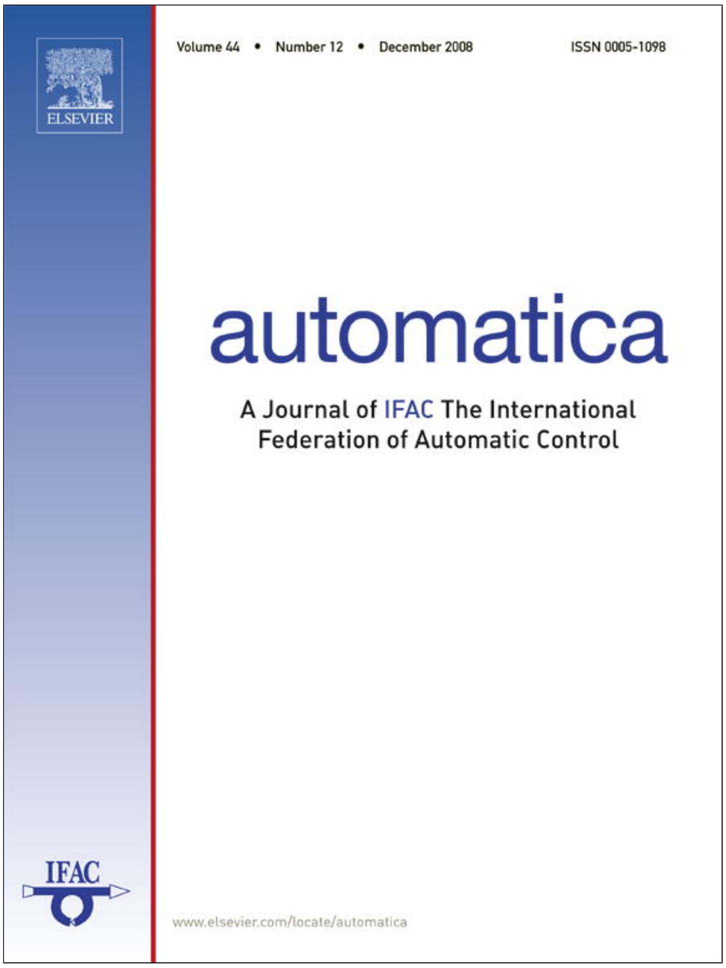
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Some remarks on static-feedback linearization for time-varying systems☆

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ABSTRACT

This work summarizes some results about static state feedback linearization for time-varying systems. Three different necessary and sufficient conditions are stated in this paper. The first condition is the one by [Sluis, W. M. (1993). A necessary condition for dynamic feedback linearization. *Systems & Control Letters*, *21*, 277–283]. The second and the third are the generalizations of known results due respectively to [Aranda-Bricaire, E., Moog, C. H., Pomet, J. B. (1995). A linear algebraic framework for dynamic feedback linearization. *IEEE Transactions on Automatic Control*, *40*, 127–132] and to [Jakubczyk, B., Respondek, W. (1980). On linearization of control systems. *Bulletin del'Academie Polonaise des Sciences. Serie des Sciences Mathematiques*, *28*, 517–522]. The proofs of the second and third conditions are established by showing the equivalence between these three conditions. The results are re-stated in the infinite dimensional geometric approach of [Fliess, M., Lévine J., Martin, P., Rouchon, P. (1999). A Lie–Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Transactions on Automatic Control*, *44*(5), 922–937].

1. Introduction

This work considers control systems of the form

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1}$$

where f is smooth with respect to its arguments, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. Let $\xi = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $\nu = (t_0, x_0, u_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$.

A local state-transformation is a local diffeomorphism $\phi : V \subset \mathbb{R} \times \mathbb{R}^n \to U \subset \mathbb{R} \times \mathbb{R}^n$, defined around ξ, such that $(t, x) \mapsto (t, z)$, where $z = \psi(t, x)$. Locally, there exists the inverse $x = \theta(t, z)$. A regular static-feedback is a local diffeomorphism $\alpha : V \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to U \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, defined around ν, such that $(t, x, u) \mapsto (t, z, v)$, where $(t, x) \mapsto (t, z)$ is a local state transformation. Locally, there exists the inverse $(t, x, u) = \alpha^{-1}(t, z, v)$. The closed loop equations are given by

$$\dot{z}(t) = \tilde{f}(t, z(t), v(t)) \tag{2}$$

where $\tilde{f}(t, z, v) = \left[\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} f(t, x, u)\right]|_{(t,x,u)=\alpha^{-1}(t,z,v)}$. The time-varying static-feedback linearization problem seeks a local regular static-feedback such that the closed loop system locally reads a controllable linear system

$$\dot{z} = Az(t) + Bv(t).$$

This problem was completely solved in its time-invariant version for affine systems (see Jakubczyk and Respondek (1980) and Hunt, Su, and Meyer (1983)). The dynamic version of this problem remains open (see Charlet, Lévine, and Marino (1989, 1991) for some sufficient conditions). Closed related to the dynamic version of this problem is the notion of flatness (see Fliess, Lévine, Martin, and Rouchon (1999)), which considers a class of transformations, called endogenous feedback. An endogenous feedback is more general than a static-feedback, but it is a particular case of dynamic feedback. The techniques of exterior calculus[1] are useful on the study of flatness and exact linearization (see Aranda-Bricaire, Moog, and Pomet (1995), Gardner and Shadwick (1992), Martin and Rouchon (1994), Shadwick (1990), Shadwick and Sluis (1994), Sluis (1992, 1993), Tilbury, Murray, and Sastry (1995) and van Nieuwstadt, Rathinam, and Murray (1998)).

This work shows the equivalence of three conditions of solvability of the time-varying static-feedback linearization problem. The conditions of Aranda-Bricaire et al. (1995) and Jakubczyk and Respondek (1980) are generalized, and the generalized versions are shown to be equivalent to the ones of Sluis (1992, 1993) and Shadwick and Sluis (1994).

The field of real numbers will be denoted by $\mathbb{R}$. The set of real matrices of n rows and m columns is denoted by $\mathbb{R}^{n\times m}$. The matrix M^{T} stands for the transpose of M. The set of natural numbers $\{1, \ldots, k\}$ will be denoted by $\lfloor k \rceil$. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}}$,

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[1] See for instance Dieudonneé (1974).

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where z_1 and z_2 are also column vectors. Let $x = (x_1, \ldots, x_n)$ be a vector of functions (or a collection of functions). Then $\{dx\}$ stands for the set $\{dx_1, \ldots, dx_n\}$.

One may associate to system (1), its infinite prolongation, called diffiety (see Fliess et al. (1999), Vinogradov (1984) and Zharinov (1992)):

$$\dot{t} = 1$$
$$\dot{x} = f(t, x, u^{(0)})$$
$$\dot{u}^{(0)} = u^{(1)}$$
$$\dot{u}^{(1)} = u^{(2)}$$
$$\vdots \quad \vdots \quad \vdots$$

This system evolves on the diffiety $S = \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^m)^\infty$ with coordinates $\{t, x, (u^{(k)} : k \in \mathbb{N})\}$. One may define on S, the Cartan Field

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(t, x, u)\frac{\partial}{x_i} + \sum_{k\in\mathbb{N}}\sum_{j=1}^{m} u_j^{(k+1)}\frac{\partial}{u_j^{(k)}}. \tag{3}$$

Given a function $\xi(t, x, u^{(0)}, \ldots, u^{(\alpha)})$, defined on S, then $\frac{\mathrm{d}}{\mathrm{d}t}\xi = \dot{\xi} = \xi^{(1)}$ stands for the Lie-derivative $L_{\frac{\mathrm{d}}{\mathrm{d}t}}\xi = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} f(t, x, u) + \sum_{k\in\mathbb{N}} \frac{\partial \xi}{\partial u^{(k)}} u^{(k+1)}$. Given a 1-form $\omega = \alpha_0 \mathrm{d}t + \sum_{i=1}^{n} \alpha_i \mathrm{d}x_i + \cdots$ defined on S, then $\frac{\mathrm{d}}{\mathrm{d}t}\omega = \dot{\omega}$ stands for the Lie-derivative $L_{\frac{\mathrm{d}}{\mathrm{d}t}}\omega = \dot{\alpha}_0 \mathrm{d}t + \sum_{i=1}^{n} [\dot{\alpha}_i \mathrm{d}x_i + \alpha_i \mathrm{d}\dot{x}_i] + \cdots$.

Recall that, if S is a manifold (or a diffiety), then $\Lambda^k(S)$ denotes the bundle of k-forms over S and $\Lambda(S) = \Lambda^0(S) \oplus \Lambda^1(S) + \cdots$ stands for the bundle of forms defined on S (see Warner (1971) and Zharinov (1992)). If $\Omega \subset \Lambda^1(S)$ is a codistribution defined on S, then (Ω) denotes the algebraic ideal generated by Ω, *i. e.*, $(\Omega) = \{\theta \in \Lambda(S) \mid \theta = \sum_{j=0}^{k} \eta_k \wedge \omega_k, \omega_k \in \Omega, \eta_k \in \Lambda(S)\}$.

2. Static-linearizability conditions

In this section we shall state the generalizations of the known conditions for the time varying static-linearization problem. In Sluis (1992, 1993), the following necessary and sufficient conditions for static-feedback linearizability are given.[2] In this section, system S stands for the diffiety with Cartan field $\frac{\mathrm{d}}{\mathrm{d}t}$ that is associated to system (1).

Theorem 1. *Consider the codistributions defined on S, given by $\Omega^{(0)} = \mathrm{span}\,\{\omega_i = \mathrm{d}x_i - f_i(t, x, u)\mathrm{d}t \mid i \in \lfloor n \rceil\}$.*[3] *Consider the derived flag $\Omega^{(k)} = \mathrm{span}\,\{\omega \in \Omega^{(k-1)} \mid \mathrm{d}\omega \in (\Omega)\}$, $k \in \mathbb{N}$. Then the system is locally static-feedback linearizable around $\xi \in S$ if and only if:*

(1) *The codistributions $\Omega^{(k)}$ are nonsingular at ξ for $k \in \mathbb{N}$.*
(2) *There exists $k^* \in \mathbb{N}$ big enough, such that $\Omega^{(k^*)} = \{0\}$.*
(3) *The codistributions $\Omega^{(k)} \oplus \mathrm{span}\{\mathrm{d}t\}$ are locally involutive around ξ, for $k \in \mathbb{N}$.*

The following conditions generalizes the ones of Aranda-Bricaire et al. (1995) for the time-varying case.

Theorem 2. *Consider the codistributions defined on S given by $\Delta_0 = \mathrm{span}\,\{\mathrm{d}t, \mathrm{d}x\}$ and $\Delta_k = \mathrm{span}\,\{\omega \in \Delta_{k-1} | \dot{\omega} \in \Delta_{k-1}\}$. Then system S is locally static-feedback linearizable around ξ if and only if:*

(1) *The codistributions Δ_k are nonsingular at ξ for $k \in \mathbb{N}$.*
(2) *There exists $k^* \in \mathbb{N}$ big enough, such that $\Delta_k = \mathrm{span}\{\mathrm{d}t\}$.*
(3) *The codistributions Δ_k are locally involutive around ξ, for $k \in \mathbb{N}$.*

The following conditions generalize the ones of Jakubczyk and Respondek (1980) for time-varying systems. These conditions are related to the ones of Fliess, Lévine, Martin, Ollivier, and Rouchon (1997) for checking controllability.

Theorem 3. *Consider the canonical coordinates $\{t, x, u^{(0)}, \ldots\}$ of S. Let $G_0 = \mathrm{span}\left\{\frac{\partial}{\partial u_i^{(k)}} \mid i \in \lfloor m \rceil, k \in \mathbb{N}\right\}$. Define $G_k = G_{k-1} + [\frac{\mathrm{d}}{\mathrm{d}t}, G_{k-1}]$. Then the system is static-feedback linearizable if and only if the codistributions $G_k^\perp$ are smooth, non-singular and involutive, and $G_{k^*} = \mathrm{span}\{\mathrm{d}t\}^\perp$ for k^* big enough.*

Remark. It is important to point out the following:

(a) In Theorem 3 above, note that $G_0^\perp = \mathrm{span}\,\{\mathrm{d}t, \mathrm{d}x\}$. In particular, $G_k^\perp \subset G_0^\perp$ is always finite dimensional.
(b) Note that our time-varying results are local in time. It is not known if a time-invariant system may admit a time-varying flat output without admitting a time-invariant one (see Pereira da Silva and Rouchon (2004) and van Nieuwstadt et al. (1998)). However, if one restricts the class of transformations to static-state feedbacks, then there is no advantage in seeking time-varying static-state feedbacks (or time-varying flat outputs) for time-invariant systems (see Pereira da Silva (1997) and van Nieuwstadt et al. (1998)). In particular, for time-invariant systems, the results are global in time.
(c) Linearization by static feedback is related to differential flatness. If one admits simultaneous time scaling (see Sampei and Furuta (1986)), then the underlying concept is *Orbital Flatness* (see Fliess et al. (1999) and Guay (1999)). It is easy to verify that the example of Sampei and Furuta (1986) does not obey the conditions of Theorems 1–3 of this work, but it is linearizable by static-feedback and simultaneous time scaling.

3. Equivalence of solvability conditions

For the proof of Theorem 1 in the form stated in this work, the reader may refer to Pereira da Silva and Corrêa Filho (2001). We shall prove Theorems 2 and 3 by showing that both are equivalent to the Theorem 1.

Proposition 1. *The conditions of Theorem 1 are equivalent to the ones of Theorem 2.*

Before proving Proposition 1 consider the following lemma[4]

Lemma 1. *Let $\Omega^{(k)}$ be the nonsingular smooth codistribution defined in the statement of Theorem 1. If $\Omega^{(k)}$ is nonsingular, with $\dim = \Omega^{(k)} = r$ and $\Omega^{(k)} \oplus \mathrm{span}\{\mathrm{d}t\}$ is involutive, then there exists a set of smooth functions $\{\theta_1, \ldots \theta_r\}$ such that, locally, $\Omega^{(k)} = \mathrm{span}\left\{\mathrm{d}\theta_1 - \dot{\theta}_1\mathrm{d}t, \ldots, \mathrm{d}\theta_r - \dot{\theta}_r\mathrm{d}t\right\}$, $\Omega^{(k)} \oplus \mathrm{span}\{\mathrm{d}t\} = \mathrm{span}\,\{\mathrm{d}t, \mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_r\}$. Furthermore*

$$\omega \in \Omega^{(k+1)} \Leftrightarrow \dot{\omega} \in \Omega^{(k)} \oplus \mathrm{span}\{\mathrm{d}t\} \tag{4}$$

[2] See Pereira da Silva (1997) for another point of view. The proof of Sluis (1992, 1993) does not consider infinite prolongations. The statement presented here is a particular case of the results of Pereira da Silva and Corrêa Filho (2001) that hold also for implicit systems.

[3] Note that $\Omega^{(0)} \subset \mathrm{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^\perp$. The forms of $\mathrm{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^\perp$ are called contact forms.

[4] See Pereira da Silva and Corrêa Filho (2001) for a similar result that holds in the context of implicit system.

Proof of Lemma 1. Note first that, for all $k \in \mathbb{N}$, $\Omega^{(k)} \subset \Omega^{(0)} \subset \text{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^{\perp}$. In particular, if $\omega \in \Omega^{(k)} \cap \text{span}\{\mathrm{d}t\}$, then $\omega = \alpha \mathrm{d}t$, and so $0 = \langle \omega, \frac{\mathrm{d}}{\mathrm{d}t}\rangle = \alpha$. Hence, $\Omega^{(k)} \cap \text{span}\{\mathrm{d}t\} = \{0\}$. If for some k, $J^{(k)} = \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$ is involutive and nonsingular, by the Frobenius Theorem, one locally has $J^{(k)} = \text{span}\{\mathrm{d}t, \mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_r\}$, for convenient smooth functions $\theta_1, \ldots, \theta_r$, where $\{\mathrm{d}t, \mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_r\}$ is locally independent. Note now that $(\Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}) \cap \text{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^{\perp} = \Omega^{(k)}$. In fact, $\omega \in (\Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}) \cap \text{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^{\perp}$ is of the form $\omega = \tilde{\omega} + \alpha \mathrm{d}t$ with $\tilde{\omega} \in \Omega^{(k)}$. Since $\Omega^{(k)} \subset \text{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^{\perp}$, it follows that $\alpha \equiv 0$. Now, it will be shown that $\Omega^{(k)} = \text{span}\left\{\mathrm{d}\theta_1 - \dot{\theta}_1 \mathrm{d}t, \ldots, \mathrm{d}\theta_r - \dot{\theta}_r \mathrm{d}t\right\}$. In fact, note that $\omega_i = \mathrm{d}\theta_i - \dot{\theta}_i \mathrm{d}t \in (\Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}) \cap \text{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\right\}^{\perp} = \Omega^{(k)}$. Hence, $\Omega^{(k)} \supset \text{span}\left\{\mathrm{d}\theta_1 - \dot{\theta}_1 \mathrm{d}t, \ldots, \mathrm{d}\theta_r - \dot{\theta}_r \mathrm{d}t\right\}$. Now, as $\dim(\Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}) = \dim \Omega^{(k)} + 1 = r + 1$, it is easy to see that the equality $\Omega^{(k)} = \text{span}\left\{\mathrm{d}\theta_1 - \dot{\theta}_1 \mathrm{d}t, \ldots, \mathrm{d}\theta_r - \dot{\theta}_r \mathrm{d}t\right\}$ holds. Now assume that $\Omega^{(k)} = \text{span}\{\mathrm{d}\theta_1 - \dot{\theta}_1 \mathrm{d}t, \ldots, \mathrm{d}\theta_r - \dot{\theta}_r \mathrm{d}t\}$. It will be shown that (4) holds.[5] For this, let $\omega_i = (\mathrm{d}\theta_i - \dot{\theta}_i \mathrm{d}t)$ and let $\omega = \sum_{i=1}^{r} \alpha_i \omega_i \in \Omega^{(k)}$. Notice that $\dot{\omega}_i = \mathrm{d}\dot{\theta}_i - \ddot{\theta}_i \mathrm{d}t$. Note also that $\dot{\omega} = \sum_{i=1}^{r} (\dot{\alpha}_i \omega_i + \alpha_i \dot{\omega}_i)$. Then $d\omega = \sum_{i=1}^{r} [\mathrm{d}\alpha_i \wedge \omega_i - \alpha_i \mathrm{d}\dot{\theta}_i \wedge \mathrm{d}t] = \sum_{i=1}^{r} [\mathrm{d}\alpha_i \wedge \omega_i - \alpha_i (\mathrm{d}\dot{\theta}_i - \ddot{\theta}_i \mathrm{d}t) \wedge \mathrm{d}t] = \sum_{i=1}^{r} [\mathrm{d}\alpha_i \wedge \omega_i - \alpha_i \dot{\omega}_i \wedge \mathrm{d}t] = \sum_{i=1}^{r} [\mathrm{d}\alpha_i \wedge \omega_i + \dot{\alpha}_i \omega_i \wedge \mathrm{d}t - \dot{\alpha}_i \omega_i \wedge \mathrm{d}t - \alpha_i \dot{\omega}_i \wedge \mathrm{d}t] = \sum_{i=1}^{r} [\mathrm{d}\alpha_i \wedge \omega_i + \dot{\alpha}_i \omega_i \wedge \mathrm{d}t - \dot{\omega} \wedge \mathrm{d}t]$. Since $\mathrm{d}\alpha_i \wedge \omega_i$ and $\dot{\alpha}_i \omega_i \wedge \mathrm{d}t$ are in the ideal $(\Omega^{(k)})$, then it follows that $\mathrm{d}\omega \bmod (\Omega^{(k)}) = \dot{\omega} \wedge \mathrm{d}t \bmod (\Omega^{(k)})$. In particular, if $\mathrm{d}\omega \in (\Omega^{(k)})$ then, $0 = \dot{\omega} \wedge \mathrm{d}t + \sum \eta_i \wedge \omega_i$ for convenient one forms η_i. As the set $\{\mathrm{d}t, \omega_1, \ldots, \omega_r\}$ is linearly independent, by the Cartan Lemma (see Warner (1971, p. 80)), it follows that $\dot{\omega} \in \text{span}\{\mathrm{d}t, \omega_1, \ldots, \omega_r\} = \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$. If $\dot{\omega} \in \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$, then $\dot{\omega} \wedge \mathrm{d}t \in (\Omega^{(k)})$, and hence, $\mathrm{d}\omega \in (\Omega^{(k)})$. □

Proof of Proposition 1. It will be shown first that the assumptions of Theorem 1 imply that $\Delta_k = \Omega^{(0)} \oplus \text{span}\{\mathrm{d}t\}$. In particular, the assumptions of Theorem 2 also hold. In fact, note that, by construction, $\Delta_0 = \Omega^{(0)} \oplus \text{span}\{\mathrm{d}t\}$. Assume by induction that $\Delta_k = \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$ and suppose that $\Omega^{(k)}$ is nonsingular and $\Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$ is involutive. Hence, by Lemma 1, $\Delta_k = \text{span}\{\mathrm{d}t, \mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_r\}$ and $\Omega_k = \text{span}\left\{\mathrm{d}\theta_1 - \dot{\theta}_1 \mathrm{d}t, \ldots, \mathrm{d}\theta_r - \dot{\theta}_r \mathrm{d}t\right\}$ and (4) holds. Now let $\tilde{\omega} \in \Delta_{k+1}$. Since $\tilde{\omega} \in \Delta_k$, then $\tilde{\omega} = \omega + \alpha \mathrm{d}t$, with $\omega \in \Omega^{(k)}$. Note that $\dot{\omega} \in \Delta_k$ is equivalent to have $\dot{\tilde{\omega}} \in \Delta_k = \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$. By (4), it follows that this is equivalent to have $\tilde{\omega} \in \Omega^{(k+1)}$. Hence $\Delta_{k+1} = \Omega^{(k+1)} \oplus \text{span}\{\mathrm{d}t\}$. Now it will be shown that the assumptions of Theorem 2 imply that $\Delta_k = \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$. In particular, the assumptions of Theorem 2 imply that the ones of Theorem 1 hold. Now assume by induction that $\Delta_k = \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$ and suppose that Δ_k is nonsingular and involutive (as seen above, it is true for $k = 0$). Let $\omega \in \Delta_{k+1}$. By definition $\dot{\omega} \in \Delta_k$. As $\omega \in \Delta_k \supset \Delta_{k+1}$, $\omega = \tilde{\omega} + \alpha \mathrm{d}t$. Then, $\dot{\omega} = \dot{\tilde{\omega}} + \dot{\alpha} \mathrm{d}t$. Hence $\dot{\omega} \in \Delta_k$ implies $\dot{\tilde{\omega}} \in \Omega^{(k)} \oplus \text{span}\{\mathrm{d}t\}$. By condition (4) of Lemma 1, it follows that $\omega \in \Delta_{k+1}$ is equivalent to have $\tilde{\omega} \in \Omega^{(k+1)}$. In particular, $\Delta_{k+1} = \Omega^{(k+1)} \oplus \text{span}\{\mathrm{d}t\}$. □

The next Proposition is an indirect proof of Theorem 3.

Proposition 2. *The conditions of Theorem 2 are equivalent to the ones of Theorem 3.*

Proof. To show that the conditions of Theorem 2 holds, it suffices to show that $\Delta_k = G_k^{\perp}$ for all $k \in \mathbb{N}$. This is true for $k = 0$. By induction, assume that this is true for some k. Let $\omega \in \Delta_{k+1}$. Then $\dot{\omega} \in \Delta_k$. To show that $\Delta_{k+1} \subset G_{k+1}^{\perp}$, it suffices to show that $\langle \omega, \tau \rangle = 0$ for all $\tau \in G_{k+1}^{\perp}$. Since $\tau \in G_{k+1}$ is of the form $\tau_1 + [\frac{\mathrm{d}}{\mathrm{d}t}, \tau_2]$ for τ_1 and τ_2 in G_k and $\Delta_{k+1} \subset \Delta_k$, it suffices to show that $\langle \omega, [\frac{\mathrm{d}}{\mathrm{d}t}, \tau] \rangle = 0$ for for all τ in G_k. In fact, this follows from the identity

$$\langle L_{\frac{\mathrm{d}}{\mathrm{d}t}} \omega, \tau \rangle = L_{\frac{\mathrm{d}}{\mathrm{d}t}} \langle \omega, \tau \rangle - \left\langle \omega, \left[\frac{\mathrm{d}}{\mathrm{d}t}, \tau\right] \right\rangle. \tag{5}$$

To show the inverse inclusion, take $\omega \in G_{k+1}^{\perp} \subset G_k^{\perp}$. Then, by the same identity, for all $\tau \in G_k$ one has $\langle \dot{\omega}, \tau \rangle = 0$. Assume now that the conditions of Theorem 2 hold. Then the nonsingularity of Δ_k implies that $\Delta_k^{\perp}$ is smooth for all $k \in \mathbb{N}$. To show that the conditions of Theorem 3 holds, it suffices to show that $\Delta_k^{\perp} = G_k$ for $k \in \mathbb{N}$. This is true for $k = 0$. By induction, assume that this is true for some k. Let $\tau \in \Delta_k^{\perp}$ and let $\omega \in \Delta_k = G_k^{\perp}$. Then, from the identity (5), it follows that $\dot{\omega} \in \Delta_k$ if and only if $\omega \in G_{k+1}^{\perp}$. □

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[5] See Eq. (A.3) in page 1947 of Pereira da Silva and Corrêa Filho (2001).