper input *of the* regular *implicit system 1 around some $\xi \in \tilde{\Delta}$ (see* (6)*). Assume that δ is the corresponding integer, whose existence is assured by Theorem 1, and let $\Gamma_{\delta+1}$ be the corresponding codistribution that is defined in that Theorem. Let $\mathcal{D}_0 = \Gamma_{\delta+1}$ and let $\mathcal{D}_k =$ span $\{\omega \in \mathcal{D}_{k-1} \,|\, \dot{\omega} \in \mathcal{D}_{k-1}\}$. Assume that*
(a) The codistributions $\mathcal{D}_k$ are nonsingular at ξ for $k \in \mathbb{N}$.
(b) There exists $k^* \in \mathbb{N}$ big enough, such that $\mathcal{D}_{k^*} = \{0\}$.
(c) The codistributions $\mathcal{D}_k + \mathcal{Y}$ are locally integrable around ξ, for $k \in \mathbb{N}$. Let (2) *be local state equations for the implicit system* (1)*. Then this state representation is linearizable by a regular static state feedback of the form $v = \mathcal{V}(t, z, \eta)$.*

Proof (Sketch): According to (Pereira da Silva and Batista, 2010), the explicit system S defined by (3) admits a strongly adapted local state representation $((x_a, z), (u_a, v))$ with local state equations given by

$$\dot{x}_a = f_a(t, \tilde{x}_a, \tilde{u}_a) \tag{11a}$$

$$\dot{z} = f_z(t, z, v, \tilde{x}_a, \tilde{u}_a) \tag{11b}$$

Recall the context of the proof of proposition 1 (see (Pereira da Silva et al., 2008)). Let $\iota : \Delta \to S$ be the Lie-Bäcklund immersion that is locally defined by $(t, z, (v^{(k)} : k \in \mathbb{N})) \mapsto (t, z, (v^{(k)} : k \in \mathbb{N}), x_a, (u_a^{(k)} : k \in \mathbb{N}))$, where $(x_a, (u_a^{(k)} : k \in \mathbb{N})) = 0$. Since the state representation is strongly adapted, then $\mathbb{Y} = \text{span}\left\{dy^{(k)}\right\} = \text{span}\left\{dx_a, (du_a^{(k)} : k \in \mathbb{N})\right\}$. Then it is easy to show that $\iota^* \mathbb{Y} = 0$. From this, it is easy to show from the fact that $\Gamma_{\delta+1} + \mathcal{Y} = \text{span}\,\{dt, dz\} + \mathcal{Y}$ (see (Pereira da Silva and Batista, 2010)) that

$$\iota^*(\Gamma_{\delta+1} + \mathcal{Y}) = \text{span}\,\{dt, dz\}\,. \tag{12}$$

From the fact that ι is Lie-Bäcklund, it follows that, locally, $L_{\partial_\Delta} \iota^* \omega = \iota^* L_{\frac{d}{dt}} \omega$ for every one-form ω defined on S, where ∂_Δ is the Cartan field of Δ and $\frac{d}{dt}$ is the Cartan-field of S. From this, and from (12) it follows that, if one defines $\mathcal{E}_k = \iota^* \mathcal{D}_k, k \in \mathbb{N}$, then it is easy to show that

$$\mathcal{E}_k = \text{span}\,\{\omega \in \mathcal{E}_{k-1} \,|\, \dot{\omega} \in \mathcal{E}_{k-1}\}, k \in \mathbb{N}.$$

Then, from the fact that ι^* is an isomorphism (pointwise) when restricted to $\mathcal{D}_0 + \text{span}\,\{dt\}$, it is easy to see that conditions (a), (b), (c) implies that
(A) The codistributions $\mathcal{E}_k$ are locally nonsingular at for $k \in \mathbb{N}$.
(B) There exists $k^* \in \mathbb{N}$ big enough, such that

[11] Recall that the components of v may depend on $t, x, u, u^{(1)}, \ldots$

[12] The existence of the integer γ is assured by the fact that a state representation is a local coordinate system.

[13] Intrinsic in the sense that it does not depend on the choice of a particular adapted state representation.

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$\mathcal{E}_{k^*} = \text{span}\{dt\}$.
(C) The codistributions $\mathcal{E}_k$ are locally integrable around ξ, for $k \in \mathbb{N}$. The desired result follows easily, since (A), (B), and (C) are the static linearizability conditions of (Pereira da Silva, 2008, Theo. 2). □

Remark 3 *It is shown*[14] *in (Pereira da Silva and Batista, 2010) that, under the assumptions of Theorem 3 (see Appendix B), the codistribution* $\tilde{\Gamma}_0 = \Gamma_0 + \mathcal{Y}$ *is given by* $\tilde{\Gamma}_0 = \text{span}\left\{dx, du^{(0)}, \ldots, du^{(\gamma)}, dv^{(0)}, \ldots, dv^{(\delta)}\right\} + \mathcal{Y}$ *and it is also shown that the codistributions* $\Gamma_k + \mathcal{Y}$, $k \in \mathbb{N}$ *do not depend on the particular (strongly) adapted state representation that is chosen. In particular, it is easy to show that the conditions of theorem 2 are also independent of this choice.*

6 Example

Consider the following academic example (see (Pereira da Silva and Batista, 2010))

$$\begin{aligned} \dot{x}_1(t) &= u_1 + 2x_3u_2 \\ \dot{x}_2(t) &= tx_3 + 2(x_1+1)u_1 + 4x_1x_3u_2 \\ \dot{x}_3(t) &= u_2(t),\ y(t) = x_1 - x_3^2 = 0 \end{aligned}$$

Let $x = (x_1, x_2, x_3)$ and $u = (u_1, u_2)$ The input candidate for this example is $v = tx_3 + u_1\left(x_1 - x_3^2\right)$. In (Pereira da Silva and Batista, 2010) it is shown fr this example that is possible to choose $x_a = y = x_1 - x_3^2$, $u_a = \dot{y} = u_1$, $x_b = (x_2, x_3)$ and $u_b = u_2$. It is also shown that $\Gamma_0 = \text{span}\{dx_b, du_b, d\dot{u}_b\}$. $\Gamma_1 = \text{span}\{dx_b, du_b\}$, and that $\Gamma_2 = \text{span}\{dx_b\}$. $\Gamma_3 = \text{span}\{dt, dx_2 - 4x_3x_1dx_3\}$. To see that $\Gamma_3 + \text{span}\{dy\}$ is integrable, it suffices to notice that $dy = dx_1 - 2x_3dx_3$, hence $\Gamma_3 + \text{span}\{dy\} = \text{span}\{dt, dx_2 - 2x_1dx_1, dy\} = \text{span}\left\{dt, d(x_2 - x_1^2), dy\right\}$. It follows that all the assumptions of Theorem 1 holds.

Further computations[15] shows that $\mathcal{D}_1 = \{0\}$. In particular the conditions of Theorem 2 holds. In this relatively simple example, it was possible to compute the state representation 2, obtaining (see (Pereira da Silva and Batista, 2010))

$$\dot{z} = v \tag{13}$$

where z is given by $x_2 - x_1^2$. It is obvious that (13) is feedback linearizable and so z is a flat output of the implicit system. It is important to say that, in some cases, it may be very hard to integrate the codistributions in order to obtain z and (2), that is, an explicit state representation of the implicit system. However, the existence conditions of Theorems 1 and 2 are always checkable.

[14] See the proof of Theorem 4.9 of (Pereira da Silva and Batista, 2010)

[15] These computations have been performed using Matlab® / Maple®.

A Diffieties and Systems

This appendix is a very brief summary of some facts about the infinite dimensional approach of (Fliess et al., 1999). A survey about this subject can be found in (Pereira da Silva et al., 2008).

$\mathbb{R}^A$-Manifolds, Diffieties and Systems.

An ordinary diffiety is an $\mathbb{R}^A$ manifold for which there exists a field $\frac{d}{dt}$, called Cartan field.

A system S is a pair (S, t), where S is an ordinary diffiety, and $t : S \to \mathbb{R}$ is a function , called time, such that $\frac{d}{dt}(t) = 1$ and such that around any point $\xi \in S$ there exists local coordinates of S of the form (t, η)[16].

State Space Representation and Outputs. A local *state representation* of a system (S, t) is a local coordinate system $\psi = \{t, x, U\}$, where $x = \{x_i, i \in \lfloor n \rceil\}$, $U = \{u_j^{(k)} | j \in \lfloor m \rceil, k \in \mathbb{N}\}$. The set of functions $x = (x_1, \ldots, x_n)$ is called *state* and the set $u = (u_1, \ldots, u_m)$ is called *input.* In these coordinates the Cartan field is locally written by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in \mathbb{N}, \\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \tag{14}$$

It follows from (14) that $L_{\frac{d}{dt}} u^{(k)} = \frac{d}{dt}(u^{(k)}) = u^{(k+1)}$. So the notation $u^{(k)}$ is consistent with the fact that, along a solution[17], it represents the differentiation of $u^{(k-1)}$ with respect to time.

A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by (x, u). An *output* y of a system S is a set of functions defined on S. A state representation is said to be *classical (or proper)* if f_i does not depend on $u^{(\alpha)}$ for $\alpha > 1$. A *control system* S is a system such that there exists a local state representation around every $\xi \in S$.

System associated to differential equations. Now assume that a control system is given by a set of equations

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x}_i &= f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n \rceil \\ y_j &= \eta_j(x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p \rceil \end{aligned} \tag{15}$$

One can always associate to these equations a diffiety S of global coordinates $\psi = \{t, x, U\}$ and Cartan field given by (14).

Solutions. A *solution* of a system S with Cartan field $\frac{d}{dt}$ is a smooth map $\sigma : (a, b) \to S$, where $(a, b) \subset \mathbb{R}$, such that $\dot{\sigma}(t) = \frac{d}{dt}(\sigma(t))$.

Subsystems. A (local) *subsystem* S_a of a system S with time notion t is a pair (S_a, π), where S_a is a system with a time notion τ_a and Cartan

[16] This is equivalent to saying that the function t is a submersion, and the fact that $\frac{d}{dt}(t) = 1$ is equivalent to saying that that the function t is Lie-Bäcklund, when $\mathbb{R}$ is regarded as a diffiety with trivial Cartan field.

[17] See the definition of solution given in this section.

field ∂_a, and π is a Lie-Bäcklund submersion $\pi : U \subset S \to S_a$ between the system $U \subset S$ and S_a such that $\tau_a \circ \pi = t$. A local state representation $x = (x_a, x_b)$, $u = (u_a, u_b)$ is said to be *adapted* to a subsystem S_a if we locally have

$$\dot{x}_a = f_a(t, x_a, u_b) \quad (16a)$$
$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \quad (16b)$$

and (x_a, u_a) is a local state representation of S_a with state equations (16a).

B Existence of strongly adapted state equations

Theorem 3 *(Pereira da Silva et al., 2008) Let S be the system with classical state representation (x, u) and classical output y, defined*[18] *by* (3). *Let $\nu \in \tilde{\Delta}$, where $\tilde{\Delta} \subset S$ is defined by* (6). *Assume that there exists a partition $y = (\bar{y}, \widehat{y})$, where $\bar{y}$ is called the* independent *part, and $\hat{y}$ is called* dependent *part of the output. Assume also that there exists some $\alpha \in \mathbb{N}$ such that, locally around ν, one has*

1. $span\{dx\} \cap span\left\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha-1)}\right\} = span\{dx\} \cap span\left\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha)}\right\}$.

2. $span\left\{dt, dx, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha-1)}\right\}$ is locally nonsingular around ξ.

3. $span\left\{dt, dx, du, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha)}\right\}$ is locally nonsingular around ξ.

4. The set $\{dt, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha)}\}$ is pointwise independent in an open neighborhood of ξ.

5. $span\left\{dy^{(0)}, \dots, dy^{(\alpha-1)}\right\} \subset span\left\{dt, dx, d\bar{y}^{(0)}, \dots, d\bar{y}^{(\alpha-1)}\right\}$.

6. $span\left\{dt, dy^{(0)}, \dots, dy^{(k)}\right\}$ is nonsingular for $k = \alpha$ and $k = \alpha - 1$.

7. $span\{dy^{(\alpha)}\} \subset span\{dt, dy^{(0)}, dy^{(1)}, \dots, dy^{(\alpha-1)}, d\bar{y}^{(\alpha)}\}$.

8. $span\{dy^{(0)}, dy^{(1)}, \dots, dy^{(k)}\}$ is nonsingular around ν for $k = \alpha - 1$ and $k = \alpha$.

Then there exists a local output subsystem Y defined around ν that admits a strongly adapted state representation $(\tilde{x}, \tilde{u})$, where $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$. Moreover:

(A) One may choose $u_a = \bar{y}^{(\alpha)}$.

(B) One may choose $x_a \subset \{y^{(0)}, \dots, y^{(\alpha-1)}\}$ such that $\{dt, dx_a\}$ is a local basis of $span\{dy^{(0)}, \dots, dy^{(\alpha-1)}\}$.

(C) One may chose x_b in a way that dx_b completes $\{dt, dx_a\}$ to a local basis of $span\{dt, dx, dy, \dots, dy^{(\alpha-1)}\}$.

(D) One may chose u_b in order to complete $\{dt, dx_a, dx_b, du_a\}$ to a basis $\{dt, dx_a, dx_b, du_a, du_b\}$ of $span\{dt, dx, du, dy, \dots, dy^{(\alpha)}\}$.

In particular, if $\tilde{\Delta}$ is nonempty and these assumptions hold around all $\xi \in \tilde{\Delta}$, then the corresponding implicit system (1) *is regular. Furthermore, $span\{dx\} + \mathcal{Y} = span\{dx_b\} \oplus \mathcal{Y}$ and $span\{dx, du\} + \mathcal{Y} = span\{dx_b, du_b\} \oplus \mathcal{Y}$.*

Proof: The proof of the theorem is an easy consequence of the proof of Theorem 5 of (Pereira da Silva et al., 2008). □

[18] The state representation (x, u) is classical if span$\{d\dot{x}\} \subset$ span$\{dt, dx, du\}$ and the output y is classical if span$\{dy\} \subset$ span$\{dt, dx, du\}$.

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