International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

On state representations of nonlinear implicit systems

Paulo Sergio Pereira da Silva [a]; Simone Batista [b]
[a] Escola Politécnica da USP, São Paulo-SP, Brazil [b] Escola de Ciência e Tecnologia da UFRN, Natal-RN 59015-001, Brazil

First Published on: 04 November 2009

International Journal of Control
2009, 1–16, iFirst

On state representations of nonlinear implicit systems

Paulo Sergio Pereira da Silva[a*] and Simone Batista[b]

[a]*Escola Politécnica da USP, Av. Luciano Gualberto trav. 03, 158-05508-900, São Paulo-SP, Brazil;* [b]*Escola de Ciência e Tecnologia da UFRN, Avenida Hermes da Fonseca, n 1111. Tirol. Natal-RN 59015-001, Brazil*

(*Received 13 June 2008; final version received 19 July 2009*)

This work considers a semi-implicit system Δ, that is, a pair (S, y), where S is an explicit system described by a state representation $\dot{x}(t) = f(t, x(t), u(t))$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, which is subject to a set of algebraic constraints $y(t) = h(t, x(t), u(t)) = 0$, where $y(t) \in \mathbb{R}^l$. An input candidate is a set of functions $v = (v_1, \ldots, v_s)$, which may depend on time t, on x, and on u and its derivatives up to a finite order. The problem of finding a (local) proper state representation $\dot{z} = g(t, z, v)$ with input v for the implicit system Δ is studied in this article. The main result shows necessary and sufficient conditions for the solution of this problem, under mild assumptions on the class of admissible state representations of Δ. These solvability conditions rely on an integrability test that is computed from the explicit system S. The approach of this article is the infinite-dimensional differential geometric setting of Fliess, Lévine, Martin, and Rouchon (1999) ('A Lie-Bäcklund Approach to Equivalence and Flatness of Nonlinear Systems', *IEEE Transactions on Automatic Control*, 44(5), (922–937)).

Keywords: nonlinear systems; state representations. implicit systems; realisation theory; DAEs; differential geometric approach; diffieties

1. Introduction

The works of van der Schaft (1987), Crouch and Lamnabhi-Lagarrigue (1988), Glad (1988), van der Schaft (1990), Liu and Moog (1994), Moog, Zheng, and Liu (2002) consider the problem of giving proper realisations of input–output equations of the form

$$y^{(n)} = \phi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}), \tag{1}$$

where the highest derivative of y appears linearly. A comparison between these works can been found in Kotta and Mullari (2005).

A proper realisation of system (1) is an equivalent system of the form

$$\dot{z}(t) = g(z(t), u(t)), \tag{2}$$

where $z(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input of the realisation.

Strongly related to the input–output realisation problem is the question of elimination of input derivatives by generalised state transformation. Let S be a nonlinear system with state $x \in \mathbb{R}^n$ and input $v \in \mathbb{R}^m$, given by

$$\dot{x}(t) = f(x(t), v(t), \ldots, v^{(\alpha)}). \tag{3}$$

One may seek a generalised (local) state transformation $z = \phi(x, v^{(0)}, \ldots, v^{(\gamma)})$ such that the system (3) is transformed into (2), where z is a new state for the system, with $\dim z = \dim x$. This transformation is invertible in the sense that one may locally write $x = \psi(z, v^{(0)}, \ldots, v^{(\delta)})$. The necessary and sufficient conditions for the existence of such a transformation are given in Delaleau and Respondek (1995). A state representation (3) is said to be *generalised* if f depends on $v^{(\alpha)}$ for $\alpha > 0$, whereas the state representation (2) is said to be *classical*, or *proper*.[1]

Recall that in the behavioural approach of Willems (1992) the input and the output are not chosen a priori. The same point of view is shared by the approach of Fliess, Lévine, Martin, and Rouchon (1999), and this fact is in accordance with what is found in physical systems. The results of Delaleau and Respondek (1995) are generalised in Pereira da Silva and Batista (2009) for the case where there is freedom to redefine the input, that is, v it is not necessarily the original input of the system.

The work of Pereira da Silva and Batista (2009) considers systems of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{4}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. A set of functions $v = (v_1, \ldots, v_s)$ is chosen, and it is called the *input candidate*. Note that each function v_i may depend on t,

*Corresponding author. Email: paulo@lac.usp.br

ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207170903193441
http://www.informaworld.com

$x,\ u^{(0)},\dots,u^{(\gamma)}$ where $\gamma\in\mathbb{N}$. The main result of that paper solves the problem of checking if (4) admits an equivalent system

$$\dot{z}(t)=g(t,z(t),v(t)), \tag{5}$$

where $z(t)\in\mathbb{R}^q$ is a new state for the system obtained by an endogenous transformation, which is much more general than the transformation ϕ considered in Delaleau and Respondek (1995). In this case, the dimension of the state x is not necessarily equal to the one of the new state z. To see this, consider the trivial example $\dot{x}_1=x_2,\ \dot{x}_2=u$. If one chooses[2] $v=x_1$, this system admits a state representation $\dot{z}_1=v$ with $z_1=x_1$. If $v=u^{(1)}$, the system admits a state representation $\dot{z}_1=z_2,\ \dot{z}_2=z_3,\ \dot{z}_3=v$, where $z_1=x_1,\ z_2=x_2,\ z_3=u_1$. This last state representation is nothing less than a dynamic extension of the original state x.

Here, the results of Pereira da Silva and Batista (2009) are generalised in order to obtain, under mild assumptions, necessary and sufficient conditions for the existence of a proper state representation of nonlinear implicit systems. As in Pereira da Silva and Batista (2009), the conditions are constructive, and provide a test that decides whether a given set of functions v is an input of the implicit system that admits a proper state representation. Furthermore, v is not necessarily coincident with the '*original*' input of the implicit system. It is important to stress that, although the solvability conditions are constructive, if one wants to compute the corresponding state representation, then one must apply the Frobenius theorem to an integrable codistribution. This process is not constructive, and it may be a very difficult task in general. Once the choice of v is made, one may test 'a posteriori' whether these conditions hold or not. However, there is no known method for selecting the suitable input candidates 'a priori'. This question may be the subject of future investigation.

At this point, we give a preliminary statement of the main problem to be considered[3] in this work.

State representation problem for implicit systems. *Let an implicit system Δ be given by*

$$\Delta:\begin{cases}\dot{x}(t)=f(t,x(t),u(t))\\ y(t)=h(t,x(t),u(t))=0,\end{cases} \tag{6}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, and given a set of function $v=(v_1,\dots,v_s)$, where $v_i=\phi_i(t,x(t),u(t),\dots,u^{(\alpha_i)})$, $\alpha_i\in\mathbb{N}$, $i\in\lfloor s\rceil$, when there exists an equivalent system[4] *(5) with a given input v?*$\diamond$

In this article, it will be shown that the solution of this problem relies on the geometric properties of the explicit system S given by

$$S:\begin{cases}\dot{x}(t)=f(t,x(t),u(t))\\ y(t)=h(t,x(t),u(t)),\end{cases} \tag{7}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$. Note that S is obtained from Δ by disregarding the algebraic constraints $y\equiv 0$ and considering them as *outputs* $y=h(t,x(t),u(t))$. In particular, $\{t,x,u^{(0)},u^{(1)},\dots\}$ are global coordinates for S, and so, those variables are independent for S (but not for Δ). Furthermore, it is clear that it is not important whether an original input of the implicit system[5] is available a priori or not. The input candidate v may be an arbitrary function of system variables and of its derivatives up to some finite order.

Our approach will follow the infinite-dimensional geometric setting introduced in control theory by Fliess, Lévine, Martin, and Rouchon (1993), Pomet (1995), Fliess et al. (1999), in combination with the ideas presented in Pereira da Silva and Corrêa Filho (2001), Pereira da Silva, Silveira, Correa Filho, and Batista (2008) and Conte, Moog, and Perdon (2007).

We use the standard notations of differential geometry in the finite and infinite-dimensional case. A brief overview of the infinite-dimensional approach of Fliess et al. (1999) is presented in Appendix A. Some notations and the definitions of Appendix A are used along this article (e.g. the definition of system as a *diffiety* and the definition of (classical) *state representation* as a local coordinate system). The survey Pereira da Silva et al. (2008) presents the results about this approach that are considered here.

The field of real numbers will be denoted by $\mathbb{R}$. The matrix M^T stands for the transpose of a matrix M. The set of natural numbers $\{1,\dots,k\}$ will be denoted by $\lfloor k\rceil$. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T,z_2^T)^T$, where z_1 and z_2 are also column vectors. Let $x=(x_1,\dots,x_n)$ be a vector of functions (or a collection of functions). For a finite set x, card x will denote the cardinal of x, that is, the number of elements of x. Let M be a module over the ring R, let $N\subset M$ be a submodule, and let $\pi: M\to M/N$ be the canonical projection. A subset $\{m_1,\dots,m_s\}\subset M$ is said to be independent *modulo N* if the set $\{\pi(m_1),\dots,\pi(m_s)\}$ is R-independent.

One may let $\{\mathrm{d}x\}$ stands for the set $\{\mathrm{d}x_1,\dots,\mathrm{d}x_n\}$. Let S be a system with Cartan field $\frac{\mathrm{d}}{\mathrm{d}t}$ (Appendix A). The Lie derivative $L_{\frac{\mathrm{d}}{\mathrm{d}t}}\eta$ of a function (or a form) η will be denoted by $\dot{\eta}$ (or $\eta^{(1)}$) and the k-fold Lie derivative $L^k_{\frac{\mathrm{d}}{\mathrm{d}t}}\eta$ of η will be denoted by $\eta^{(k)}$. If $\eta=(\eta_1,\dots,\eta_m)$ is a set of functions (or forms), then $\eta^{(k)}$ stands for $\eta^{(k)}=(\eta_1^{(k)},\dots,\eta_m^{(k)})$. A codistribution $\Omega=\mathrm{span}\{\omega_i, i\in A\}$ is said to be *integrable* if the exterior derivatives of each ω_i can be expressed as $\mathrm{d}\omega_i=\sum_{j\in F}\eta_j\wedge\omega_j$ for convenient one-forms η_j, $j\in F$ and F is a finite set.[6] Cartan's version of the Frobenius theorem can be used in the context of diffieties for finite-dimensional integrable distributions Γ, when Γ is nonsingular (Pomet 1995; Pereira da Silva 2008).

This article is organised as follows. Section 2 presents some preliminary results about implicit systems. Section 3 introduces some important results about derived flags. The main results are presented in Section 4. Some worked examples are developed in Section 5. Conclusions and a comparison with the results of Pereira da Silva and Batista (2009) are stated in Section 6. Some auxiliary results and their proofs are presented in Appendices A–E. Finally, Appendix F presents an algorithm that summarises the main results from a computational viewpoint.

2. Some facts about implicit systems

Consider an implicit system Δ of the form (6), and suppose that all the functions defining (6) are smooth. One will call $x(t) \in \mathbb{R}^n$ the 'pseudo-state' and $u(t) \in \mathbb{R}^m$ will be called the 'pseudo-input', a terminology that will be justified later. Recall from (7) that S is obtained from Δ by disregarding the constraints $y \equiv 0$. Furthermore, the functions $y = h(t, x, u)$ are considered to be outputs of S. Throughout this article, the system Δ is the implicit system defined by (6), and S is the explicit system given by (7).

System S can be viewed as a diffiety with Cartan field $\frac{d}{dt}$ and output[7] $y = h(t, x, u)$, in the framework of Fliess et al. (1999) that is briefly summarised in Appendix A. Then $y^{(k)}$ stands for the function $L^k_{\frac{d}{dt}} y = \frac{d^k}{dt^k} y$ defined on S, which may depend on $x, u^{(0)}, u^{(1)}, \ldots$.

It must be stressed out that u is not necessarily the input of the implicit system, since the constraints $y^{(k)} \equiv 0$ may induce relations[8] among the components of x, $u^{(0)}$, $u^{(1)}$ etc. For instance, the implicit system $\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ y = x_1 + x_2 = 0$ is equivalent to the explicit system $\dot{x}_1 = u_1$ (and the relations $x_2 = -x_1, u_2 = -u_1$). A possible state of the implicit system is x_1 and a possible input is u_1. This explains why x and u are called, respectively, pseudo-input and pseudo-state of (6).

The following codistribution, defined on the system (diffiety) S given by (7), is used in the sequel

$$\mathcal{Y} = \operatorname{span}\left\{dt, (dy^{(k)} : k \in \mathbb{N})\right\}. \tag{8}$$

Definition 2.1: A local output subsystem[9] Y for the explicit system S defined (7), with output y defined by (7), is a diffiety Y and a Lie–Bäcklund submersion $\pi : U \subset S \to Y$, where $U \subset S$ is an open subset, such that $\pi^*(T^*_\xi Y) = \mathcal{Y}|_\xi$ for all $\xi \in U$. A local state representation $((x_a, x_b), (u_a, u_b))$ of S is said to be strongly adapted[10] to the output subsystem Y if:

(A) The corresponding Lie–Bäcklund submersion π is locally given by

$$\pi(t, x_a, x_b, (u_a^{(j)}, u_b^{(j)} : j \in \mathbb{N})) = (t, x_a, (u_a^{(j)}, j \in \mathbb{N})).$$

(B) The local state equations of S are of the form

$$\dot{x}_a = f_a(t, x_a, u_a) \tag{9a}$$

$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b), \tag{9b}$$

where (9a) are the local state equations[11] for Y.

(C) $\operatorname{span}\{dx_a, (du_a^{(k)} : k \in \mathbb{N})\} = \operatorname{span}\left\{dy^{(k)} : k \in \mathbb{N}\right\}$.

(D) The set of functions $\{x_a, u_a^{(k)} \cdot k \in \mathbb{N})\}$ is contained in the set $\{y^{(k)} : k \in \mathbb{N}\}$.

Remark 2.2: It is important to point out that the components of the input (u_a, u_b) are redefined, that is, they are not necessarily a reordering of the original input u of S. The same remark applies to the components of (x_a, x_b), with respect to the original state x. A common abuse of notation occurs in the definition above. The same name x_a stands for sets of coordinate functions defined on S and also on Y. It would be more precise to let x_a be a subset of local coordinates of Y, and $\tilde{x}_a$ be a subset of local coordinates of S such that $\tilde{x}_a = x_a \circ \pi$. The same remark applies to $u_a^{(k)}, k \in \mathbb{N}$.

Let $\tilde{\Delta}$ be the subset of the explicit system (diffiety) S defined by the points of S for which all the Lie derivatives $y^{(k)} = \frac{d}{dt}^k y$ vanish

$$\tilde{\Delta} = \{\xi \in S \mid y^{(k)}(\xi) = 0, k \in \mathbb{N}\}. \tag{10}$$

Definition 2.3: An implicit system Δ (defined by (6)) is regular if

- $\tilde{\Delta} \neq \emptyset$.
- There exists a local output subsystem Y for system S with output y around all $\xi \in \tilde{\Delta}$.
- Around all $\xi \in \tilde{\Delta}$, system S admits local state equations that are strongly adapted to Y.

Sufficient conditions for showing that a given implicit system is regular are given in Pereira da Silva and Corrêa Filho (2001), Pereira da Silva et al. (2008). They are essentially linked to the notion of zero dynamics that appears in the decoupling theory. For completeness, these results are re-stated in Appendix E.

The following definition regards an implicit system as an immersed submanifold.

Definition 2.4: Consider an implicit system Δ defined by (6) and the explicit system S defined by (7). A diffiety Γ is said to be equivalent to the implicit system Δ if:

- There exists a Lie–Bäcklund immersion $\iota : \Gamma \to S$.
- For every solution $\sigma(t)$ of Δ, there exists a solution $\nu(t)$ of Γ such that $\sigma(t) = \iota \circ \nu(t)$.

The equivalent system Γ is said to be canonical if $\Gamma \subset S$, the topology of Γ is the subset topology, ι is the insertion map and Γ is a control system.[12]

The concept defined above is compatible with the notion of equivalence by endogenous feedback. In fact, it is shown in Pereira da Silva et al. (2008) that, if an implicit system Δ is equivalent to Γ_1 and Δ is also equivalent to Γ_2, then Γ_1 and Γ_2 are equivalent by endogenous feedback. It can be shown that a regular implicit system defined by (6) is equivalent to an immersed system in the explicit system S defined by (7). This results in Proposition 2.5.

Proposition 2.5 (Pereira da Silva and Corrêa Filho 2001; Pereira da Silva et al. 2008)**:** *Let Δ be a regular implicit system defined by* (6) *and let S be the explicit system associated with* (7). *Then the subset $\tilde{\Delta} \subset S$ defined by* (10) *has a canonical structure of immersed* (*embedded*) *submanifold of S such that the canonical insertion $\iota : \tilde{\Delta} \to S$ is a Lie–Bäcklund immersion.*[13] *Furthermore, $\tilde{\Delta}$ admits a local classical state representation around every point $\xi \in \tilde{\Delta}$. In particular, $\tilde{\Delta}$ is canonically equivalent to Δ.*

The idea of the proof of Proposition 2.5 is to consider the local state representation (9) that is strongly adapted to the output subsystem. It is shown that $\{t, x_a, x_b, U_a, U_b\}$ and $\{t, x_b, U_b\}$ are, respectively, local coordinates[14] for S and Δ, where $U_a = \{u_a^{(j)} : j \in \mathbb{N}\}$ and $U_b = \{u_b^{(j)} : j \in \mathbb{N}\}$. In these coordinates $\iota(t, x_b, U_b) = (t, 0, x_b, 0, U_b)$.

It must be pointed out that the proof of Proposition 2.5 also shows that the local state equations for Δ are given by $\dot{x}_b = f_b(t, 0, x_b, 0, u_b)$. In particular, the implicit system is equivalent to a kind of 'zero dynamics', as pointed out in Byrnes and Isidori (1991), Krishnan and McClamroch (1994). Note that these notions of equivalence allow one to define a state representation of an implicit system Δ as being a state representation of an equivalent system $\tilde{\Delta}$.

3. Adapted projections and derived flags

The main results of this article are based on derived flags, defined on the explicit system S given by (7), considering the quotient with respect to the codistribution $\mathcal{Y}$ defined in (8). Such kind of construction is called relative derived flag and it plays an important role in the theory of implicit systems (Pereira da Silva and Corrêa Filho 2001).

Given any state representation $(\tilde{x}, \tilde{u})$ defined on $U \subset S$, with $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$ of S that is adapted to the output subsystem Y, it is clear that the $C^\infty(U)$-module $\mathcal{U} = \operatorname{span}\{\mathrm{d}t, \mathrm{d}x_a, \mathrm{d}x_b, (\mathrm{d}u_a^{(k)}, \mathrm{d}u_b^{(k)} : k \in \mathbb{N})\}$ is locally decomposed as $\mathcal{U} = \mathcal{B} \oplus \mathcal{Y}$, where $\mathcal{Y} = \operatorname{span}\left\{\mathrm{d}t, \mathrm{d}x_a, (\mathrm{d}u_a^{(j)} : j \in \mathbb{N})\right\}$ and $\mathcal{B} = \operatorname{span}\{\mathrm{d}x_b, (\mathrm{d}u_b^{(j)} : j \in \mathbb{N})\}$. Here one may regard $\mathcal{U} = \mathcal{B} \oplus \mathcal{Y}$, $\mathcal{B}$ and $\mathcal{Y}$ as modules over $C^\infty(U)$. Another possible point of view is to work pointwise at some $\xi \in U$. Then, $\mathcal{U}|_\xi$, $\mathcal{B}|_\xi$, $\mathcal{Y}|_\xi$ become vector spaces over $\mathbb{R}$. Recall that a one-form defined on U can be written as

$$\omega = \alpha_0\,\mathrm{d}t + \sum_{i=1}^{n_a} \alpha_i\,\mathrm{d}x_{a_i} + \sum_{i=1}^{n_b} \beta_i\,\mathrm{d}x_{b_i} + \sum_{j=0}^{m_a}\sum_{k=0}^{\infty} \gamma_{jk}\,\mathrm{d}u_{a_j}^{(k)} + \sum_{j=0}^{m_b}\sum_{k=0}^{\infty} \epsilon_{jk}\,\mathrm{d}u_{b_j}^{(k)} \tag{11}$$

for adequate smooth functions α_i, β_i, γ_{jk}, ϵ_{jk}. Around any $\nu \in U$, a state representation defines a local coordinate system, so there exists some open neighbourhood V_ν of ν, such that only a finite subset of those functions could be nonzero on V_ν. However, for the sake of defining our projection θ, one may consider the previous infinite sum without any problem.

One may define locally the projection $\theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B}$, called *adapted projection*, which associates a one-form ω to its projection

$$\theta(\omega) = \sum_{i=1}^{n_b} \beta_i\,\mathrm{d}x_{b_i} + \sum_{k=0}^{\infty}\sum_{j=1}^{m_b} \epsilon_{jk}\,\mathrm{d}u_{b_j}^{(k)}. \tag{12}$$

This projection is clearly a module morphism (or a linear map between vector spaces, when one works pointwise). Let $\Gamma = \operatorname{span}\{\omega_i, i \in \Lambda\}$ be a given codistribution and define $\theta\Gamma = \operatorname{span}\{\theta(\omega_i), i \in \Lambda\}$. Then, by construction

$$\Gamma + \mathcal{Y} = \theta\Gamma \oplus \mathcal{Y}. \tag{13}$$

Let $\tilde{\theta} : \tilde{\mathcal{B}} \oplus \mathcal{Y} \to \tilde{\mathcal{B}}$ be another projection (locally) constructed from other adapted state representation that is also defined on U. Since $\theta(\omega) - \omega \in \mathcal{Y}$ and $\tilde{\theta}(\omega) - \omega \in \mathcal{Y}$, it follows that $\theta(\omega) - \tilde{\theta}(\omega) \in \mathcal{Y}$. Let $\pi : \mathcal{U} \to \mathcal{U}/\mathcal{Y}$ be the canonical projection, i.e. the map $\omega \mapsto \omega \bmod \mathcal{Y}$. Then

$$\pi \circ \tilde{\theta} = \pi \circ \theta. \tag{14}$$

Note that $\mathcal{B} = \theta(\mathcal{U})$ and $\tilde{\mathcal{B}} = \tilde{\theta}(\mathcal{U})$. Furthermore, $\pi|\mathcal{B} \to \mathcal{U}/\mathcal{Y}$ and $\pi|\tilde{\mathcal{B}} \to \mathcal{U}/\mathcal{Y}$ are isomorphisms[15] such that $\pi(\theta\omega) = \pi(\tilde{\theta}\omega)$. In particular, if $\theta\Gamma$ is a nonsingular finite-dimensional codistribution defined on S, then

$$\dim \tilde{\theta}(\Gamma) = \dim \theta(\Gamma). \tag{15}$$

Proposition 3.1: *Let θ be any adapted projection defined on an open set $U \subset S$. Let Ω_0 be a codistribution defined on U and let $\Gamma_0 = \theta\Omega_0$. Let $\tilde{\Gamma}_0 = \Omega_0 + \mathcal{Y}$. Define the relative derived flags*[16]

$$\Gamma_k = \operatorname{span}\{\omega \in \Gamma_{k-1} | \dot{\omega} \in \Gamma_{k-1} + \mathcal{Y}\}, \quad k \in \mathbb{N}$$

and

$$\tilde{\Gamma}_k = \text{span}\{\omega \in \tilde{\Gamma}_{k-1} | \dot{\omega} \in \tilde{\Gamma}_{k-1}\}, \quad k \in \mathbb{N}.$$

Then, one has $\theta\tilde{\Gamma}_k = \Gamma_k$ *and* $\Gamma_k \oplus \mathcal{Y} = \tilde{\Gamma}_k$, *for all* $k \in \mathbb{N}$. *Furthermore*[17]

$$\Gamma_k = \{\omega \in \Gamma_{k-1} \mid \theta\dot{\omega} \in \Gamma_{k-1}\}. \tag{16}$$

Proof: It is clear that $\Gamma_{k+1} \subset \Gamma_0$, for $k = 0, 1, \ldots$. As $\Gamma_0 \cap \mathcal{Y} = \{0\}$ it follows that $\Gamma_k \cap \mathcal{Y} = \{0\}$ for $k = 0, 1, \ldots$. By definition, from (13), the properties hold for $k = 0$. By induction, assume that $\theta\tilde{\Gamma}_k = \Gamma_k$ and $\Gamma_k \oplus \mathcal{Y} = \tilde{\Gamma}_k$. Let $\omega \in \Gamma_{k+1}$. Then it will be shown that $\theta(\omega) = \omega$ and $\omega \in \tilde{\Gamma}_{k+1}$. In particular, one concludes that $\Gamma_{k+1} \subset \theta\tilde{\Gamma}_{k+1}$. In fact, the first statement is a consequence of the fact that $\theta|\mathcal{B}$ is the identity map, where $\mathcal{B} = \text{im}\,\theta$ and $\Gamma_{k+1} \subset \Gamma_0 \subset \theta\Omega_0$. Now let $\omega \in \Gamma_{k+1}$. It follows that $\dot{\omega} \in \Gamma_k + \mathcal{Y} = \tilde{\Gamma}_k$. Then $\omega \in \tilde{\Gamma}_{k+1}$.

Now let $\tilde{\omega} \in \tilde{\Gamma}_{k+1}$. Then, $\dot{\omega} \in \tilde{\Gamma}_k$. Since $\theta\omega = \omega + \eta$ for some $\eta \in \mathcal{Y}$, then $\frac{\mathrm{d}}{\mathrm{d}t}(\theta\omega) = \dot{\omega} + \dot{\eta}$. As $\dot{\eta} \in \mathcal{Y}$ and $\Gamma_k \oplus \mathcal{Y} = \tilde{\Gamma}_k$, then $\frac{\mathrm{d}}{\mathrm{d}t}(\theta\omega) \in \Gamma_k + \mathcal{Y}$. Hence $\theta\omega \in \Gamma_{k+1}$ and so, $\Gamma_{k+1} \supset \theta\tilde{\Gamma}_{k+1}$. Now, as $\Gamma_{k+1} = \theta\tilde{\Gamma}_{k+1}$, by (13), it follows that $\theta\tilde{\Gamma}_{k+1} + \mathcal{Y} = \Gamma_{k+1} \oplus \mathcal{Y} = \tilde{\Gamma}_{k+1} + \mathcal{Y}$. Since $\tilde{\Gamma}_0 \supset \mathcal{Y}$, it is easy to see that $\tilde{\Gamma}_k \supset \mathcal{Y}$ for $k \in \mathbb{N}$. In fact, this follows from the fact that $\omega \in \mathcal{Y}$ implies that $\dot{\omega} \in \mathcal{Y}$. Hence, $\Gamma_{k+1} \oplus \mathcal{Y} = \tilde{\Gamma}_{k+1}$. Equation (16) can be proved in a similar way and is left to the reader. □

Given a control system S, assume that $\Gamma = \text{span}\{\omega_1, \ldots, \omega_s\}$ is a smooth codistribution defined on an open set $U \subset S$. One may define the first term of the derived flag in two different forms. The first one, when one regards Γ as a $C^\infty(U)$-submodule

$$\Omega_1 = \text{span}\{\omega \in \Gamma \mid \dot{\omega} \in \Gamma\}. \tag{17}$$

The second one is when one regards things pointwise. At a point $\nu \in U$, $\Gamma|_\nu$ is a $\mathbb{R}$-subspace of $T^*_\nu U$ and then

$$\Gamma_1|_\nu = \text{span}\{\omega|_\nu \in \Gamma \mid \dot{\omega}|_\nu \in \Gamma|_\nu\}. \tag{18}$$

The following proposition states that, under some regularity assumptions, the pointwise definition (18) coincides (locally) with (17). The technique that is used in its proof is useful for the computation of derived flags.

Proposition 3.2: *Define* $\Gamma_1|_\nu$ *for* $\nu \in U$ *by* (18). *Assume that the family* $\{\omega_1, \ldots, \omega_s\}$ *is pointwise independent on* U. *Let* $\xi \in U$ *be a regular point of* Γ_1. *Then,* Γ_1 *is locally smooth around* ξ *and the two definitions* (18) *and* (17) *are locally equivalent.*

Proof: The proof is deferred to Appendix D. □

Now let $\theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B}$ be an adapted projection. Let $\Gamma = \text{span}\{\omega_1, \ldots, \omega_s\} \subset \mathcal{B}$. Then, analogously to the definitions above, one may define the relative derived flags in two different ways. The first one regards codistributions as $C^\infty(U)$-modules

$$\Omega_1 = \text{span}\{\omega \in \Gamma \mid \dot{\omega} \in \Gamma \oplus \mathcal{Y}\}. \tag{19}$$

The second one is the pointwise definition. At a point $\nu \in U$, $\Gamma|_\nu$ is a $\mathbb{R}$-subspace of $T^*_\nu U$ and then

$$\Gamma_1|_\nu = \text{span}\{\omega|_\nu \in \Gamma \mid \dot{\omega}|_\nu \in \Gamma|_\nu \oplus \mathcal{Y}|_\nu\}. \tag{20}$$

The following proposition generalises Proposition 3.2 to relative derived flags. As the proof is essentially the same, it furnishes an algorithm for computing relative derived flags.

Proposition 3.3: *Define* $\Gamma_1|_\nu$ *for* $\nu \in U$ *by* (20). *Assume that the family* $\{\omega_1, \ldots, \omega_s\}$ *is pointwise independent on* U. *Let* $\xi \in U$ *be a regular point of* Γ_1. *Then,* Γ_1 *is locally smooth around* ξ *and the two definitions* (20) *and* (19) *are locally equivalent.*

Proof: It is not difficult to show from (16) that the proof of the proposition can be obtained from the proof of Proposition 3.2 if one replaces $\dot{\omega}|_\nu$ by $\theta\dot{\omega}|_\nu$ in that proof. □

4. Classical state representations of implicit systems

The state representation problem studied in this article is now stated in a precise manner.

Definition 4.1 (State representation problem for implicit systems)**:** Assume that a implicit system (6) is regular. Consider the explicit system S with output y defined by (7). Let $v = (v_1, \ldots, v_s)$ be a set of functions defined around a point $\nu \in \tilde{\Delta} \subset S$, where $\tilde{\Delta}$ is defined by (10).[18] The set v is the *input candidate* of the implicit system. The *state representation problem for an implicit system with input* v around some point $\nu \in \tilde{\Delta}$ is the problem of finding a proper state representation $((z_a, z), (v_a, v))$ for the explicit system S, defined around ν, that is strongly adapted to subsystem Y, if one exists.

Remark 4.2: Roughly speaking, v is the data of the problem, and the question is to verify the existence of such z. By Definition 2.3, system S admits (locally) a state representation $((x_a, x_b), (u_a, u_b))$ with state representation (9). If u_b coincides with v in (9), then the problem is solved. However, this is not necessarily the case.

In fact, the problem is locally solvable around some $\nu \in \tilde{\Delta}$, according to Definition 2.1, if and only if S admits local state equations that are strongly adapted to Y, given by

$$\dot{z}_a = f_a(t, z_a, v_a) \tag{21a}$$

$$\dot{z} = f_b(t, z_a, z, v_a, v). \tag{21b}$$

Furthermore, it follows by the proof of Proposition 2.5, that $\tilde{\Delta}$ is locally equivalent to the implicit system (6), (z, v) is a local state representation of $\tilde{\Delta}$ and the implicit system admits local proper state equations of the form

$$\dot{z} = f_b(t, 0, z, 0, v). \tag{22}$$

The following theorem gives necessary and sufficient conditions for the solution of the *state representation problem for regular implicit systems*.

Theorem 4.3: *Let $((x_a, x_b), (u_a, u_b))$ be a local proper state representation of system S defined by* (7) *that is adapted to Y. Let θ be the associated adapted projection* (*Section* 3). *Let $\mathcal{Y}$ be the codistribution, defined on S by* (8). *Let γ be the least nonnegative integer*[19] *such that one may locally write that* $\text{span}\{\text{d}v\} \subset \text{span}\{\text{d}x_b, \text{d}u_b^{(0)}, \ldots, \text{d}u_b^{(\gamma)}\} \oplus \mathcal{Y}$.

Then the state representation problem for the implicit system (6) *with input v is solvable around some $\nu \in S$ if and only if there exists a nonnegative integer δ such that, for the codistribution Γ_0 defined by*

$$\Gamma_0 = \text{span}\left\{\text{d}x_b, \text{d}u_b^{(0)}, \ldots, \text{d}u_b^{(\gamma)}, \theta\left(\text{d}v^{(0)}\right), \ldots, \theta\left(\text{d}v^{(\delta)}\right)\right\} \tag{23}$$

and for the codistributions Γ_k defined by

$$\Gamma_k = \text{span}\{\omega \in \Gamma_{k-1} | \dot{\omega} \in \Gamma_{k-1} + \mathcal{Y}\} \tag{24}$$

one has:

(i) *Γ_k is finite-dimensional and nonsingular and $\dim \Gamma_{k-1} - \dim \Gamma_k = \dim v$ for $k = 1, \ldots, \delta + 1$.*
(ii) *$\Gamma_{\delta+1} \oplus \mathcal{Y}$ is integrable.*
(iii) *$\Gamma_0 = \Gamma_1 \oplus \text{span}\{\theta\ (\text{d}v^{(\delta)})\}$.*
(iv) *The set $\mathbb{V}^{(k)} = \{\theta(\text{d}v^{(k)})\}$ is locally linearly independent for $k = 0, \ldots, \delta$.*

The proof of Theorem 4.3 is based on the following auxiliary results.

Lemma 4.4: *Let S be the system* (7). *Let $\mathcal{Y}$ be the codistribution defined by* (8). *Let $(\tilde{z}, \tilde{v}) = ((z_a, z), (v_a, v))$ and $(\tilde{x}, \tilde{u}) = ((x_a, x_b), (u_a, u_b))$ be two local proper state representations of S that are strongly adapted to subsystem Y around some point $\xi \in S$.*

(1) *If one has*[20] $\text{span}\{\text{d}v\} \subset \text{span}\{\text{d}x_b, \text{d}u_b, \ldots, \text{d}u_b^{(\alpha)}\} \oplus \mathcal{Y}$ *for* $\alpha \in \mathbb{N}$ *then* $\text{span}\{\text{d}z\} \subset \text{span}\{\text{d}x_b, \text{d}u_b, \ldots, \text{d}u_b^{(\alpha-1)}\} \oplus \mathcal{Y}$.
(2) *Let $\beta \in \mathbb{N}$ be the smallest integer for which one may locally write*, $\text{span}\{\text{d}\tilde{u}\} \subset \text{span}\{\text{d}t, \text{d}\tilde{z}, d\tilde{v}^{(0)}, \ldots, \text{d}\tilde{v}^{(\beta)}\}$. *If* $\text{span}\{\text{d}\tilde{z}, d\tilde{v}\} \subset \text{span}\{\text{d}t, \text{d}\tilde{x}, \text{d}\tilde{u}^{(0)}, \ldots, \text{d}\tilde{u}^{(\gamma)}\}$ *then $\beta \leq n + m\gamma$, where $n = \text{card}\,\tilde{x}$ and $m = \text{card}\,\tilde{u}$.*

Proof: See Appendix B. □

The following corollary[21] depends on the technical conditions for the existence of strongly adapted state equations given in Theorem E.2 of Appendix E.

Corollary 4.5: *Let S be the explicit system with state representation (x, u) defined by* (7). *Assume that the conditions of Theorem* E.2 *hold. Let $x = (x_a, x_b)$ and $u = (u_a, u_b)$ be strongly adapted state representation constructed by Theorem* E.2. *Let $((z_a, z),(v_a, v))$ be a local proper state representation around $\xi \in S$ that is also strongly adapted to the output subsystem Y. Let $\alpha \in \mathbb{N}$ be an integer such that* $\text{span}\{\text{d}v\} \subset \text{span}\{\text{d}t, \text{d}x, \text{d}u, \ldots, \text{d}u^{(\alpha)}\} \oplus \mathcal{Y}$. *Then* $\text{span}\{\text{d}z\} \subset \text{span}\{\text{d}t, \text{d}x, \text{d}u, \ldots, \text{d}u^{(\alpha-1)}\} \oplus \mathcal{Y}$.

Proof: By Theorem E.2, $\text{span}\{\text{d}x\} + \mathcal{Y} = \text{span}\{\text{d}x_b\} \oplus \mathcal{Y}$ and $\text{span}\{\text{d}x, \text{d}u\} + \mathcal{Y} = \text{span}\{\text{d}x_b, \text{d}u_b\} \oplus \mathcal{Y}$. By differentiation of the last condition, it follows that $\text{span}\left\{\text{d}x, \text{d}u, \ldots, \text{d}u^{(\alpha)}\right\} + \mathcal{Y} = \text{span}\{\text{d}x_b, \text{d}u_b, \ldots, \text{d}u_b^{(\alpha)}\} \oplus \mathcal{Y}$ for $\alpha \in \mathbb{N}$. The corollary then follows from Lemma 4.4. Note that the (local) existence of such α is implied by the fact that $\{t, x_a, x_b, (u_a^{(k)}, u_b^{(k)} : k \in \mathbb{N})\}$ is a local coordinate system for which $\mathcal{Y} = \text{span}\left\{t, x_a, (u_a^{(k)} : k \in \mathbb{N})\right\}$. □

Lemma 4.6: *Assume that the local state representation $((z, z_a), (v, v_a))$ of S is a solution of the state representation problem for a given implicit system (S, y). Let $\delta \in \mathbb{N}$, and let $\mathcal{Y}$ be given by* (8). *Define $\tilde{\Gamma}_0 = \text{span}\left\{\text{d}z, \text{d}v^{(0)}, \ldots, \text{d}v^{(\delta)}\right\} \oplus \mathcal{Y}$ and $\tilde{\Gamma}_k = \{\omega \in \tilde{\Gamma}_k \mid \dot{\omega} \in \tilde{\Gamma}_k\}$, for $k \in \mathbb{N}$. Then,*

$$\tilde{\Gamma}_k = \text{span}\left\{\text{d}z, \text{d}v^{(0)}, \ldots, \text{d}v^{(\delta-k)}\right\} \oplus \mathcal{Y} \quad \textit{for } 0 \leq k \leq \delta. \tag{25a}$$

$$\tilde{\Gamma}_{\delta+1} = \text{span}\{\text{d}z\} \oplus \mathcal{Y}. \tag{25b}$$

Now, let θ be an arbitrary adapted projection (*Section* 3). *Let $\Gamma_k = \theta\tilde{\Gamma}_k$, and $L_k = \{\theta(\text{d}z), \theta(\text{d}v^{(0)}), \ldots, \theta(\text{d}v^{(\delta-k)})\}$. Then L_k is linearly independent, $\Gamma_k = \text{span}\{L_k\}$ and $\tilde{\Gamma}_k = \Gamma_k \oplus \mathcal{Y}$ for $k = 0, 1, \ldots, \delta + 1$.*

Proof: Some calculations show (25a) and (25b). Let $\tilde{\theta}$ be the adapted projection associated with the adapted state representation $((z, z_a), (v, v_a))$. Then $\tilde{\theta}(\text{d}z) = \text{d}z$ and $\tilde{\theta}(\text{d}v^{(k)}) = \text{d}v^{(k)}$ for $k \in \mathbb{N}$. In particular, the set $\tilde{L}_k = \{\tilde{\theta}(\text{d}z), \tilde{\theta}(\text{d}v^{(0)}), \ldots, \tilde{\theta}(\text{d}v^{(\delta-k)})\}$ is linearly independent. The proof may be completed using (15) and Proposition 3.1. □

Lemma 4.7: *Let $\theta : \mathcal{B} \oplus \mathcal{Y} \to \mathcal{B}$ be an adapted projection associated with some* (*strongly*) *adapted state representation $((x_a, x_b), (u_a, u_b))$ that is defined in some open neighbourhood U of ξ. Assume that $\Gamma \subset \mathcal{B}$ is nonsingular and finite-dimensional and suppose that $\Gamma \oplus \mathcal{Y}$ is integrable. Then there exists a set*

$z=\{z_1,\dots,z_s\}$ of s smooth functions that are locally defined on some open neighbourhood $V\subset U$ of ξ such that, on V one has $\Gamma=\theta(\mathrm{span}\{\mathrm{d}z\})$, $\Gamma\oplus\mathcal{Y}=\mathrm{span}\{\mathrm{d}z\}\oplus\mathcal{Y}$, and the set $\{\theta\mathrm{d}z_1,\dots,\theta\mathrm{d}z_s\}$ is linearly independent pointwise.

Proof: Since Γ is nonsingular and finite dimensional, one locally has $\Gamma=\mathrm{span}\{\omega_1,\dots,\omega_s\}$ for convenient one-forms ω_i defined on U. To say that $\Gamma\oplus\mathcal{Y}$ is integrable is equivalent to writing

$$\mathrm{d}\omega_i=\sum_{p=1}^{s}\eta_p\wedge\omega_p+\sum_{j=1}^{n_a}\epsilon_j\wedge\mathrm{d}x_{a_j}+\sum_{k=1}^{m_a}\sum_{l=1}^{\beta}\gamma_{kl}\wedge\mathrm{d}u_{a_k}^{(l)}+\zeta\wedge\mathrm{d}t$$

for convenient one-forms η_p, ϵ_j, γ_{kj} and ζ defined on U, for $p=1,\dots,s$, $j=1,\dots,n_a$, $k=1,\dots,m_a$ and $l=1,\dots,\beta$. This means that the finite-dimensional codistribution $\Gamma\oplus\mathcal{Y}_\beta$ is integrable, where $\mathcal{Y}_\beta=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_a,\mathrm{d}u_b^{(0)},\dots,\mathrm{d}u_b^{(\beta)}\}$. An application of finite-dimensional Frobenius theorem constructs a local basis $\mathbb{W}=\{\mathrm{d}w_1,\dots,\mathrm{d}w_r\}$ of $\Gamma\oplus\mathcal{Y}_\beta$, where the w_i are smooth functions defined on some open neighbourhood of ξ. Hence we may locally complete the set $\{\mathrm{d}t,\mathrm{d}x_a,\mathrm{d}u_a^{(0)},\dots,\mathrm{d}u_a^{(\beta)}\}$ with elements $\{\mathrm{d}w_{i_1},\dots,\mathrm{d}w_{i_s}\}\subset\mathbb{W}$, forming a local basis of $\Gamma\oplus\mathcal{Y}_\beta$. In particular, one may chose $z=(w_{i_1},\dots,w_{i_s})$.

Note that $\mathrm{span}\{\mathrm{d}z\}\oplus\mathcal{Y}_\beta=\Gamma\oplus\mathcal{Y}_\beta$. In particular one has $\dim\Gamma=\dim\mathrm{span}\,\{\mathrm{d}z\}=\mathrm{card}\,z$ and $\mathrm{span}\{\mathrm{d}z\}+\mathcal{Y}=\Gamma\oplus\mathcal{Y}$. Since $\Gamma\subset\mathcal{B}$, $\theta|\mathcal{B}$ is the identity map, and $\ker\theta=\mathcal{Y}$, then $\theta(\Gamma\oplus\mathcal{Y})=\Gamma$. Note also that $\theta(\mathrm{span}\{\mathrm{d}z\}+\mathcal{Y})=\theta\,\mathrm{span}\{\mathrm{d}z\}$. Hence $\Gamma=\theta\,\mathrm{span}\{\mathrm{d}z\}$. Since $\dim\Gamma=\dim\mathrm{span}\{\mathrm{d}z\}=\mathrm{card}\,z$, it follows that the set $\{\theta\mathrm{d}z_1,\dots,\theta\mathrm{d}z_s\}$ is independent. To conclude the proof, note that $\sum_{i=1}^{s}\alpha_i\,\mathrm{d}z_i+\eta_i=0$ for some $\eta_i\in\mathcal{Y}$ implies that $\sum_{i=1}^{s}\alpha_i\theta\,\mathrm{d}z_i=0$. Hence $\mathrm{span}\{\mathrm{d}z\}\cap\mathcal{Y}=\{0\}$. □

Lemma 4.8: *Let $(x,u)=((x_a,x_b),(u_a,u_b))$ be a state representation that is defined in some open neighbourhood U of ξ and assume that it is strongly adapted to the output subsystem Y. Let z, v be sets of smooth functions defined on U such that*

(1) $\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}$ *is linearly independent modulo $\mathcal{Y}$ at every point ξ of U.*
(2) $\mathrm{span}\{\mathrm{d}\dot{z}\}\subset\mathrm{span}\{\mathrm{d}z,\mathrm{d}v\}\oplus\mathcal{Y}$.
(3) $\mathrm{span}\{\mathrm{d}x_b\}\subset\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta-1)}\}\oplus\mathcal{Y}$ *and* $\mathrm{span}\{\mathrm{d}u_b\}\subset\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}\oplus\mathcal{Y}$ *for some* $\delta\in\mathbb{N}$.

Let $c\in\mathbb{N}$. Let $\tilde{x}_a=(x_a,u_a^{(0)},\dots,u_a^{(c-1)})$ and $\tilde{u}_a=u_a^{(c)}$. There exists a convenient $c\in\mathbb{N}$ and an open neighbourhood $V\subset U$ of ξ such that, $((\tilde{x}_a,z),(\tilde{u}_a,v))$ is also a local state representation that is (strongly) adapted to Y on V with local state equations given by

$$\dot{x}_a=f_a(t,\tilde{x}_a,\tilde{u}_a)\tag{26a}$$
$$\dot{z}=f_z(t,z,v,\tilde{x}_a,\tilde{u}_a).\tag{26b}$$

Proof: See Appendix C. □

Proof (of Theorem 4.3)**:**
(Necessity). By Definition 4.1, there exists a local proper state representation $((z_a,z),(v_a,v))$ of S that is strongly adapted to Y. In particular $\{t,z,z_a,(v_a^{(k)},\ v^{(k)}:k\in\mathbb{N})\}$ is a local coordinate system of the diffiety S. Let $\mathcal{Y}$ be given by (8). Then, by part (1) of Lemma 4.4, one has $\mathrm{span}\{\mathrm{d}x_b\}\subset\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\beta-1)}\}\oplus\mathcal{Y}$, and $\mathrm{span}\{\mathrm{d}u_b\}\subset\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\beta)}\}\oplus\mathcal{Y}$, for β big enough. In the same way, one may locally write[22] $\mathrm{span}\{\mathrm{d}z\}\subset\mathrm{span}\{\mathrm{d}x_b,\mathrm{d}u_b^{(0)},\dots,\mathrm{d}u_b^{(\gamma-1)}\}\oplus\mathcal{Y}$ and $\mathrm{span}\{\mathrm{d}v\}\subset\mathrm{span}\{\mathrm{d}x_b,\mathrm{d}u_b^{(0)},\dots,\mathrm{d}u_b^{(\gamma)}\}\oplus\mathcal{Y}$ for some convenient γ. Now take $\delta=\beta+\gamma$. Since

$$\mathrm{span}\{\mathrm{d}\dot{x}_b\}\subset\mathrm{span}\{\mathrm{d}x_b,\mathrm{d}u_b\}\oplus\mathcal{Y},\tag{27}$$

by derivation, it follows that

$$\mathrm{span}\left\{\mathrm{d}x_b,\mathrm{d}u_b^{(0)},\dots,\mathrm{d}u_b^{(\gamma)}\right\}\oplus\mathcal{Y}\subset\mathrm{span}\left\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\right\}\oplus\mathcal{Y}.\tag{28}$$

Let $\mathcal{B}=\mathrm{span}\{\mathrm{d}x_b,(\mathrm{d}u_b^{(k)}:k\in\mathbb{N})\}$. Let $\theta:\mathcal{B}\oplus\mathcal{Y}\to\mathcal{B}$ be the corresponding adapted projection. Recall that $\theta|\mathcal{B}$ is the identity map and $\theta(\mathcal{Y})=0$. Let Γ_0 be defined by (23). Then it is clear that $\theta(\Gamma_0\oplus\mathcal{Y})=\Gamma_0$.

Now define $\Omega_0=\mathrm{span}\{\mathrm{d}x_b,\mathrm{d}u_b,\dots,\mathrm{d}u_b^{(\gamma)},\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}$. It is clear that $\Gamma_0=\theta\Omega_0$. By Proposition 3.1 for $k=0$, $\Gamma_0\oplus\mathcal{Y}=\Omega_0+\mathcal{Y}$. By (28), one has $\Omega_0+\mathcal{Y}=\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}\oplus\mathcal{Y}$. By Definition 4.1 (see also Definition 2.1), one notes that $\{t,\,z,\,z_a,\,(v^{(k)},\,v_a(k):k\in\mathbb{N})\}$ is a local coordinate system and $\mathcal{Y}=\mathrm{span}\{t,\ z_a,\ (v_a(k)\ :\ k\in\mathbb{N})\}$. It follows that $\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}\cap\mathcal{Y}=0$. Then $\Omega_0+\mathcal{Y}=\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}\oplus\mathcal{Y}$. In particular, from Proposition 3.1 for $k=0$, one shows that, for γ and δ previously constructed, then $\Gamma_0\oplus\mathcal{Y}=\mathrm{span}\{\mathrm{d}z,\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}\oplus\mathcal{Y}$.

The proof of necessity can be reached using Lemma 4.6.

(Sufficiency). It will be shown first that

$$\theta\big(\mathrm{span}\left\{\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta-k)}\right\}\big)\subset\Gamma_k,\quad k=0,\dots,\delta\tag{29a}$$

and that

$$\Gamma_k=\Gamma_{k+1}\oplus\theta\big(\mathrm{span}\left\{\mathrm{d}v^{(\delta-k)}\right\}\big),\quad k=0,\dots,\delta.\tag{29b}$$

Take $\Omega_0=\mathrm{span}\{\mathrm{d}x_b,\mathrm{d}u_b^{(0)},\dots,\mathrm{d}u_b^{(\gamma)},\mathrm{d}v^{(0)},\dots,\mathrm{d}v^{(\delta)}\}$. Note that $\Gamma_0=\theta\Omega_0$. Then, the condition (29a) is a straightforward consequence of the definition of (23), (24) and Proposition 3.1. Equation (29b) will be shown by induction. Note that (29b) coincides with (iii) for $k=0$. Assume that it holds for some k, with $0 \le k \le \delta-1$. Let $V_{\delta-k-1} = \theta(\text{span}\{dv^{(\delta-k-1)}\})$. By contradiction, assume that $\{\Gamma_{k+2} \cap V_{\delta-k-1}\}|_\nu \neq \{0\}$ for some ν in the neighbourhood of definition of the state representation $((x_a, x_b), (u_a, u_b))$. By (iv), (29a) and the nonsingularity of Γ_{k+1}, one may construct a local basis of Γ_{k+1} of the form $\{\omega_1, \dots, \omega_s, \theta(dv^{(\delta-k-1)})\}$. Let $\omega \in \Gamma_{k+2} + V_{\delta-k-1}$ be a smooth one form locally defined on S such that $\omega_\nu = \omega(\nu) \neq 0$ and $\omega_\nu \in \{\Gamma_{k+2} \cap V_{\delta-k-1}\}|_\nu$. Since $\Gamma_{k+2} \subset \Gamma_{k+1}$, this is equivalent to say that $\omega = \sum_{i=1}^{s} \alpha_i \omega_i + \sum_{j=1}^{\dim v} \beta_j \theta(dv_j^{(\delta-k-1)})$, where $\alpha_i|_\nu = 0$ for all $i = 1, \dots, s$, but some $\beta_i|_\nu \neq 0$. It follows from (24) that

$$\dot{\omega} = \sum_{i=1}^{s} [\dot{\alpha}_i \omega_i + \alpha_i \dot{\omega}_i] + \sum_{i=1}^{\dim v} [\dot{\beta}_j \theta(dv_j^{(\delta-k-1)}) + \beta_j \theta(dv_j^{(\delta-k)})] \in \Gamma_{k+1} + \mathcal{Y}.$$

Note that $\sum_{i=1}^{s} \dot{\alpha}_i \omega_i + \sum_{i=1}^{\dim v} \dot{\beta}_j \theta\big(dv_j^{(\delta-k-1)}\big) \in \Gamma_{k+1}$. It follows that $\eta|_\nu = \big\{\sum_{i=1}^{\dim v} \beta_j \theta(dv_j^{(\delta-k)})\big\}|_\nu \in \Gamma_{k+1}|_\nu \oplus \mathcal{Y}|_\nu$. But $\eta|_\nu$ is in the image of θ and hence it is easy to show[23] that $\eta|_\nu \in \Gamma_{k+1}|_\nu$. Hence, $\eta|_\nu \in \Gamma_{k+1}|_\nu \cap V_{\delta-k}|_\nu$, with $\eta|_\nu \neq 0$ and this contradicts the induction hypothesis.

Now it is easy to see that (29b) is a consequence of (29a), of (iv), of the fact that $\Gamma_{k+1} \cap \text{span}\{\theta(dv^{\delta-k})\} = 0$, and of the fact that $\dim \Gamma_k - \dim \Gamma_{k+1} = \dim v$, for $k = 0, 1, \dots, \delta+1$. Applying Lemma 4.7 to $\Gamma_{\delta+1}$, one may write $\Gamma_{\delta+1} = \theta\ (\text{span}\{dz\})$, where the set $\theta(dz) = \{\theta(dz_1), \dots, \theta(dz_s)\}$ is linearly independent. From (29b) it follows that the set $L_k = \{\theta(dz), \theta(dv^{(0)}), \dots, \theta(dv^{(\delta-k)})\}$, $k = 0, \dots, \delta$ is linearly independent and $\Gamma_k = L_k$ for $k = 0, \dots, \delta$.

By (23), one may write

$$\text{span}\{du_b\} \subset \Gamma_0 = L_0. \tag{30}$$

Now, by (27) and (24) with $k=1$, it follows that

$$\text{span}\{d\dot{x}_b\} \subset \Gamma_1 = L_1. \tag{31}$$

Let $H_k = \text{span}\{dz, dv^{(0)}, \dots, dv^{(\delta-k)}\}$. Hence $L_k = \theta H_k$ and from (13), it follows that

$$H_k + \mathcal{Y} = L_k \oplus \mathcal{Y}, \quad k = 0, \dots, \delta+1. \tag{32}$$

Now note that

(1) Since L_0 is linearly independent,[24] then $\{dz, dv^{(0)}, \dots dv^{(\delta)}\}$ is linearly independent *modulo* $\mathcal{Y}$.
(2) $\text{span}\{d\dot{z}\} \subset \text{span}\{dz, dv\} \oplus \mathcal{Y}$ (by the definition of Γ_k in (i), from the fact that $\Gamma_k = L_k$, and from (32)).
(3) $\text{span}\{dx_b\} \subset \text{span}\{dz, dv^{(0)}, \dots, dv^{(\delta-1)}\} \oplus \mathcal{Y}$ and $\text{span}\, du_b \subset \text{span}\{dz, dv^{(0)}, \dots, dv^{(\delta)}\} \oplus \mathcal{Y}$ (by (30)–(32)).

The proof of sufficiency then follows from Lemma 4.8. □

The next result is an 'intrinsic' version of Theorem 4.3. It is not suitable for computations, because it deals with rather abstract, infinite-dimensional objects. However, it is useful in order to compare the main result with the ones of Pereira da Silva and Batista (2009) (Section 6).

Theorem 4.9: *Assume that the conditions of Theorem* E.2 *hold* (*Appendix E*). *Consider the explicit system* (6). *Let* $\mathcal{Y}$ *be the codistribution, defined on S by* (8). *There exists* $\gamma \in \mathbb{N}$ *such that* γ *is the least integer for which* $\text{span}\{dv\} \subset \text{span}\{dt, dx, du, \dots, du^{(\gamma)}\} \oplus \mathcal{Y}$. *Then the state representation problem for the implicit system* (6) *with input v is solvable if and only if there exists a non-negative integer* δ *such that, for the codistribution* $\tilde{\Gamma}_0$ *defined on S by*

$$\tilde{\Gamma}_0 = \text{span}\big\{dx, du^{(0)}, \dots, du^{(\gamma)}, dv^{(0)}, \dots, dv^{(\delta)}\big\} + \mathcal{Y}$$

and for $\tilde{\Gamma}_k = \text{span}\big\{\omega \in \tilde{\Gamma}_{k-1} \mid \dot{\omega} \in \tilde{\Gamma}_{k-1}\big\}$, *we have*

(i) $\tilde{\Gamma}_k/\mathcal{Y}$ *is finite dimensional and nonsingular for* $k = 0, \dots, \delta+1$, *and* $\dim \frac{\tilde{\Gamma}_{k-1}}{\mathcal{Y}} - \dim \frac{\tilde{\Gamma}_k}{\mathcal{Y}} = \dim v$ *for* $k = 1, \dots, \delta$.
(ii) $\tilde{\Gamma}_{\delta+1}$ *is integrable.*
(iii) $\frac{\tilde{\Gamma}_0}{\mathcal{Y}} = \frac{\tilde{\Gamma}_1}{\mathcal{Y}} \oplus \tilde{V}^{(\delta)}$, *where* $\tilde{V}^{(\delta)} = \text{span}\big\{dv^{(\delta)} \bmod \mathcal{Y}\big\}$.
(iv) *The set* $\{dv^{(k)} \bmod \mathcal{Y}\}$ *is locally linearly independent for* $k = 0, \dots, \delta$.

Proof: If the conditions of Theorem E.2 hold, then one may construct the strongly adapted state representation $((x_a, x_b), (u_a, u_b))$ following the steps (A), (B), (C) and (D) at the end of that theorem. By construction, $\text{span}\{dx\} + \mathcal{Y} = \text{span}\{dx_b\} \oplus \mathcal{Y}$ and $\text{span}\{dx, du\} + \mathcal{Y} = \text{span}\{dx_b, du_b\} \oplus \mathcal{Y}$. By differentiation, $\text{span}\big\{dx, du^{(0)}, \dots, du^{(k)}\big\} + \mathcal{Y} = \text{span}\{dx_b, du_b^{(0)}, du_b^{(k)}\} \oplus \mathcal{Y}$. The existence of γ is a simple consequence of the last equality and the fact that $\{t, x_a, x_b, (u_a^{(k)}, u_b^{(k)} : k \in \mathbb{N})\}$ is a local coordinate system such that $\mathcal{Y} = \text{span}\big\{dt, x_a, (u_a^{(k)} : k \in \mathbb{N})\big\}$ (see the Proof of Corollary 4.5). The result follows easily from the proof of Theorem 4.3, Proposition 3.1 and the properties of adapted projections (Section 3). □

Remark 4.10: The conditions of the Theorems 4.3 and 4.9 are equivalent. However, the conditions of Theorem 4.9 are more intrinsic than the ones of Theorem 4.3, since they do not rely on a particular choice of the adapted state representation, but only on some geometric properties of the original state representation (x, u) of S. Theorem 4.3 is more suitable for the computations of a given example, to verify the solvability of the state representation problem, and to construct solution, if one can integrate $\Gamma_{\delta+1} + \mathcal{Y}$. Note also that part (2) of Lemma 4.4 may furnish a bound for δ. In fact, in the proof of Theorem 4.3, one takes $\delta=\beta+\gamma$, and so as β is bounded by Lemma 4.4, then δ is also bounded by $n+(m+1)\gamma$, where $n=\dim x$ and $m=\dim u$.

5. Examples

Example 1: Consider the implicit system

$$\begin{aligned}\dot{x}_1(t) &= u_1 + 2x_3u_2\\ \dot{x}_2(t) &= tx_3 + 2(x_1+1)u_1 + 4x_1x_3u_2\\ \dot{x}_3(t) &= u_2(t)\\ y(t) &= x_1 - x_3^2 = 0.\end{aligned}$$

Let $x=(x_1,x_2,x_3)$ and $u=(u_1,u_2)$. The input candidate for this example is $v=tx_3+u_1\big(x_1-x_3^2\big)$. First apply Theorem E.1 of Appendix E. In fact, if one writes differentials in the basis $\{\mathrm{d}t,\mathrm{d}x,\mathrm{d}u\}$, representing them as row vectors, one may compute the 6×9 matrix $M_1=[(\mathrm{d}y)^T(\mathrm{d}\dot{y})^T(\mathrm{d}t)^T\mathrm{d}x^T]^T$.

It can be shown that the assumptions of Theorem E.1 hold for $\alpha=1$. In fact, (2) is consequence of the fact that the submatrix of M_1 formed, respectively, by 1st, 3rd, 4th, 5th and 6th rows has constant rank and equal to 4. Note that (3) is consequence of the fact that M_1 has constant rank, equal to 7. It is easy to deduce from the rows of M_1 that (4) follows. Furthermore, note that $\dim\big(\mathrm{span}\{\mathrm{d}x\}\cap\mathrm{span}\{\mathrm{d}t,\mathrm{d}y\}\big)-\dim\big(\mathrm{span}\{\mathrm{d}x\}\cap\mathrm{span}\{\mathrm{d}t,\mathrm{d}y,\mathrm{d}\dot{y}\}\big)=\dim(\mathrm{span}\{\mathrm{d}x\}+\dim(\mathrm{span}\{\mathrm{d}t,\mathrm{d}y\})-\dim(\mathrm{span}\{\mathrm{d}t,\mathrm{d}x,\mathrm{d}y\})-\dim\big(\mathrm{span}\{\mathrm{d}x\}\big)-\dim\big(\mathrm{span}\{\mathrm{d}t,\mathrm{d}y,\mathrm{d}\dot{y}\}\big)+\dim\big(\mathrm{span}\{\mathrm{d}t,\mathrm{d}x,\mathrm{d}y,\mathrm{d}\dot{y}\}\big)=3+2-4-3-3+5=0$. Hence, (1) holds. Following the steps (A), (B), (C) and (D) at the end of Theorem E.1, it is possible to choose $x_a=y=x_1-x_3^2$, $u_a=\dot{y}=u_1$, $x_b=(x_2,x_3)$ and $u_b=u_2$. If one replaces u_b by v, then condition (D) will be not respected any more. Then one may proceed with the algorithm of Appendix F.

In order to perform the next computations, it is useful to write the one-forms in the basis $\mathbb{B}_2=\{\mathrm{d}t,\mathrm{d}x_b,\mathrm{d}u_b,\mathrm{d}\dot{u}_b,\mathrm{d}\ddot{u}_b,\mathrm{d}x_a,\mathrm{d}u_a,\mathrm{d}\dot{u}_a,\mathrm{d}\ddot{u}_a\}$, instead of considering the original basis $\mathbb{B}_1=\{\mathrm{d}t,\mathrm{d}x,\mathrm{d}u,\mathrm{d}\dot{u},\mathrm{d}\ddot{u}\}$. This allows one to compute the adapted projection θ concerning the adapted state representation $((x_a,x_b),(u_a,u_b))$.

Writing $\mathrm{d}v$ in the basis $\mathbb{B}_2$, one obtains $\mathrm{d}v=x_3\,\mathrm{d}t+t\,\mathrm{d}x_{b_2}$. From this, one concludes that $\gamma=0$. Now choose $\delta=2$, and construct[25]

$$\Gamma_0=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_b,\mathrm{d}u_b,\theta\,\mathrm{d}v,\theta\,\mathrm{d}\dot{v},\theta\,\mathrm{d}\ddot{v}\}$$

For a one-form ω, the adapted projection $\theta\omega$ is obtained by writing ω in the basis $\{\mathrm{d}t,\mathrm{d}x_b,(\mathrm{d}u_b^{(k)}:k\in\mathbb{N}),\mathrm{d}x_a,(\mathrm{d}u_a^{(k)}:k\in\mathbb{N})\}$ and by deleting the components in $\mathrm{d}t,\mathrm{d}x_a,(\mathrm{d}u_a^{(k)}:k\in\mathbb{N})$ (Section 3). Note that $\theta\,\mathrm{d}v=tx_{b_1}$, $\theta\,\mathrm{d}\dot{v}=\mathrm{d}x_{b_2}+t\,\mathrm{d}u_b$, and $\theta\,\mathrm{d}\ddot{v}=2\mathrm{d}u_b+t\,\mathrm{d}\dot{u}_b$.

In particular, for[26] $t\neq 0$, one obtains $\Gamma_0=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_b,\mathrm{d}u_b,\mathrm{d}\dot{u}_b\}$. From (9), it is clear that $\mathrm{span}\{\mathrm{d}\dot{x}_b\}\bmod\mathcal{Y}\subset\mathrm{span}\{\mathrm{d}x_b,\mathrm{d}u_b\}\bmod\mathcal{Y}$. It follows from (16) that $\Gamma_1=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_b,\mathrm{d}u_b\}$, and that $\Gamma_2=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_b\}$. To compute Γ_3, note that: $\theta\,\mathrm{d}x_{b_1}=\phi\,\mathrm{d}x_{b_2}+4x_3x_1\,\mathrm{d}u_b$, and that $\theta\,\mathrm{d}x_{b_2}=\mathrm{d}u_b$. Then, one may show from (16) that $\Gamma_3=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_{b_1}-4x_3x_1\mathrm{d}x_{b_2}\}=\mathrm{span}\{\mathrm{d}t,\mathrm{d}x_2-4x_3x_1\mathrm{d}x_3\}$. Now, as $\Gamma_3+\mathrm{span}\{\mathrm{d}y\}$ is integrable, all the assumptions of Theorem 4.3 hold. To see that $\Gamma_3+\mathrm{span}\{\mathrm{d}y\}$ is integrable, it suffices to notice that $\mathrm{d}y=\mathrm{d}x_1-2x_3\,\mathrm{d}x_3$, hence $\Gamma_3+\mathrm{span}\big\{\mathrm{d}y\big\}=\mathrm{span}\big\{\mathrm{d}t,\mathrm{d}x_2-2x_1\,\mathrm{d}x_1,\mathrm{d}y\big\}=\mathrm{span}\big\{\mathrm{d}t,d(x_2-x_1^2),\mathrm{d}y\big\}$. Thus, one may choose $z=x_2-x_1^2$. Now, after some computations[27], it follows that $\dot{z}=\dot{y}+v-y\dot{y}$. Remember that, by the proof of Theorem 4.3, $((z,x_a),(v,u_a))$ is a strongly adapted state representation with state equations given by

$$\begin{aligned}\dot{x}_a &= u_a\\ \dot{z} &= v+x_a-x_au_a.\end{aligned}$$

Hence, a state representation of the implicit system is obtained by taking $y=x_a=0$ and $\dot{y}=u_a=0$ (Remark 4.2). In particular, one gets the state representation

$$\dot{z}=v.$$

Example 2: Consider the input–output equations (with output $w=(w_1,w_2)$ and input $v=(v_1,v_2)$) given by:

$$\begin{aligned}-v_1\dot{w}_1+\big(e^{\dot{w}_1}+1\big)\ddot{w}_1-w_1\dot{v}_1+\big(e^{\dot{w}_1}+\dot{w}_1-w_1v_1\big)^2+v_2=0\\ \dot{w}_2+w_2\big(e^{\dot{w}_1}+\dot{w}_1-w_1v_1\big)v_1=0.\end{aligned}$$

One may convert these equations into a system of the form (6) in the following way. Choose $x=(x_1,\dots,x_8)=(w_1,\dot{w}_1,\ddot{w}_1,w_2,\dot{w}_2,v_1,\dot{v}_1,v_2)$ and let $u=(u_1,u_2,u_3,u_4)=(w_1^{(3)},\dot{w}_2,v_1^{(2)},\dot{v}_2)$. The input–output equations can be written as

$$\begin{aligned}&\dot{x}=f(x,u)\\ &y_1=-x_6x_2+(e^{x_2}+1)x_3-x_1x_7+(e^{x_2}+x_2-x_1x_6)^2+x_8\\ &\quad=h_1(x,u)=0\\ &y_2=x_5+x_4x_6(e^{x_2}+x_2-x_1x_6)=h_2(x,u)=0\end{aligned}$$

with $f(x,u)=(x_2x_3u_1x_5u_2x_7u_3\ u_4)^T$ which is in the form (6). Note that $w_1=x_1$, $w_2=x_4$, $v_1=x_6$ and $v_2=x_8$. Now, after some symbolic computations,[28] one can show that the assumptions of Theorem E.1 hold for $\alpha=1$. Since $\frac{\partial y_1}{x_8}=1$ and $\frac{\partial y_2}{x_5}=1$, it is possible to choose $x_a=y=(y_1,y_2)$, $u_a=\dot{y}$ and $x_b=(x_1,x_2,x_3,x_4,x_6,x_7)$. Since $\frac{\partial\dot{y}_1}{u_4}=1$ and $\frac{\partial\dot{y}_2}{u_2}=1$, then one can choose

$u_b = (u_1, u_3)$. Since o u_b cannot be replaced by v (while respecting the construction of the step (D) of Theorem E.1), one must proceed with Algorithm F. As in the previous example, it is easy to show that $\gamma = 0$. One will choose[29] $\delta = 2$. Denote $x_b = (x_{b_1}, x_{b_2}, x_{b_3}, x_{b_4}, x_{b_5}, x_{b_6})$ and $u_b = (u_{b_1}, u_{b_2})$. After some symbolic computations, one obtains

$$\Gamma_0 = \mathrm{span}\{\mathrm{d}x_b, \mathrm{d}u_b, [(-1-e^{x_2})\mathrm{d}\dot{u}_{b_1} + x_1\,\mathrm{d}\dot{u}_{b_1}]\}$$
$$\Gamma_1 = \mathrm{span}\{\mathrm{d}x_b, [(-1-e^{x_2})\mathrm{d}u_{b_1} + x_1\,\mathrm{d}u_{b_1}]\}$$
$$\Gamma_2 = \mathrm{span}\{\mathrm{d}x_{b_1}, \mathrm{d}x_{b_2}, \mathrm{d}x_{b_4}, \mathrm{d}x_{b_5}, [(-1-e^{x_2})\mathrm{d}x_{b_3} + x_1\,\mathrm{d}x_{b_6}]\}$$
$$\Gamma_3 = \mathrm{span}\{\mathrm{d}x_{b_1}, \mathrm{d}x_{b_4}, [(-1-e^{x_2})\mathrm{d}x_{b_2} + x_1\,\mathrm{d}x_{b_5}]\}.$$

Note that $\Gamma_3 = \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_4, [(-1-e^{x_2})]\ \mathrm{d}x_2 + x_1\mathrm{d}x_6]\} = \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_4,\ \mathrm{d}(-x_2 - e^{x_2} + x_1x_6)\}$. In particular, Γ_3 is integrable.[30] One may take $z = (z_1, z_2, z_3) = (x_1,\ -x_2 - e^{x_2} + x_1x_6,\ x_4)$. Note that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $x_2 \mapsto [x_2 + e^{x_2}]$ is invertible with inverse $\psi : \mathbb{R} \to \mathbb{R}$. In particular, one may write $x_2 = \psi(-z_2 + x_1x_6)$. After some computations one obtains the strongly adapted state representation: $\dot{x}_a = u_a,\ \dot{z}_1 = \psi(-z_2 + z_1v_1),\ \dot{z}_2 = z_2^2 + v_2 - x_{a_1},\ \dot{z}_3 = z_2z_3v_1 + x_{a_2}$. Taking $y = x_a \equiv 0$, one obtains the following (classical) realisation for the original input-output equations (with input (v_1, v_2) and output (w_1, w_2)):

$$\begin{aligned}
\dot{z}_1 &= \psi(-z_2 + z_1v_1)\\
\dot{z}_2 &= z_2^2 + v_2\\
\dot{z}_3 &= z_2z_3v_1\\
w_1 &= z_1\\
w_2 &= z_3.
\end{aligned}$$

Example 3: Consider the implicit system:

$$\begin{aligned}
\dot{x}_1 &= u_1 + ax_4 + ax_6 + (ax_1 - ax_3)^2 + (ax_2 - ax_6)u_3\\
\dot{x}_2 &= e^{(ax_3+ax_5)}u_1 + u_2 + u_4\\
\dot{x}_3 &= -u_1 + ax_4 + ax_6 + (ax_1 - ax_3)^2 + (ax_2 - ax_6)u_3\\
\dot{x}_4 &= -e^{(ax_3+ax_5)}u_1 + u_2 + u_4\\
\dot{x}_5 &= u_1 + ax_4 + ax_6 - (ax_1 - ax_3)^2 - (ax_2 - ax_6)u_3\\
\dot{x}_6 &= e^{(ax_3+ax_5)}u_1 + u_2 - u_4\\
\dot{x}_7 &= ax_1 - ax_3 + (ax_2 - ax_4)u_2\\
y_1 &= ax_1 - ax_3 = 0\\
y_2 &= ax_2 - ax_4 = 0,
\end{aligned}$$

where $a = 1/2$. Let $v = [u_3, u4]$. Computations with MATLAB/MAPPLE shows that the assumptions of Theorem E.1 holds for $\alpha = 3$ (but they hold neither for $\alpha = 1$ nor for $\alpha = 2$). From steps (A) to (D) at the end of that theorem, it is possible to show that one may choose $x_b = (x_5, x_6, x_7)$, $u_b = v = [u_3, u_4]$, $x_a = y$ and $u_a = \dot{y}$. In particular, the problem of state representation with input v for this implicit system is solvable and one may take $z = x_b = (x_5, x_6, x_7)$ (Remark 4.2).

Example 4: Consider the same system of Example 3, but include the new component $y_3 = x_7$ to the output. Now, choose $\bar{y} = (y_1, y_2)$ and $\widehat{y} = y_3$. Since $\dim\mathrm{span}\{\mathrm{d}t, \mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y}\} = \dim\mathrm{span}\{\mathrm{d}t, \mathrm{d}\bar{y}, \mathrm{d}\dot{\bar{y}}, \mathrm{d}\ddot{\bar{y}}\}$, it is easy to show that the assumptions of Theorem E.2 hold. One may show that, in this case, the steps (A) to (D) at the end of that theorem are satisfied for $x_b = (x_5, x_6)$, $u_b = v$, $x_a = \bar{y}$, $u_a = \bar{y}^{(1)}$. In particular, from Remark 4.2 the problem of state representation with Input v for this implicit system is solvable and one may take $z = x_b = (x_5, x_6)$.

Example 5: Recall that the proof of Theorem 4.3 shows that system (7) admits (locally) a strongly adapted state representation (21). From this, one may take $z_a \equiv 0$ and $v_a \equiv 0$, obtaining the state representation of the implicit system given by (22).

The present example shows that the conditions of Theorem 4.3 are not necessary for the existence of a classical state representation of the implicit system with input v. In fact an implicit system (6) may admit a classical state representation (5) that is not associated to any strongly adapted state representation (21) of (7).

For instance, consider the system $\dot{x}_1 = x_2, \dot{x}_2 = x_3 + u_2, \dot{x}_3 = u_1, \dot{x}_4 = x_3 + x_1u_1^2, y = x_1 - \epsilon = 0,$ with $\epsilon \in \mathbb{R}$. Considering the explicit system S (by disregarding the constraint $y \equiv 0$), let $x_a = (x_1, x_2)$, $v_a = \dot{x}_2 = x_3 + u_2$, $x_b = (x_3, x_4)$, $u_b = u_1$. Then it is possible to show that Theorem E.1 holds with $\alpha = 2$. So $((x_a, x_b), (u_a, u_b))$ is a strongly adapted state representation of S with state equations given by

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= v_a\\
\dot{x}_3 &= u_1\\
\dot{x}_4 &= x_3 + x_1u_1^2.
\end{aligned}$$

One concludes that the implicit system admits the state representation

$$\begin{aligned}
\dot{x}_3 &= u_1\\
\dot{x}_4 &= x_3 + \epsilon u_1^2.
\end{aligned}$$

Considering $\epsilon = 0$, simple computations show that the implicit system admits a proper state representation with input $v = x_3$, namely, $\dot{x}_4 = v$. However, it is not difficult to show that the assumptions of Theorem 4.3 are not satisfied for any $\delta \in \mathbb{N}$ (for any $\epsilon \in \mathbb{R}$).

6. Conclusions

The main result of this article may be interpreted in the following way. Recall that the conditions obtained in

Pereira da Silva and Batista (2009) rely on an integrability test based on a derived flag obtained from the codistribution $\Gamma_0 = \{dt, dx, du^{(0)}, \ldots, du^{(\gamma)}, dv^{(0)}, \ldots, du^{(\delta)}\}$, where γ and δ are convenient integers. Theorem 4.3 of this article can be viewed as a generalisation of Pereira da Silva and Batista (2009, Theorem 1), by taking quotients with respect to the codistribution $\mathcal{Y} = \text{span}\{dt, (dy^{(k)} : k \in \mathbb{N}\}$, that is generated by the differentials of time and the constraint functions y and their derivatives. When there are no constraints, then $\mathcal{Y} = \text{span}\{dt\}$ and Theorem 4.3 reduces to Theorem 1 of Pereira da Silva and Batista (2009).

The class of systems of the form (6) include input–output equations (Example 2) and implicit systems of the form[31]

$$F(t, w(t), \dot{w}(t)) = 0. \tag{33}$$

In fact, given the system (33), let $x = w$ and $u = \dot{w}$. Then the system $\dot{x}(t) = u(t)$, $y(t) = F(x(t), u(t)) = 0$ is of the form (6).

Acknowledgements

Paulo Sergio Pereira da Silva was partially supported by CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico (Brazil), Grant Number 308465/2006-7. The authors are indebted to Felipe M. Pait for many interesting suggestions.

Notes

1. See Appendix A for a definition of classical state representation in the context of diffieties that holds for time-varying systems.
2. This choice may be regarded as a virtual input, as in the context of back-stepping, see Krstic, Kanellakopoulos, and Kokotovic (1995).
3. The statement as it stands is imprecise in the sense that some technical assumptions are deferred to Section 4 and also because the word 'equivalent' is yet to be defined (Definition 2.4).
4. The class of state representations of Δ that are considered in this article are restricted to the ones that are induced from a state representations of S that is *strongly adapted* to the output subsystem Y (Definitions 2.1, 4.1 and Remark 4.2).
5. The original input is not necessarily coincident with v.
6. This means that the differential ideal generated by Ω is differentially closed (Warner 1971).
7. It must be pointed out again that $y = h(t, x, u)$ is regarded as an output rather than a constraint.
8. The definition of state representation given in Appendix A considers that $\{t, x, u^{(0)}, u^{(1)}, \ldots\}$ is a local coordinate system, and so the variables $t, x, x, u^{(0)}, u^{(1)}, \ldots$ must not be linked by any relation.
9. See Pereira da Silva and Corrêa Filho (2001), Pereira da Silva et al. (2008).
10. The weaker definition of *adapted* state equations considered in Theorem 4.3 of Pereira da Silva and Corrêa Filho (2001) is obtained if one replaces the assumptions (C) and (D) by the only assumption that $\mathcal{Y} = \text{span}\left\{dt, dx_a, (du_a^{(k)} : k \in \mathbb{N})\right\}$. This last theorem also shows that the output subsystem is locally unique up to local Lie–Bäcklund isomorphisms.
11. Using (C) and (D), one may show that f_a does not depend on t.
12. By definition, Γ is a control system if it locally admits a state representation around every point $\gamma \in \Gamma$.
13. Since ι_* injective, it can be shown that Cartan field $\partial_{\tilde{\Delta}}$ of $\tilde{\Delta}$ may be canonically defined by $\iota_* \partial_{\tilde{\Delta}} = \frac{d}{dt} \circ \iota$, where $\frac{d}{dt}$ is the Cartan field of S.
14. Using the same name of x_b as a set of local coordinate functions of $\tilde{\Delta}$ and S is an abuse of notation. One could write for instance $\tilde{x}_b$ and consider that $\tilde{x}_b = x_b \circ \iota$.
15. Isomorphisms of modules, or isomorphism of vector spaces, depending on the case.
16. In the context of exterior differential systems, derived flags are defined by $\tilde{\Gamma}_k = \{\omega \in \tilde{\Gamma}_k \mid d\omega \in (\tilde{\Gamma}_k)\}$. Such derived flags are considered in Pereira da Silva and Corrêa Filho (2001). Under the assumption of the integrability of the members of the derived flag, Equation A.4 of that paper shows the equivalence between the last definition and the one considered in this work. See also Pereira da Silva (2008) for a similar situation.
17. Equation (16) is a suitable form for computations, since it refers only to finite-dimensional objects.
18. Recall that the components of v may depend on t, x, u, $u^{(1)}, \ldots$.
19. The existence of the integer γ is assured by the fact that a state representation is a local coordinate system.
20. If $\alpha = 0$, then $\text{span}\{dx_b, du_b, \ldots, du_b^{(\alpha-1)}\}$ stands for $\text{span}\{dx_b\}$.
21. This corollary is used only in Theorem 4.9.
22. If $\gamma = 0$, one assumes that $\text{span}\{dz\} \subset \text{span}\{dx_b\} \oplus \mathcal{Y}$.
23. Since $\Gamma_{k+1} \subset \mathcal{B}$, $\theta|\mathcal{B}$ is the identity map, $\ker\theta = \mathcal{Y}$, $\theta(\eta|_v) = \eta|_v \in \theta(\Gamma_{k+1}|_v + \mathcal{Y}|_v) = \Gamma_{k+1}|_v$.
24. If $\omega = \sum_i \alpha_i dz_i + \sum_{j,k} \beta_{jk} dv_j^{(k)} + \eta = 0$, where $\eta \in \mathcal{Y}$, for convenient smooth functions α_i, β_{ij}, then $\theta\omega = 0$ and $\ker\theta = \mathcal{Y}$ implies that L_0 is linearly dependent.
25. The reader may verify that the assumption (iii) of Theorem 4.3 fails for $\delta = 0$ and $\delta = 1$.
26. Note that every point of S such that $t = 0$ is a singular point of Γ_0.
27. These computations were performed by Matlab/Maple®.
28. These symbolic computations were also performed by Matlab/Maple®.
29. The reader may verify that the assumption (iii) of Theorem 4.3 does not hold for $\delta = 0$ and $\delta = 1$.
30. Without the need of summation with $\mathcal{Y}$.
31. Explicit systems can be also converted to the form (33) (Lévine 2006).
32. This is equivalent to saying that the function t is a submersion, and the fact that $\frac{d}{dt}(t) = 1$ is equivalent to saying that that the function t is Lie–Bäcklund, when $\mathbb{R}$ is regarded as a diffiety with trivial Cartan field.
33. See the definition of solution given in this section.
34. Since $\{t, x_a, u_a(u_a^{(k)}, u_b^{(k)} : k \in \mathbb{N})\}$ is a local coordinate system, such α always exists (locally).
35. Complete locally the independent one-forms of $\mathbb{B}$, regarded as column vectors, to a local basis of $T_v^* U$ forming a locally invertible matrix T (one may need to

restrict the open set V_ξ). The coefficients a_i^j and b_l^j may be computed from the first $s+k$ components of $T^{-1}\dot{\omega}_{j+k}$, locally around ξ.

36. From (7), it is clear that the state representation (x, u) is classic (that is, $\mathrm{span}\{\mathrm{d}\dot{x}\} \subset \mathrm{span}\{\mathrm{d}t, \mathrm{d}u, \mathrm{d}x\}$) and the output is also classic (that is $\mathrm{span}\{\mathrm{d}y\} \subset \mathrm{span}\{\mathrm{d}t, \mathrm{d}u, \mathrm{d}x\}$).
37. As in Theorem E.1, the state representation (x, u) is classical and $\mathrm{span}\{\mathrm{d}y\} \subset \mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}u\}$.
38. It shows that the result of Theorem 4.3 is independent of the adapted state representation (9) that is chosen.
39. In this algorithm, one will call the problem of state representation for the implicit system Δ with input v simply by *problem*.
40. As Theorem E.1 is a particular case of Theorem E.2, and the assumptions of Theorem E.2 are generically implied by the properties of the dynamic extension algorithm, if those theorems do not work for $\alpha \leq n$, then ξ is not a generical point (Pereira da Silva et al. 2008). In this case it is not known if it is useful to try to choose $\alpha > n$.
41. See Remark 4.2.
42. See Remark 4.10.

References

Byrnes, C.I., and Isidori, A. (1991), 'Asymptotic Stabilization of Minimum Phase Nonlinear Systems', *IEEE Transactions on Automatic Control*, 36(10), 1122–1137.

Conte, G., Moog, C.H., and Perdon, A.M. (2007), *Algebraic Methods for Nonlinear Control Systesms* (2nd ed.), Communications and Control Engineering Series, London: Springer-Verlag London Ltd.

Crouch, P.E., and Lamnabhi-Lagarrigue, F. (1988), 'State Space Realizations of Nonlinear Systems Defined by Input–output Equations', *Lecture Notes in Control and Information Science*, 111, 138–149.

Delaleau, E., and Respondek, W. (1995), 'Lowering the Orders of Derivatives of Controls in Generalized State Space Systems', *Journal of Mathematical Systems, Estimation and Control*, 5(3), 375–378.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1993), 'Linéarisation Par Bouclage Dynamique Et Transformations De Lie–Bäcklund', *Comptes Rendus de l' Academie des Sciences Paris Series I Mathematique*, 317, 981–986.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1999), 'A Lie-Bäcklund Approach to Equivalence and Flatness of Nonlinear Systems', *IEEE Transactions on Automatic Control*, 44(5), 922–937.

Glad, S.T. (1988), 'Nonlinear State Space and Input Output Descriptions using Differential Polynomials', in *New Trends in Nonlinear Control Theory*, eds. J. Descusse, M. Fliers, and A. Isidori, Berlin: Springer-Verlag, pp. 182–189.

Kotta, U., and Mullari, T. (2005), 'Equivalence of Different Realization Methods for High Order Nonlinear Input-output Differential Equations', *European Journal of Control*, 11(3), 185–193.

Krishnan, H., and McClamroch, N.H. (1994), 'Tracking in Nonlinear Differential-algebraic Control Systems with Applications to Constrained Robot Systems', *Automatica Journal of IFAC*, 30, 1885–1897.

Krstic, M., Kanellakopoulos, I., and Kokotovic, P. (1995), *Nonlinear and Adaptive Control Design*, New York: John Wiley and Sons.

Lévine, J. (2006), 'On Necessary and Sufficient Conditions for Differential Flatness', Technical Report, CAS – Ećole des Mines de Paris, http://arxiv.org/abs/math/0605405

Liu, P., and Moog, C.H. (1994), 'A Local Minimal Realization Algorithm for SISO Nonlinear Systems', in *Proceedings of the American Control Conference*, Baltimore, pp. 551–552.

Moog, C.H., Zheng, Y., and Liu, P. (2002), 'Input–output Equivalence of Nonlinear Systems and their Realizations', in *Proceedings of the CD of 15th IFAC World Congress*, pp. 1–6.

Pereira da Silva, P.S. (2008), 'Some Remarks on Static-feedback Linearization for Time-varying Systems', *Automatica Journal of IFAC*, 44(12), 3219–3221.

Pereira da Silva, P.S., and Batista, S. (2009), 'On State Representations of Time-varying Nonlinear Systems', *Automatica*, 45(5), 1260–1264, doi:10.1016/j.automatica.2008.12.025

Pereira da Silva, P.S., and Corrêa Filho, C. (2001), 'Relative Flatness and Flatness of Implicit Systems', *SIAM Journal on Control and Optimization*, 39, 1929–1951.

Pereira da Silva, P.S., Silveira, H.B., Correa Filho, C., and Batista, S. (2008), 'An Infinite Dimensional Differential-Geometric Approach for Nonlinear Systems: Part II – System Theory', submitted, www.lac.usp.br/~paulo/

Pomet, J.B. (1995), 'A Differential Geometric Setting for Dynamic Equivalence and Dynamic Linearization', in *Geometry in Nonlinear Control and Differential Inclusions*, ed. B. Jackubczyk, Warsaw: Banach Center Publications, pp. 319–339.

van der Schaft, A.J. (1987), 'On Realization of Nonlinear Systems Described by Higher-order Differential Equations', *Mathematical Systems Theory*, 19, 239–275.

van der Schaft, A.J. (1990), 'Transformations and Representations of Nonlinear Systems', in *Perspectives in Control Theory*, ed. B. Jacubczyk, Boston: Birkäuser, pp. 297–314.

Warner, F.W. (1971), *Foundations of Differentiable Manifolds and Lie Groups*, Glenview, Illinois: Scott, Foresman and Company.

Willems, J.C. (1992), 'Paradigms and Puzzles in the Theory of Dynamical Systems', *IEEE Transactions on Automatic Control*, 36, 259–294.

Zharinov, V.V. (1992), *Geometrical Aspects of Partial Differentials Equations*, Singapore: World Scientific.

Appendix A. Diffieties and systems

This appendix is a very brief summary of some facts about the infinite-dimensional approach of Fliess et al. (1993), Pomet (1995), Fliess et al. (1999). A survey about this subject can be found in Pereira da Silva et al. (2008).

$\mathbb{R}^A$-Manifolds, diffieties and systems. The infinite-dimensional approach of Fliess et al. (1999) relies on $\mathbb{R}^A$ manifolds. For an introduction to this kind of manifolds the reader may refer to Zharinov (1992).

An ordinary diffiety is an $\mathbb{R}^A$-manifold for which there exists a field $\frac{\mathrm{d}}{\mathrm{d}t}$, called Cartan field.

A system S is a pair (S, t), where S is an ordinary diffiety and $t: S \to \mathbb{R}$ is a function, called time, such that $\frac{\mathrm{d}}{\mathrm{d}t}(t) = 1$ and such that around any point $\xi \in S$ there exists local coordinates of S of the form (t, η).[32]

State space representation and outputs. A local *state representation* of a system (S, t) is a local coordinate system $\psi = \{t, x, U\}$, where $x = \{x_i, i \in \lfloor n \rceil\}$, $U = \{u_j^{(k)} | j \in \lfloor m \rceil, k \in \mathbb{N}\}$. The set of functions $x = (x_1, \ldots, x_n)$ is called *state* and the set $u = (u_1, \ldots, u_m)$ is called *input*.

In these coordinates the Cartan field is locally written by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{\substack{k \in \mathbb{N},\\ j \in \lfloor m \rceil}} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}}. \tag{A1}$$

It follows from (A1) that $L_{\frac{\mathrm{d}}{\mathrm{d}t}} u^{(k)} = \frac{\mathrm{d}}{\mathrm{d}t}(u^{(k)}) = u^{(k+1)}$. So the notation $u^{(k)}$ is consistent with the fact that, along a solution,[33] it represents the differentiation of $u^{(k-1)}$ with respect to time.

A state representation of a system S is completely determined by the choice of the state x and the input u and will be denoted by (x, u). An *output y* of a system S is a set of functions defined on S. A state representation is said to be *classical* (*or proper*) if f_i does not depend on $u^{(\alpha)}$ for $\alpha > 1$. A *control system S* is a system such that there exists a local state representation around every $\xi \in S$.

System associated with differential equations. Now assume that a control system is given by a set of equations

$$\begin{aligned} \dot{t} &= 1 \\ \dot{x}_i &= f_i(t, x, u, \ldots, u^{(\alpha_i)}),\ i \in \lfloor n \rceil \\ y_j &= \eta_j(x, u, \ldots, u^{(\alpha_j)}),\ j \in \lfloor p \rceil. \end{aligned} \tag{A2}$$

One can always associate with these equations, a diffiety S of global coordinates $\psi = \{t, x, U\}$ and Cartan field given by (A1).

Solutions. A *solution* of a system S with Cartan field $\frac{\mathrm{d}}{\mathrm{d}t}$ is a smooth map $\sigma: (a, b) \to S$, where $(a, b) \subset \mathbb{R}$, such that $\dot{\sigma}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\sigma(t))$.

Subsystems. A (local) *subsystem* S_a of a system S with time notion t is a pair (S_a, π), where S_a is a system with a time notion τ_a and Cartan field ∂_a, and π is a Lie–Bäcklund submersion $\pi: U \subset S \to S_a$ between the system $U \subset S$ and S_a such that $\tau_a \circ \pi = t$. A local state representation $x = (x_a, x_b)$, $u = (u_a, u_b)$ is said to be *adapted* to a subsystem S_a if we locally have

$$\dot{x}_a = f_a(t, x_a, u_b) \tag{A3a}$$

$$\dot{x}_b = f_b(t, x_a, x_b, u_a, u_b) \tag{A3b}$$

and (x_a, u_a) is a local state representation of S_a with state equations (A3a).

Equivalence. Two systems S_1 and S_2 with time notions, respectively, given by τ_1 and τ_2 are said to be *equivalent by endogenous feedback* if there exists a Lie–Bäcklund diffeomorphism $\phi : S_1 \to S_2$ (also called Lie–Bäcklund isomorphism), such that $\tau_1 = \tau_2 \circ \phi$. Recall that this notion does not imply input–output equivalence. If y_i and u_i are, respectively, an output and an input for S_i, $i = 1, 2$, then ϕ *preserves the output and the input* if $y_1 = y_2 \circ \phi$ and $u_1 = u_2 \circ \phi$. Clearly, if S_1 and S_2 are equivalent by endogenous feedback and ϕ preserves the output and the input, then S_1 and S_2 are also input–output equivalent. In fact, a solution $\sigma(t)$ of S_1, corresponding to an initial condition $\sigma(t_0)$ and an input $u_1 \circ \sigma(t)$, is transformed into a solution $\phi \circ \sigma(t)$ of S_2 corresponding to the initial condition $\phi \circ \sigma(t_0)$ and the same input–output behaviour.

Appendix B. Proof of Lemma 4.4

To prove part (1), let $a \in \mathbb{N}$ and let H_a stand for $\mathrm{span}\{\mathrm{d}x_b, \mathrm{d}u_b^{(0)}, \ldots, \mathrm{d}u_b^{(a)}\}$. If $\mathrm{span}\{\mathrm{d}z\} \subset \mathrm{span}\{\mathrm{d}x_b\} \oplus \mathcal{Y}$, one may take $\alpha = 0$. So assume that $\mathrm{span}\{\mathrm{d}z\} \not\subset \mathrm{span}\{\mathrm{d}x_b\} \oplus \mathcal{Y}$. Let $\beta \in \mathbb{N}$ be such that $\mathrm{span}\{\mathrm{d}z\} \subset H_\beta \oplus \mathcal{Y}$, but $\mathrm{span}\{\mathrm{d}z\} \not\subset H_{\beta-1} \oplus \mathcal{Y}$, where $\mathcal{Y}$ is defined by (8). Since $\{t, x_a, x_b, (u_a^{(k)}, u_b^{(k)} : k \in \mathbb{N})\}$ is a local coordinate system and $\mathcal{Y} = \mathrm{span}\left\{\mathrm{d}t, \mathrm{d}x_a, (\mathrm{d}u_a^{(k)} : k \in \mathbb{N})\right\}$, as one assumes that and $\mathrm{span}\{\mathrm{d}z\} \not\subset \mathrm{span}\{\mathrm{d}x_b\} \oplus \mathcal{Y}$, it is clear that there exists such β (locally). As $((z_a, z), (v_a, v))$ is proper, then $\mathrm{span}\{\mathrm{d}\dot{z}\} \subset \mathrm{span}\{\mathrm{d}z, \mathrm{d}v\} \oplus \mathcal{Y}$. Note that

$$\mathrm{d}z = \sum_{i=1}^{n_b} \gamma_i\, \mathrm{d}x_{b_i} + \sum_{j=0}^{\beta} \sum_{k=1}^{m_b} \epsilon_{jk}\, \mathrm{d}u_{b_k}^{(j)} + \eta,$$

where $\eta \in \mathcal{Y}$ and γ_i, ϵ_{jk} are smooth functions defined on $U \subset S$. Since $\mathrm{span}\{\mathrm{d}z\} \not\subset H_{\beta-1} \oplus \mathcal{Y}$, then some function $\epsilon_{\beta j}$ is not identically zero for some $j \in \{1, \ldots, m_b\}$. So,

$$\mathrm{d}\dot{z} = \sum_{i=1}^{n_b} (\dot{\gamma}_i\, \mathrm{d}x_{b_i} + \gamma_i\, \mathrm{d}\dot{x}_{b_i}) + \sum_{j=0}^{\beta} \sum_{k=1}^{m_b} \left(\dot{\epsilon}_{jk}\, \mathrm{d}u_{b_k}^{(j)} + \epsilon_{jk}\, \mathrm{d}u_{b_k}^{(j+1)} \right) + \dot{\eta}.$$

The properness of the state representation implies that $\mathrm{span}\{\mathrm{d}\dot{x}_b\} \subset \mathrm{span}\{\mathrm{d}x_b, \mathrm{d}u_b\} \oplus \mathcal{Y}$. It follows that $\mathrm{span}\{\mathrm{d}\dot{z}\} \subset H_{\beta+1} \oplus \mathcal{Y}$, but $\mathrm{span}\{\mathrm{d}\dot{z}\} \not\subset H_\beta \oplus \mathcal{Y}$. Since $\mathrm{span}\{\mathrm{d}\dot{z}\} \subset \mathrm{span}\{\mathrm{d}z, \mathrm{d}v\} \oplus \mathcal{Y}$, it follows that $\mathrm{span}\{\mathrm{d}v\} \not\subset H_\beta \oplus \mathcal{Y}$.

Now assume that for some[34] $\alpha \in \mathbb{N}$, one has $\mathrm{span}\{\mathrm{d}v\} \subset H_\alpha \oplus \mathcal{Y}$. Assume by contradiction that $\mathrm{span}\{\mathrm{d}z\} \not\subset H_{\alpha-1} \oplus \mathcal{Y}$. Then $\alpha \le \beta$, with β defined above. Then, from the reasoning above, $\mathrm{span}\{\mathrm{d}v\} \not\subset H_\beta$ and so $\mathrm{span}\{\mathrm{d}v\} \not\subset H_\alpha (\subset H_\beta)$, which is an absurd. Part (2) is a direct consequence of Pereira da Silva et al. (2008, Lemma 1, part 2).

Appendix C. Proof of Lemma 4.8

The following two results are instrumental for the proof of Lemma 4.8.

Lemma C.1 (Lemma 2 of Pereira da Silva et al. (2008)**:** *Let (x, u) be a local proper state representation of a system S around some $\xi \in S$ and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_s)$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_m)$ be sets of functions defined on the diffiety S. Suppose that* $\mathrm{span}\{\mathrm{d}\bar{x}, \mathrm{d}\bar{v}\} \subset \mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}u\}$. *Then $(\bar{x}, \bar{v})$ is a local state representation of S around ξ if and only if there exist $\alpha \in \mathbb{N}$ such that*

- *The set $\mathbb{S} = \{\mathrm{d}t, \mathrm{d}\bar{x}, \mathrm{d}\bar{v}, \ldots, \mathrm{d}\bar{v}^{(\alpha)}\}$ is linearly independent pointwise in an open neighbourhood of ξ.*

- *One has* $\text{span}\{\mathrm{d}x\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}\bar{x}, \mathrm{d}\bar{v}^{(0)}, \ldots, \mathrm{d}\bar{v}^{(\alpha-1)}\}$, *in an open neighbourhood of* ξ.
- *One has* $\text{span}\{\mathrm{d}\dot{\bar{x}}, \mathrm{d}u\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}\bar{x}, \mathrm{d}\bar{v}^{(0)}, \ldots, \mathrm{d}\bar{v}^{(\alpha)}\}$ *in an open neighbourhood of* ξ.

Lemma C.2: *Let S be a $\mathbb{R}^A$ manifold and let $\xi \in S$. Let $\psi = \{(x_i, i \in A), (y_j, j \in B)\}$ be a local coordinate system defined in an open neighbourhood V_ξ of ξ. Let $\mathcal{Y} = \text{span}\{\mathrm{d}y_j, j \in B\}$ and let $\{\omega_1, \ldots, \omega_s\}$ be a set of one-forms defined on V_ξ. Let ω be a one-form such that $\omega \in \text{span}\{\omega_1, \ldots, \omega_s\} + \mathcal{Y}$. Then there exists a finite subset $F \subset B$ such that $\omega \in \text{span}\{\omega_1, \ldots, \omega_s\} +$ $\text{span}\{\mathrm{d}y_j, j \in F\}$.*

Proof: One may locally write

$$\omega_l = \tilde{\omega}_l + \sum_{j=1}^{\gamma_l} \alpha_j^i \,\mathrm{d}y_j, \qquad \tilde{\omega}_l = \sum_{i=1}^{\epsilon_l} \beta_i^l \,\mathrm{d}x_i. \tag{A4}$$

In particular, $\omega \in \text{span}\{\tilde{\omega}_1, \ldots, \tilde{\omega}_s\} + \mathcal{Y}$ as $C^\infty(V_\xi)$-module. Hence, for convenient smooth functions a_l, $l = 1, \ldots, s$ and b_j, $j \in B$, one may write

$$\omega = \sum_{l=1}^{s} a_l \tilde{\omega}_l + \sum_{j \in B} b_j \,\mathrm{d}y_j = \sum_{l=1}^{s} \sum_{i=1}^{\epsilon_l} (a_l \beta_i^l) \mathrm{d}x_i + \sum_{j \in B} b_j \,\mathrm{d}y_j. \tag{A5}$$

One may locally write, perhaps after restricting the open neighbourhood of ξ to some $W_\xi \subset V_\xi$:

$$\omega = \sum_{i \in F_1} e_i \,\mathrm{d}x_i + \sum_{j \in F_2} \tilde{b}_j \,\mathrm{d}y_j \tag{A6}$$

for convenient finite subsets $F_1 \subset A$ and $F_2 \subset B$. On W_ξ, as ψ is a local coordinate system, the expression (A6) is unique. In particular, subtracting (A6) from (A5) on W_ξ gives zero. From the independence of the differentials of a local coordinate system, one concludes that $\omega = \sum_{l=1}^{s} a_l \tilde{\omega}_l + \sum_{j \in F_2} b_j \,\mathrm{d}y_j$. The proof is concluded from the last equation and (A4). □

Proof (of Lemma 4.8)**:** From Assumptions 1–3 and Lemma C.2 it is clear that there exists $c \in \mathbb{N}$ big enough such that

(A) $\text{span}\{\mathrm{d}\dot{z}\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}z, \mathrm{d}v^{(0)}, \mathrm{d}x_a, \mathrm{d}u_a^{(0)}, \ldots, \mathrm{d}u_a^{(c)}\}$.

(B) $\text{span}\{\mathrm{d}x_b\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}z, \mathrm{d}v^{(0)}, \ldots, \mathrm{d}v^{(\delta-1)}, \mathrm{d}x_a, \mathrm{d}u_a^{(0)}, \ldots, \mathrm{d}u_a^{(c-1)}\}$.

(C) $\text{span}\{\mathrm{d}u_b\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}z, \mathrm{d}v^{(0)}, \ldots, \mathrm{d}v^{(\delta)}, \mathrm{d}x_a, \mathrm{d}u_a^{(0)}, \ldots, \mathrm{d}u_a^{(c)}\}$.

By Assumption 1 and from the fact that $\mathcal{Y} = \text{span}\{\mathrm{d}t, \mathrm{d}x_a, (\mathrm{d}u_a^{(k)} : k \in \mathbb{N})\}$, it also follows that

(D) The set $\{\mathrm{d}t, \mathrm{d}z, \mathrm{d}v^{(0)}, \ldots, \mathrm{d}v^{(\delta)}, \mathrm{d}x_a, \mathrm{d}u_a^{(0)}, \ldots, \mathrm{d}u_a^{(c)}\}$ is locally linearly independent.

Now let $\bar{x} = \{z, v^{(0)}, \ldots, v^{(\delta-1)}, x_a, u_a^{(0)}, \ldots, u_a^{(c-1)}\}$ and $\bar{v} = (v^{(\delta)}, u_a^{(c)})$. By (A), (B), (C), (D) it is clear that

- $\text{span}\{\mathrm{d}\dot{\bar{x}}\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}\bar{x}, \mathrm{d}\bar{v}\}$.
- $\text{span}\{\mathrm{d}x\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}\bar{x}\}$ and $\text{span}\{\mathrm{d}u\} \subset \text{span}\{\mathrm{d}t, \mathrm{d}\bar{x}, \mathrm{d}\bar{u}\}$
- $\{\mathrm{d}t, \mathrm{d}\bar{x}, \mathrm{d}\bar{v}\}$ is linearly independent.

Hence, by Lemma C.1, it follows that $(\bar{x}, \bar{v})$ is also a local state representation around ξ. Since $\{(\tilde{x}_a, z), (\tilde{u}_a^{(k)}, v^{(k)}, k \in \mathbb{N})\}$ is a local coordinate system around ξ, it is then clear that $((\tilde{x}_a, z), (\tilde{u}_a, v))$ is also a local state representation. By (A) it follows that the state equations are of the form (26). As the original state representation$((x_a, x_b), (u_a, u_b))$ is (strongly) adapted to the subsystem Y, it follows easily from Definition 2.1 that $((\tilde{x}_a, z), (\tilde{u}_a, v))$ is also (strongly) adapted to the subsystem Y. □

Appendix D. Proof of Proposition 3.2

Proof: Write the one-forms ω_i and $\dot{\omega}_i$ in local coordinates $x = \{x_i, i \in A\}$ defined around ξ. One obtains $\omega_i = \sum_{i \in F} \alpha_i(x) \mathrm{d}x_i$, $\dot{\omega}_i = \sum_{i \in F} \beta_i(x) \,\mathrm{d}x_i$, $i = 1, \ldots, s$. The subset F can be chosen finite for a convenient open neighbourhood V_ξ of ξ. Without loss of generality, assume that $F = \{1, \ldots, k\}$. Hence one may identify $\omega_i(\nu)$ and $\dot{\omega}_i(\nu)$, respectively, with the row vectors $\omega_i(\nu) = (\alpha_1(\nu), \ldots, \alpha_k(\nu))$ and $\dot{\omega}_i(\nu) = (\beta_1(\nu), \ldots, \beta_k(\nu))$. Define the $2s \times k$ matrix

$$N_\nu = \begin{bmatrix} \omega_1(\nu) \\ \vdots \\ \omega_s(\nu) \\ \dot{\omega}_1(\nu) \\ \vdots \\ \dot{\omega}_s(\nu) \end{bmatrix}$$

and let $M_\nu = N_\nu^T$, the transpose of M_ν. Let $\pi_1 : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^s$ the projection $(z_1, z_2) \mapsto z_1$. To say that $\omega_\nu \in \Gamma_1|_\nu$ is equivalent to say that $\omega_\nu = \sum_{i \in F} \alpha_i \omega_i(\nu)$, where $(\alpha_1, \ldots, \alpha_s)^T \in \pi_1(\ker M_\nu)$. Hence, Γ_1 is nonsingular around $\xi \in u$ if and only if $\pi_1(\ker M_\nu)$ is locally constant dimensional around ξ. As the family $\{\omega_1, \ldots, \omega_s\}$ is pointwise independent, then $\ker M_\nu \cap \ker \pi_1 = \{0\}$. Hence, the map π_1 restricted to $\ker M_\nu$ is an isomorphism into its image and so $\dim \ker M_\nu = \dim \pi_1(\ker M_\nu)$. Hence ξ is also a regular point of $\ker M_\nu$. As $\dim \text{im } M_\nu + \dim \ker M_\nu = 2s$, ξ is also a regular point of $\text{im}\, M_\nu$. Note that one may identify $\text{im } M_\nu$ with $\Theta|_\nu = \text{span}\{\Gamma + \dot{\Gamma}\}|_\nu = \text{span}\{\omega_1, \ldots, \omega_s, \dot{\omega}_1, \ldots, \dot{\omega}_s\}|_\nu$. Without loss of generality, one may construct a local basis $\mathbb{B} = \{\omega_1, \ldots, \omega_s, \dot{\omega}_1, \ldots, \dot{\omega}_r\}$ of Θ, where $s + k = \dim \text{im}\, M_\nu = \dim \Theta_\nu$, $\nu \in V_\xi$. In particular

$$\dot{\omega}_{k+j} = \sum_{i=1}^{s} a_i^j \omega_i + \sum_{l=1}^{k} b_l^j \dot{\omega}_i, \quad j = k+1, \ldots, s,$$

where the coefficients a_i^j and β_l^j depend smoothly[35] on the point ν.

Note now that $\dim \Gamma_1|_\nu = \dim \ker M_\nu = 2s - s - k = s - k$. In particular, it follows from a dimensional argument, that the set $\{\eta_1, \ldots, \eta_{s-k}\}$ is a local basis of Γ_1, where

$$\eta_j = \omega_{k+j} - \sum_{l=1}^{k} \beta_l^k \omega_l.$$

This shows the smoothness of Γ_1 on some open neighbourhood W_ξ of ξ.

Now, it is clear from smoothness of η_j that $\eta_j \in \Omega_1$ (when one restricts the open neighbourhood of definition to W_ξ) and so $\Gamma_1 \subset \Omega_1$. By (17) and (18) it follows that $\Omega_1 \subset \Gamma_1$. □

Appendix E. Existence of strongly adapted state equations

The following result gives sufficient conditions for the existence of output subsystems and strongly adapted state equations in the invertible case. It generalises previous results whose proofs were based on the properties of the dynamic extension algorithm (Pereira da Silva and Corrêa Filho 2001). Theorem E.2 is a generalisation of this result for the noninvertible case. These two results allows also the application of Theorem 4.3 from the algorithmic point of view (Appendix F).

Theorem E.1 (Existence of strongly adapted state equations – invertible case)**:** *Let S be a system with state representation (x, u) and output y defined*[36] *by* (7). *Assume that there exists some $\alpha \in \mathbb{N}$ such that, locally around some $\nu \in S$, one has*

(1) $\mathrm{span}\{\mathrm{d}x\} \cap \mathrm{span}\{\mathrm{d}t, \mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha-1)}\} = \mathrm{span}\{\mathrm{d}x\} \cap \mathrm{span}\{\mathrm{d}t, \mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha)}\}$.
(2) $\mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha-1)}\}$ *is locally nonsingular around* ξ.
(3) $\mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}u, \mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha)}\}$ *is locally nonsingular around* ξ.
(4) *The set* $\{\mathrm{d}t, \mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha)}\}$ *is pointwise independent in an open neighbourhood of* ξ.

Then there exists a local output subsystem Y defined around ν that admits a strongly adapted state representation $(\tilde{x}, \tilde{u})$, where $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$. Moreover:

(A) *One may choose* $u_a = y^{(\alpha)}$.
(B) *One may choose* $x_a \subset \{y^{(0)}, \ldots, y^{(\alpha-1)}\}$ *such that* $\{\mathrm{d}t, \mathrm{d}x_a\}$ *is a local basis of span* $\{\mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha-1)}\}$.
(C) *One may chose* x_b *in a way that* $\mathrm{d}x_b$ *completes* $\{\mathrm{d}t, \mathrm{d}x_a\}$ *to a local basis of span* $\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \ldots, \mathrm{d}y^{(\alpha-1)}\}$.
(D) *One may chose* u_b *in order to complete* $\{\mathrm{d}t, \mathrm{d}x_a, \mathrm{d}x_b, \mathrm{d}u_a\}$ *to a basis* $\{\mathrm{d}t, \mathrm{d}x_a, \mathrm{d}x_b, \mathrm{d}u_a, \mathrm{d}u_b\}$ *of span* $\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}u, \mathrm{d}y, \ldots, \mathrm{d}y^{(\alpha)}\}$.

In particular, if $\tilde{\Delta}$ is nonempty and these assumptions hold around all $\xi \in \tilde{\Delta}$, then the corresponding implicit system (6) *is regular. Furthermore,* $\mathrm{span}\{\mathrm{d}x\} + \mathcal{Y} = \mathrm{span}\{\mathrm{d}x_b\} \oplus \mathcal{Y}$ *and* $\mathrm{span}\{\mathrm{d}x, \mathrm{d}u\} + \mathcal{Y} = \mathrm{span}\{\mathrm{d}x_b, \mathrm{d}u_b\} \oplus \mathcal{Y}$.

Proof: The proof of the theorem is an easy consequence of the proof of Theorem 3 of Pereira da Silva et al. (2008). □

The next theorem is a generalisation of Theorem E.1. One considers that the output y is partitioned into two parts $(\bar{y}, \hat{y})$ in a way that the given system S with output $\bar{y}$ obeys the assumptions of Theorem E.1, that is, $\bar{y}$ is such that the system with input u and output $\bar{y}$ is right invertible, that is the output rank coincides with card $\bar{y}$. The subset $\hat{y}$ of the output y represents the 'dependent' part of the output, that is, it does not contribute to the output rank.

Theorem E.2 (Existence of strongly adapted state equations – non-invertible case)**:** *Let S be the system with state representation (x, u) and output y, defined*[37] *by* (7). *Let $\nu \in \tilde{\Delta}$, where $\tilde{\Delta} \subset S$ is defined by* (10). *Assume that there exists a partition $y = (\bar{y}, \hat{y})$, where $\bar{y}$ is called the* independent *part and $\hat{y}$ is called* dependent *part of the output. Assume also that there exists some $\alpha \in \mathbb{N}$ such that, locally around ν, one has*

(1) $\mathrm{span}\{\mathrm{d}x\} \cap \mathrm{span}\{\mathrm{d}t, \mathrm{d}\bar{y}^{(0)}, \ldots, \mathrm{d}\bar{y}^{(\alpha-1)}\} = \mathrm{span}\{\mathrm{d}x\} \cap \mathrm{span}\{\mathrm{d}t, \mathrm{d}\bar{y}^{(0)}, \ldots, \mathrm{d}\bar{y}^{(\alpha)}\}$.
(2) $\mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}\bar{y}^{(0)}, \ldots, \mathrm{d}\bar{y}^{(\alpha-1)}\}$ *is locally nonsingular around* ξ.
(3) $\mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}u, \mathrm{d}\bar{y}^{(0)}, \ldots, \mathrm{d}\bar{y}^{(\alpha)}\}$ *is locally nonsingular around* ξ.
(4) *The set* $\{\mathrm{d}t, \mathrm{d}\bar{y}^{(0)}, \ldots, \mathrm{d}\bar{y}^{(\alpha)}\}$ *is pointwise independent in an open neighbourhood of* ξ.
(5) $\mathrm{span}\{\mathrm{d}y^{(0)},\ldots,\mathrm{d}y^{(\alpha-1)}\} \subset \mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}\bar{y}^{(0)},\ldots,\mathrm{d}\bar{y}^{(\alpha-1)}\}$
(6) $\mathrm{span}\{\mathrm{d}t, \mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(k)}\}$ *is nonsingular for* $k = \alpha$ *and* $k = \alpha - 1$.
(7) $\mathrm{span}\{\mathrm{d}y^{(\alpha)}\} \subset \mathrm{span}\{\mathrm{d}t, \mathrm{d}y^{(0)}, \mathrm{d}y^{(1)}, \ldots, \mathrm{d}y^{(\alpha-1)}, \mathrm{d}\bar{y}^{(\alpha)}\}$
(8) $\mathrm{span}\{\mathrm{d}y^{(0)}, \mathrm{d}y^{(1)}, \ldots, \mathrm{d}y^{(k)}\}$ *is nonsingular around* ν *for* $k = \alpha - 1$ *and* $k = \alpha$.

Then there exists a local output subsystem Y defined around ν that admits a strongly adapted state representation $(\tilde{x}, \tilde{u})$, where $\tilde{x} = (x_a, x_b)$ and $\tilde{u} = (u_a, u_b)$. Moreover:

(A) *One may choose* $u_a = \bar{y}^{(\alpha)}$.
(B) *One may choose* $x_a \subset \{y^{(0)}, \ldots, y^{(\alpha-1)}\}$ *such that* $\{\mathrm{d}t, \mathrm{d}x_a\}$ *is a local basis of* $\mathrm{span}\{\mathrm{d}y^{(0)}, \ldots, \mathrm{d}y^{(\alpha-1)}\}$.
(C) *One may chose* x_b *in a way that* $\mathrm{d}x_b$ *completes* $\{\mathrm{d}t, \mathrm{d}x_a\}$ *to a local basis of* $\mathrm{span}\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \ldots, \mathrm{d}y^{(\alpha-1)}\}$.
(D) *One may chose* u_b *in order to complete* $\{\mathrm{d}t, \mathrm{d}x_a, \mathrm{d}x_b, \mathrm{d}u_a\}$ *to a basis* $\{\mathrm{d}t, \mathrm{d}x_a, \mathrm{d}x_b, \mathrm{d}u_a, \mathrm{d}u_b\}$ *of span* $\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}u, \mathrm{d}y, \ldots, \mathrm{d}y^{(\alpha)}\}$.

In particular, if $\tilde{\Delta}$ is nonempty and these assumptions hold around all $\xi \in \tilde{\Delta}$, then the corresponding implicit system (6) *is regular. Furthermore,* $\mathrm{span}\{\mathrm{d}x\} + \mathcal{Y} = \mathrm{span}\{\mathrm{d}x_b\} \oplus \mathcal{Y}$ *and* $\mathrm{span}\{\mathrm{d}x, \mathrm{d}u\} + \mathcal{Y} = \mathrm{span}\{\mathrm{d}x_b, \mathrm{d}u_b\} \oplus \mathcal{Y}$.

Proof: The proof of the theorem is an easy consequence of the proof of Theorem 5 of Pereira da Silva et al. (2008). □

Appendix F. Algorithmic issues

This section is devoted to the algorithmic aspects of the main result, namely, Theorem 4.3. It is important to stress that Theorem 4.9 is an '*intrinsic*' interpretation[38] of Theorem 4.3, and it is not suitable for computations. One may summarise the theoretical results presented in this article in the following algorithm,[39] which verifies the solvability conditions of Theorem 4.3.

Algorithm.
Preparation process I. Let ξ be a point of S defined by (7). Verify for $\alpha \in \{0, 1, \ldots, n\}$ if the assumptions[40] of Theorem E.1 hold. Construct x_a, x_b, u_a, u_b as described in the steps (A)–(D) in end of the Theorem E.1. If one may choose $u_b = v$, then the Problem is solvable, then **stop**. (Remark 4.2). If it is not the case, then continue.

Preparation process II. If the assumptions of Theorem E.1 do not hold for $\alpha \in \{0, 1, \ldots, n\}$ then choose a partition $(\bar{y}, \hat{y})$ of y, and verify if the assumptions of Theorem E.2 hold for some $\alpha \in \{0, 1, \ldots, n\}$. If it is the case, construct x_a, x_b, u_a, u_b as described in the steps (A)–(D) in end of the Theorem 4.2. If one may choose[41] $u_b = v$, so the Problem is solvable with $z_a = x_a$, $v_a = u_a$ and $z = x_b$. **Stop**. If it is not the case, then continue.

Step 0. Let $\delta = 0$. Let ω be a one-form given by (11) and let θ be the map defined by (12).
Step 1. Compute γ in the following way. Let

$$\theta(\mathrm{d}v) = \sum_{i=1}^{n_b} \beta_i \, \mathrm{d}x_{b_i} + \sum_{j=0}^{m_b} \sum_{k=0}^{\kappa_j} \epsilon_{jk} \, \mathrm{d}u_{b_j}^{(k)}.$$

Then $\gamma = \max_{j \in \{0,1,\ldots,m_b\}} \{\kappa_j\}$.

Step 3. Compute Γ_0 given by (23).
Step 4. Compute the relative derived flag Γ_k for $k = 1, \ldots, \delta + 1$, using (16) and the idea of the proof of Proposition 3.3.
Step 5. Verify the assumptions (iii), and (iv) of Theorem 4.3 by direct computation. If these conditions hold, go to step 6. If at least one of these condition fails and $\delta < n + \gamma(m+1)$, then increment δ and go to step 1. If at least one of these conditions fails and $\delta = n + \gamma(m+1)$, then the assumptions of Theorem 4.3 fail[42] for all possible values of δ. **Stop.**
Step 6. Verify the assumption (i) of Theorem 4.3 by computing the dimensions of the (finite dimensional) codistributions Γ_k. If this condition holds, go to step 7. If this condition fails and $\delta < n + \gamma(m+1)$, then increment δ and go to step 1. If this condition fails and $\delta = n + \gamma(m+1)$, then the assumptions of Theorem 4.3 fail for all possible values of δ. **Stop.**
Step 7. Verify the assumption (ii) by applying the idea of the proof of Lemma 4.7. If the codistribution $\Gamma_{\delta+1} + \mathcal{Y}$ is integrable, it may be possible to compute z by the idea of the proof of Lemma 4.7. If this condition holds, go to step 8. If this condition fails and $\delta < n + \gamma(m+1)$, then increment δ and go to step 1. If this condition fails and $\delta = n + \gamma(m+1)$, then the assumptions of Theorem 4.3 fail for all possible values of δ. **Stop.**
Step 8. The problem is solvable by choosing $z_a = (x_a, u_a^{(0)}, \ldots, u_a^{(c-1)})$ and $v_a = u_a^{(c)}$, where $c \in \mathbb{N}$ can be computed as in the proof of Lemma 4.8. **Stop.**