An infinite dimensional differential-geometric approach for nonlinear systems : Part I - $I\!R^A$-manifolds, diffieties and Systems

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Abstract

This paper is the first part of a survey about an infinite dimensional differential geometric approach of nonlinear control systems. It summarizes the basic definitions and the fundamental results about $I\!R^A$-manifolds, diffieties and control systems that are used in the second part of this work.
Keywords. Differential Geometry, $I\!R^A$-manifolds, Diffieties, differential-geometric approach.

1 Introduction

The aim of the Part I of this paper is to present an exposition of the main facts about $I\!R^A$ manifolds and diffieties that are needed in Part II of this work (Pereira da Silva 2008). The text is written in a very elementary style, and is conceived for students and researchers that have some knowledge about finite dimensional manifolds, but are not acquainted with the difficulties and particularities that appears in the context of infinite dimensional manifolds. The classical constructions using projective (and inductive) limits have been avoided. Instead of using those elegant tools, one will find explicit constructions of the topologies and maps. This point of view is suitable for such a basic introduction to $\mathbb{R}^A$-manifolds and diffieties. The authors believe that the great majority of the results of the first part of this work are obvious for specialists on diffieties. However, many results that have been stated in the literature without proof are certainly not so trivial for the novice on the subject. For more advanced expositions, one may refer to (Alekseevskij, Vinogradov & Lychagin 1991, Anderson & Ibragimov 1979, Golubitsky & Guillemin 1973, Ibragimov 1985, Krasil'shchik, Lychagin & Vinogradov 1986, Olver 1993, Vinogradov 1984, Zharinov 1992, Tsujishita 1990).

1.1 Organization

1.2 Notations

The field of real numbers will be denoted by $\mathbb{R}$. The set of natural numbers $\{0, 1, 2, \ldots\}$ will be denoted by $\mathbb{N}$, the set $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}^*$, and the set of integers is denoted by $\mathbb{Z}$. If H is a finite set then $\operatorname{card} H$ stands for the cardinal of H. If H is finite, then $\operatorname{card} H$ is the number of elements of H. We will use the standard notations of differential geometry in the finite and infinite dimensional case (Warner 1971, Zharinov 1992). If M is a matrix (or a vector), then M^T stands for its transpose. Let z_1 and z_2 be column vectors. For simplicity, we abuse notation, letting (z_1, z_2) stand for the column vector $(z_1^T, z_2^T)^T$.

2 The space $\mathbb{R}^A$

Following (Bernšteĭn & Rosenfel′d 1973), a smooth infinite-dimension manifold is usually understood to be a space obtained by pasting together open subsets of a "model" topological vector space (most frequently a Banach or a Hilbert space) by means of isomorphisms satisfying certain smoothness conditions. In what follows, one is interested in a geometry that is adapted to infinite prolongations of differential equations, that is, the geometry of infinite jets (Saunders 1989, Krasil'shchik et al. 1986, Zharinov 1992). Hence one must consider manifolds whose "model" is a Fréchet space, or a $\mathbb{R}^A$-space, that is also denoted by $\mathbb{R}^\infty$. This kind of linear vector spaces are equipped with a topology (the Fréchet topology) that is not defined from a norm, and so, those manifolds do not present all the nice features of the manifolds that are modeled by Banach spaces (Lang 1995, Abraham & Marsden 1988). In fact, many difficulties arise when one is trying to consider versions of the inverse function theorem, the flow-box theorem etc (Zharinov 1992).

2.1 $\mathbb{R}^A$-spaces, continuity and smoothness

{sFrechet}

Let A be a countable set. The space $\mathbb{R}^A = \prod_{\alpha \in A} E_\alpha$, where $E_\alpha = \mathbb{R}$ is the set of functions $\xi : A \to \mathbb{R}$. A point $\xi \in \mathbb{R}^A$ may be denoted by $(x^\alpha, \alpha \in A)$. The coordinate function $x_\alpha : \mathbb{R}^A \to \mathbb{R}$ maps $\xi = (x^\alpha, \alpha \in A)$ to $x^\alpha = \xi(\alpha)$.

The space $\mathbb{R}^A$ is a $\mathbb{R}$-linear vector space with the following definition

- $(\xi + \zeta)(\alpha) = \xi(\alpha) + \zeta(\alpha)$ for all $\alpha \in A$, $\xi, \zeta \in \mathbb{R}^A$.
- $(c\,\xi)(\alpha) = c\,\xi(\alpha)$, for all $\alpha \in \mathbb{R}^A$ and $c \in \mathbb{R}$.

In other words, these operation are defined in the usual, componentwise way, that is, $(x^\alpha, \alpha \in A) + (z^\alpha, \alpha \in A) = (x^\alpha + z^\alpha, \alpha \in A)$, and $c(x^\alpha, \alpha \in A) = (cx^\alpha, \alpha \in A)$.

The set $\mathbb{R}^A$ can be endowed with the Fréchet topology $\mathfrak{F}$ (it is the product, or Tychonoff topology). A basis $\mathfrak{B}$ of this topology is the collection of subsets of $\mathbb{R}^A$ of the form $\mathcal{B} = \{\xi \in \mathbb{R}^A \mid \xi = (x^\alpha, \alpha \in A), |x^\alpha - \delta^\alpha| < \epsilon^\alpha, \alpha \in F\}$, where F is a finite subset of A , $\delta^\alpha \in \mathbb{R}$ and ϵ^α is a positive real number for $\alpha \in F$. Remember that an open set U of $\mathfrak{F}$ is an arbitrary union $\bigcup_{i \in \Lambda} \mathcal{B}_i$ of basic open sets $\mathcal{B}_i \in \mathfrak{B}$.

It can be shown that $\mathbb{R}^A$ is a projective limit and the Fréchet topology is the corresponding projective limit topology.

Let $\sigma : A \to \mathbb{N}^*$ be a bijection. One denotes $\mathbb{R}^{\mathbb{N}^*} = \mathbb{R}^\infty$, and this bijection induces an isomorphism between $\mathbb{R}^A$ and $\mathbb{R}^\infty$, namely, $(x^\alpha : \alpha \in A) \mapsto (x^i : i \in \mathbb{N}^*)$, where $x^i = x^{\sigma(i)}$. In particular one may denote a point ξ of $\mathbb{R}^A$ by an infinite vector $(x^1, x^2, x^3, \ldots)$, where $x^i = \xi(i), i \in \mathbb{N}^*$. In particular, coordinate functions may be denoted by x_i. A common abuse of notation[1] is to consider $x_i(\xi) = x_i$.

The projection π_k stands for the map $(x_1, x_2, x_3, \ldots, x_k, x_{k+1}, \ldots) \mapsto (x_1, x_2, x_3, \ldots, x_k)$. It is a simple exercise to show that π_k is an open map, *i. e.* , $\pi_k(U)$ is an open subset of $\mathbb{R}^k$ for every open set $U \subset \mathbb{R}^A$.

The following definition is the classical definition of continuity in General Topology.

Definition 1 *(Continuous Map) Let $U \subset \mathbb{R}^A$, where U is an open set of $\mathbb{R}^A$, and $V \subset \mathbb{R}^B$. A map $g : U \subset \mathbb{R}^A \to V \subset \mathbb{R}^B$ is continuous, if $g^{-1}(W)$ is an open set of $\mathbb{R}^A$ for every open set W of $\mathbb{R}^B$.*

{pContinuousMap}

Proposition 1 *Let A,B be countable sets. Let $y^j, j \in \mathbb{N}^*$ be the coordinate functions of $\mathbb{R}^B$. Let $U \subset \mathbb{R}^A$ and $V \subset \mathbb{R}^B$. A map $g : U \subset \mathbb{R}^A \to V \subset \mathbb{R}^B$ is continuous, if and only if the "*component functions*" $g_j : U \to \mathbb{R}$ defined by $g_j = y_j \circ g$ are continuous for $j \in \mathbb{N}^*$.*

Proof. The proof is an easy exercise that is left to the reader. ☐

[1] A more precise notation should be $x_i(\xi) = x^i$. Note that x^i is a real number, whereas x_i is the coordinate function $x_i : \mathbb{R}^A \to \mathbb{R}$.

{dSmooth}

Definition 2 *(Smooth Map) Let A,B be countable sets. Let U be an open set of $I\!R^A$. A map $f : U \to I\!R$ is* smooth *if, for every $\xi \in U$ there exists some open neighborhood $V \subset U$ of ξ such that $f|_V = \tilde{f} \circ \pi_k$, for a convenient smooth*[2] *function $\tilde{f} : W \subset I\!R^k \to I\!R$, where $W = \pi_k(V)$. Let $y^j, j \in I\!N^*$ be the coordinate functions of $I\!R^B$. A map $g : U \to I\!R^B$ is* smooth *if $y_j \circ g : U \to I\!R$ is smooth.*

{r1}

Remark 1 *The following affirmations are straightforward to verify:*

- *A map $g : U \to I\!R^B$ is smooth if and only if, for every smooth function $\phi : I\!R^B \to I\!R$ then $\phi \circ g$ is smooth (and this may be an alternate definition of smooth maps).*
- *Let S be a topological space and let $\{B_\lambda : \lambda \in \Lambda\}$ be a basis of this topology. Since every open set of S is an arbitrary union of basic open sets, when one wants to show that a map $\phi : T \to S$ is continuous, it suffices to show that $\phi^{-1}(B_\lambda)$ is an open subset of T for every $\lambda \in \Lambda$.*
- *Let $U = \bigcup_{\lambda \in \Lambda} V_\lambda$, where V_λ is an open set for all $\lambda \in \Lambda$. Note now that, if $\phi : U \subset I\!R^A \to I\!R^B$ is such that $\phi|_{V_\lambda}$ is continuous, then $\phi : U \subset I\!R^A \to I\!R^B$ is continuous. In fact, if $\phi_\lambda = \phi|_{V_\lambda}$, for every open set $W \subset I\!R^B$, $\phi^{-1}(W) = \bigcup_{\lambda \in \Lambda} \phi_\lambda^{-1}(W)$.*

Locally speaking, a smooth function must depend only on a finite number of variables. Hence, around a point ξ, there exists a neighborhood such that this number of variables is minimal. This is the idea of the next definition.

{dMinimalIndex}

Definition 3 *(*Minimal Index and Minimal Neihghborhood of a Smooth Function*) Let $\{x_i : i \in I\!N\}$ be the set of canonical coordinate functions of $I\!R^A$. Let $U \subset I\!R^A$ be an open set. Let $\phi : U \to I\!R$ be a smooth function. Let $\xi \in U$ and let V_ξ be an open neighborhood of ξ such that $f|V_\xi = \tilde{\phi} \circ \pi_k$. Among all the open neighborhoods V_ξ with this property, there exists V_ξ^*, called* a minimal neighborhood of ϕ at ξ, *such that the $k = k^*$ is minimal. Such k^* is called the* minimal index of ϕ at ξ. *For a constant function on V_ξ^* one defines $k^* = 0$.*

Remark 2 *Note that $k^* \in I\!N$ is a unique, well defined integer. However any open subset of V_ξ^* is also a* minimal neighborhood of ϕ at ξ. *This explains why one says* a minimal neighborhood at ξ *instead of* the minimal neighborhood at ξ. *If V_ξ^* and W_ξ^* are minimal neighborhoods, then it is clear that $V_\xi^* \bigcup W_\xi^*$ is also a minimal neighborhood at ξ. Note that the minimal index is a property of the germ*[3] *of ϕ.*

Definition 4 *The union of all minimal neighborhoods of ϕ at ξ is called* the minimal neighborhood at ξ.

[2] Smooth in the usual sense of finite dimensional analysis.
[3] See section 2.3 for the definition of germ of a function.

Proposition 2 *Let $X = \{x_i :\in I\!N\}$ be the set of canonical coordinate functions of $I\!R^A$. Let $\bar{X} = \{i_1, \ldots, i_k\} \subset I\!N$ with $0 < i_1 < i_2 < \ldots < i_k$. Define the projection $\pi_{\bar{X}} : U \subset I\!R^A \to I\!R^k$ by $x \mapsto (x_{i_1}(x), \ldots, x_{i_k}(x))$. Then $\pi_{\bar{X}}$ is continuous.*

Proof. Take a basic open set $V = \{w \in I\!R^k \mid |w_j - \delta_j| < \epsilon_j, j = 1, \ldots, k\}$ of $I\!R^k$ and observe that $\pi_{\bar{X}}^{-1}(V) = U \bigcap \mathcal{B}$, where $\mathcal{B} = \{x \in I\!R^A \mid |x_{i_j}(x) - \delta_j| < \epsilon_j, j = 1, \ldots, k\}$ is a basic open set of $I\!R^A$. □

Proposition 3 *Let $U \subset I\!R^A$ be an open set. Every smooth function $\phi : U \subset I\!R^A \to I\!R$ is continuous.*

Proof. Let $\xi \in U$ and let $V_\xi \subset U$ an open neighborhood of ξ such that $\phi|_{V_\xi} = \tilde{f} \circ \pi_k|_{V_\xi}$. Then $\phi|_{V_\xi}$ is continuous since it is the composition of continuous maps. As $U = \bigcup_{\xi \in U} V_\xi$, then ϕ is continuous (see Remark 1). □

Proposition 4 *Let A, B be countable sets (finite or infinite). Let $U \subset I\!R^A$ be an open set. Every smooth map $\phi : U \subset I\!R^A \to I\!R^B$ is continuous.*

Proof. Consider the case where both A, B are infinite (other cases are left to the reader). Denote the coordinate functions of $I\!R^B$ by $y_j, j \in I\!N^*$. Let $V \subset I\!R^B$ be the basic open set $V = \{y \in I\!R^B \mid |y_j(y) - \delta_j| < \epsilon_j, j \in F\}$, where $F \subset I\!N^*$ is a finite subset. Let $\phi_j = y_j \circ \phi$. Then $\phi^{-1}(V) = \bigcap_{j \in F} \phi_j^{-1}[(\delta_j - \epsilon_j, \delta_j + \epsilon_j)]$. By the last proposition, each ϕ_j is continuous and this concludes the proof. □

Remark 3 *One may show that the Fréchet topology is the weaker one such that the projections $\pi_{\bar{X}}$ are continuous. In (Bernšteĭn & Rosenfel′d 1973) one may found an equivalent definition of smooth function and smooth maps. Such elegant definition is based on projective limits.*

Proposition 5 *Let $f : U \subset I\!R^A$ and $g : V \subset I\!R^B \to U$ be smooth maps, where U and V are open sets. Then $f \circ g : V \to I\!R^A$ is smooth.*

Proof. Let $\xi \in V$ and $\nu = g(\xi) \in U$. Since the notion of smoothness is a componentwise notion, without loss of generality, assume that f is a function $f : U \to I\!R$, and write $f|_W = \tilde{f} \circ \pi_k|_W$, where W is an open neighborhood of ν (see Definition 2). As g is continuous, then $Y = g^{-1}(W)$ is open. On Y, one may locally write[4], $f \circ g(x) = \tilde{f} \circ \pi_k \circ g(x) = \tilde{f} \circ (g_1(x), \ldots, g_k(x))$. Now, let $l = \max\{k_1^*, \ldots, k_k^*\}$, where k_i^* is the minimal index of g_i at ξ. Let V_i^* be the cooresponding minimal neighborhods of g_i at ξ. Let $Z = \bigcap_{i=1}^k V_i^*$. By using the idea of Remark 4, on Z, one may write $\tilde{f} \circ \pi_k \circ g(x) = \tilde{f}(g_1(x), \ldots, g_k(x)) =$

[4] On $g^{-1}(W)$.

$\tilde{f}(\tilde{g}_1 \circ \pi_l(x), \ldots, \tilde{g}_k \circ \pi_l(x))$. Defining $\tilde{g} : \pi_l(Z) \subset I\!R^l \to I\!R^k$ by $s \mapsto (\tilde{g}_1(s) \circ \pi_l(s), \ldots, \tilde{g}_k \circ \pi_l(s))$, one gets $f \circ g = (\tilde{f} \circ \tilde{g}) \circ \pi_l$, which is the composition of the smooth function $(\tilde{f} \circ \tilde{g})$ with π_l. Hence, by definition, $f \circ g$ it is smooth. □

{rX}

Remark 4 *Let k^* be the minimal index of a smooth function $\phi : U \subset I\!R^A \to I\!R$ at ξ, and let V_ξ be a minimal neighborhood of ϕ at ξ. Then one may write $\phi|_{V_\xi} = \tilde{\phi} \circ \pi_{k^*}|_{V_\xi}$. So for every $k \geq k^*$ one obtains $\phi|_{V_\xi} = (\tilde{\phi} \circ \pi_{k,k^*}) \circ \pi_k|_{V_\xi}$, where π_{k,k^*} is the map defined by $(x_1, \ldots, x_{k^*}, \ldots, x_k) \mapsto (x_1, \ldots, x_{k^*})$. In particular, for all $k \geq k^*$ there exists a smooth map $\hat{\phi} : W \subset I\!R^k$, where $\hat{\phi} = \tilde{\phi} \circ \pi_{k,k^*}$, and $W = \pi_k(V_\xi)$, such that $\phi|_{V_\xi} = \hat{\phi} \circ \pi_k$.*

{dDifferential}

Definition 5 *Let $U \subset I\!R^A$ and let $f : U \to I\!R$ be a smooth function. The differential $df : I\!R^A \to I\!R$ is the linear map defined in the following way. As f is smooth, one may locally write $f = \tilde{f} \circ \pi_k$ (see definition 2). Then, for $x \in I\!R^A$ one sets $df(x) = d\tilde{f}(\pi_k(x)) \circ \pi_k$ where $d\tilde{f} : I\!R^k \to I\!R$ is the standard differential of the smooth function $\tilde{f}$ that is defined in finite-dimensional analysis. Let $\phi : U \to I\!R^B$ be a smooth map. Let $x \in U$. Let $y_j : j \in B$ bee the coordinate functions of $I\!R^B$. Define*[5] *$d\phi(x) : I\!R^A \to I\!R^B$ by $y_j \circ d\phi(x) = d(y_j \circ \phi)(x)$.*

Remark 5 *It is easy to show that the last definition does not depend on a particular representation of a function,* i. e. , *if a function admits two different representations $f = \tilde{f}_1 \circ \pi_1 = \tilde{f}_2 \circ \pi_2$, then the differential defined with these different representations must coincide (see Appendix A).*

The proof of the following proposition is straightforward and is left to the reader.

{pMinimal}

Proposition 6 *Let $\phi : Z : I\!R^A \to I\!R$ be an smooth function, where Z is a minimal neighborhood of ϕ at ξ. Let k^* be the minimal index of ϕ at ξ.*

Let $\xi = (\xi_1, \xi_2, \xi_3, \ldots) \in Z$ and let $\Psi : I\!R \to I\!R^A$ be such that $\Psi(t) = (\xi_1, \xi_2, \xi_3, \ldots, \xi_{l-1}, \xi_l + \alpha t, \xi_{l+1}, \xi_{l+2}, \ldots)$. By continuity of Ψ, there exists $\epsilon > 0$ such that that $\Psi[(-\epsilon, \epsilon)] \subset Z$. If $\phi \circ \Psi(t)$ is not constant for $|t| < \epsilon$, then $k^* > l$.

2.2 The tangent bundle $TI\!R^A$ and fields

Definition 6 *Let $U \subset I\!R^A$ be an open subset. The tangent bundle TU is the triple $(U \times I\!R^A, U, \pi)$ where $\pi : U \times I\!R^A \to U$ is the canonical projection.*

Definition 7 *A field τ on an open set $U \subset I\!R^A$ is a smooth section of the bundle T, that is, a map $\tau : U \to U \times I\!R^A$ of the form $x \mapsto (x, \bar{\tau}(x))$, where the map $\bar{\tau} : U \to I\!R^A$ is smooth.*

Remark 6 *For convenience, one may identify τ with $\bar{\tau}$.*

[5] One may give a more intrinsic definition using projective limits (see (Bernšteĭn & Rosenfel′d 1973)).

Let τ be a vector field, and let $f : U \to I\!R$ be a smooth function. Let $(x_i, i \in I\!R^{I\!N^*})$ be the canonical coordinate functions of $I\!R^A$. Then $\tau(x) = (x, (\tau_i(x) : i \in I\!N^*))$. If $f : U \to I\!R$ is is smooth, for all $\xi \in U$ there exists an open neighborhood V_ξ of ξ and a local representation $f = \tilde{f} \circ \pi_k$ defined on V_ξ.

Definition 8 *The Lie-derivative $L_\tau f$, also denoted by $\tau(f)$, is the smooth function $L_\tau f : U \to I\!R$ defined by the rule*

$$L_\tau f(x) = \sum_{i=1}^{k} \tau_i(x) \frac{\partial \tilde{f}}{x_i}(\pi_k(x)) \tag{1}$$ {eLtau}

It is a simple exercise to show that the definition above do not depend on the particular representation of f that is chosen.

The set of fields on U has a structure of $C^\infty(U)$-module defined by the operations $(\tau_1 + \tau_2)(x) = (x, \bar{\tau}_1(x) + \bar{\tau}_2(x))$ and $(f\tau)|_x = (x, f(x)\bar{\tau}(x))$, where τ_1 and τ_2 are arbitrary fields on an open subset $U \subset I\!R^A$ and $f : U \to I\!R$ is a smooth function.

2.3 Derivations and the tangent space $T_x I\!R^A$

{ssTangent}

Let $x \in I\!R^A$. Let $C^\infty\{x\}$ be the set of smooth functions $f : V_x \to I\!R$, where $V_x \subset I\!R^A$ is an open neighborhood of x. In $C^\infty\{x\}$ one may define the following equivalence relation, denoted by $\sim$. Given two functions $f_1, f_2 \subset C^\infty\{\xi\}$, then $f_1 \sim f_2$ if $f_1|_{W_\xi} = f_2|_{W_\xi}$, where W_ξ is some open neighborhood of x. The equivalent class of $f \in C^\infty\{x\}$ is denoted by $[f]$, and is called the *germ* of f at x. The set of all germs at x is denoted by $\mathcal{C}^\infty(x)$. Clearly, one may define a structure of $I\!R$-vector space on $\mathcal{C}^\infty(x)$. If $f_1 : U_x \to I\!R$ and $f_2 : V_x \to I\!R$, let $W = U_x \bigcap V_x$. Then define $[f_1] + [f_2] = [f_1|_W + f_2|_W]$ and for $\alpha \in I\!R$, let $\alpha[f_1] = [\alpha f_1]$. Note also that $\mathcal{C}^\infty(x)$ has also a structure of a ring, if one defines $[f_1][f_2] = [(f_1|_W)(f_2|_W)]$.

Definition 9 *A* derivation *at $x \in I\!R^A$ is a map $v_x : \mathcal{C}^\infty(x) \to I\!R$ such that*

- *The map v_x is linear, that is, $v_x([f_1] + [f_2]) = v_x([f_1]) + v_x([f_2])$ and $v_x(a[f]) = av_x([f]), a \in I\!R$.*
- *$v_x([f_1][f_2]) = f_2(x)v_x([f_1]) + f_1(x)v_x([f_2])$.*

By convenience one may let $v_x(f)$ stands for $v_x([f])$. In this way one may regard v_x as a map from $C^\infty\{x\}$ to $I\!R$. A derivation v_x is also called a tangent vector *at x. Clearly, the set $\mathcal{T}_x U$ of all tangent vectors at x is a $I\!R$-vector space with the operations $(v_x^1 + v_x^2)(f) = v_x^1(f) + v_x^2(f)$ and $(\alpha v_x)(f) = \alpha(v_x(f)), \alpha \in I\!R$.*

Let $T_x U$ stand for the elements τ_x of $TU = U \times I\!R^A$ of the form

$$\tau_x = (x, (\tau_1, \tau_2, \tau_3, \ldots)).$$

Given $f \in C^\infty\{x\}$, it is easy to verify that one may regard τ_x as a tangent vector v_x at x, with the action of v_x defined (see Definition 1), by:

$$v_x([f]) = \sum_{i=1}^{k} \tau_i \frac{\partial \tilde{f}}{x_i}(\pi_k(x)) \qquad (2)$$ {eRule}

The following theorem shows that one may identify $\mathcal{T}_xU$ with T_xU by the rule (2).

{t1}

Theorem 1 *The map $\alpha : T_xU \to \mathcal{T}_x$ such that $\tau_x \mapsto v_x$, where the action of v_x is defined by* (2)*, is an isomorphism of $I\!R$-vector spaces.*

Proof. The linearity of v_x is clear. Now, given two functions f_1 and f_2 belonging to $C^\infty\{x\}$, if $k = \max\{k_1, k_2\}$, where k_i is the minimal index of $f_i, i = 1, 2$ at x, and $V = V_1 \bigcap V_2$, where V_i is the minimal index of f_i at x, one may write $f_i|_V = \tilde{f}_i \circ \pi_k|_V$. Note that $f_1 f_2|_V = (\tilde{f}_1 \tilde{f}_2) \circ \pi_k|_V$. then, one may apply the last formula for the product $[f_1][f_2]$ showing that $v_x([f_1][f_2]) = f_2(x)v_x([f_1]) + f_1(x)v_x([f_2])$. This shows that τ_x may be regarded as a tangent vector, and so α is well defined.

Now one has to show that any tangent vector $v_x \in \mathcal{T}_xU \in$ is the image $\alpha(\tau_x)$ for a convenient $\tau_x \in T_xU$. Let $\{x_i, i \in I\!N^*\}$ be canonical coordinates for $I\!R^A$. Let $\tau_i = v_x(x_i)$ and define $\tau_x = (\tau_1, \tau_2, \tau_3, \ldots)$. It will be shown that $\alpha(\tau_x) = v_x$, showing our claim.

Note first that, if $f = f_1 f_2$ with $f_1(x) = f_2(x) = 0$ then $v_x(f) = f_1(x)v_x(f_2) + f_2(x)v_x(f_1) = 0$. If $f(x) = 1$, for every x, then $v_x(f^2) = f(x)v_x(f) + f(x)v_x(f) = 2v_x(f)$ implies that $v_x(f) = 0$ for every x. By linearity, for every constant function f, one has $v_x(f) = 0$.

Let $\tilde{f} : B_\epsilon(\tilde{x}) \subset I\!R^n \to I\!R$, be a smooth function, where $B_\epsilon(\tilde{x}) = \{x \in I\!R^n \,|\, \|x - \tilde{x}\| < \epsilon\}$. Then, by the fundamental theorem of Calculus and the chain rule, one may write for all $x \in B_\epsilon(\tilde{x})$

$$\begin{aligned} \tilde{f}(x) - \tilde{f}(\tilde{x}) &= \tilde{f}(\tilde{x}) + \int_0^1 \frac{d}{dt}\left\{\tilde{f}\left[\tilde{x} + t(x - \tilde{x})\right]\right\} dt \\ &= \tilde{f}(\tilde{x}) + \sum_{i=1}^{n}\left(\int_0^1 \frac{\partial \tilde{f}}{\partial x_i}[\tilde{x} + t(x - \tilde{x})]dt\right)(x_i - \tilde{x}_i) \end{aligned}$$

In particular

$$\tilde{f}(x) - \tilde{f}(\tilde{x}) = \frac{\partial \tilde{f}}{\partial x_i}|_{\tilde{x}}(x_i - \tilde{x}_i) + \sum_{i=1}^{n} f_1^i(x) f_2^i(x) \qquad (3)$$ {eProductZero}

where $f_i^1 = \left\{\left(\int_0^1 \frac{\partial \tilde{f}}{\partial x_i}[\tilde{x} + t(x - \tilde{x})]dt\right) - \frac{\partial \tilde{f}}{\partial x_i}|_{\tilde{x}}\right\}$ and $f_i^2 = (x_i - \tilde{x}_i)$.

Now, given $f \in \mathcal{C}^\infty(x)$, let $f|_V = \tilde{f} \circ \pi_k|_V$. Let $\tilde{x} = \pi_k(x)$. Abusing notation, one may write $f|_V = \tilde{f}(x_1, \ldots, x_k)$. From (3), one may write $\tilde{f}(x_1, \ldots, x_k) = \sum_{i=1}^k a_i(x_i - \tilde{x}_i) + \sum_{i=1}^n f_1^i f_2^i$, where $f_1^i(\tilde{x}) = 0$ and $f_2^i = (\tilde{x}) = 0$ and $a_i = \frac{\partial \tilde{f}}{\partial x_i}|_{\tilde{x}}$.

From this, one gets $v_x([f]) = \sum_{i=1}^k v_x(x_i) a_i$ and, by (2), this concludes the proof (compare also with (1)). $\square$

Let $x_i, i \in \mathbb{N}$ be canonical coordinate functions of $\mathbb{R}^A$. By the Proof of Theorem 1, one may define the tangent vector $\frac{\partial}{\partial x_i}|_x \in \mathcal{T}_x$ such that $\frac{\partial}{\partial x_i}|_x(x_j) = \delta_{ij}$, where $\delta_{ij} = 0$, if $i \neq j$ and $\delta_{ij} = 1$, if $i = j$. Given $\tau_x \in T_x U$, with $v_x = \alpha(\tau_x)$, one may identify $\tau_x \in T_x U$ with v_x.

Abusing notation, one writes

$$\tau_x = \sum_{i=1}^{\infty} \tau_i \frac{\partial}{\partial x_i}|_x$$

where

$$\tau_i = \tau_x(x_i)$$

From Theorem 1, one may regard TU as the union of all tangent spaces $\bigcup_{x \in U} T_x U$, where $T_x U$ is the linear space of tangent vectors (or derivations) at x. In finite dimensional theory, the set $\{\frac{\partial}{\partial x_i}|_x, i \in n\}$ form a basis of $T_x U$. In our infinite dimensional setting, one may say that every tangent vector τ_x is an infinite[6] sum $\sum_{i=1}^{\infty} \tau_i \frac{\partial}{\partial x_i}|_x$. Note that the action of τ_x on a function is always a finite sum, and so this notation has a precise sense, without the need of establishing any convergence result.

Using this notation, one may prove the following straightforward consequence.

{eFieldRA}

Proposition 7 *Let $U \subset \mathbb{R}^A$ and let $\tau : U \to TU$ be a field. Let $x_i, i \in \mathbb{N}$ be canonical coordinate functions of $\mathbb{R}^A$. Then*

$$\tau(x) = \sum_{i=1}^{\infty} \tau_i(x) \frac{\partial}{\partial x_i}|_x \tag{4a}$$ {eTauxA}

where

$$\tau_i(x) = \tau(x)(x_i) \tag{4b}$$ {eTauxB}

and $\tau_i : U \to \mathbb{R}$ are smooth functions. Conversely, every section τ defined on U by (4a)*, where τ_i are smooth functions, is a field.*

The inverse β of the linear map α may be regarded as a map that associates $v_x \in \mathcal{T}_x \mathbb{R}^A$ to a vector of $\mathbb{R}^A$. This map induces an isomorphism between $\mathbb{R}^A$ and $T_x \mathbb{R}^A \cong \mathcal{T}_x \mathbb{R}^A$. Hence one may write the following obvious result.

{pIdentifyTxRA}

Proposition 8 *The map $\beta : T_x \mathbb{R}^A \to \mathbb{R}^A$ defined by*

$$\sum_{i=1}^{\infty} \tau_i \frac{\partial}{\partial x_i}|_x \mapsto (\tau_1, \tau_2, \tau_3, \ldots)$$

is an isomorphism. In particular one may endow $T_x \mathbb{R}^A$ with the Fréchet topology.

[6] One may note that an infinite sum of vectors is not a linear combination in linear algebra. Hence $\{\frac{\partial}{\partial x_i}, i \in \mathbb{N}^*\}$ is not a basis from the linear-algebraic viewpoint.

The next definition establishes the concept of tangent maps. This concept will be generalized to $I\!R^A$-manifolds (see Definition 22). For the moment, only the following definition is needed.

{dTangentMapRA}

Definition 10 *Let $g : U \subset I\!R^A \to I\!R^B$ be a smooth map. The tangent map $g_*(x) : T_x U \to T_{g(x)} S$ is the $I\!R$-linear map*[7] *defined by $g_*(x)(v_x)(\lambda) = v_x(\lambda \circ g)$.*

Let $(x_i, i \in I\!N^*)$ and $(y_j, j \in I\!N^*)$ be canonical coordinates respectively of $I\!R^A$ and $I\!R^B$. Now, it is easy to obtain the expression of $g_* v_x$. In fact, if $v_x = \sum_{i=0}^{\infty} a_i \frac{\partial}{\partial x_i}|_x$, then, by (4a)-(4b), it is easy to show that

{eGEstrela}

$$g_*(x) v_x = \sum_{j=0}^{\infty} b_j \frac{\partial}{\partial y_j}|_{g(x)} \tag{5a}$$ {eGEstrelaA}

where

$$b_j = (g_*(x) v_x)(y_j) = v_x(y_j \circ g) = v_x(g_j) \tag{5b}$$ {eGEstrelaB}

2.4 Lie Derivatives and Lie-Brackets on $I\!R^A$

{sLieRA}

In finite dimensional differential-geometry, a field τ is associated to a flow $\phi_t(x)$ by the flow box theorem. For instance, the Lie-derivative $L_\tau \theta$ of the field θ may be defined by (or at least interpreted) as

$$L_\tau \theta|_x = \lim_{t \to 0} \frac{(\phi_{-t})_*(\theta(\phi_t(x)) - \theta(x))}{t},$$

which is a nice geometrical way of regarding the Lie-Bracket (Warner 1971). However, a field on $I\!R^A$ is not necessarily associated to a flow (see (Zharinov 1992)). Hence, Lie-derivatives and Lie-brackets may not be interpreted as limits, and the definitions of such objects are purely algebraic, although those definitions implies, at least in some situations, the usual properties that are found in finite dimensional geometry.

{d9}

Definition 11 *(Lie-derivative of a function, and Lie-brackets of fields) Let τ and θ be fields on $I\!R^A$, and let $f : I\!R^A \to I\!R$ be a smooth function.*

- *The Lie-derivative $L_\tau f : I\!R^A \to I\!R$ is the smooth function defined by $L_\tau f(x) = \tau(f)(x)$.*
- *The Lie-bracket $[\tau, \theta]$ (also denoted by $L_\tau \theta$) is the field on $I\!R^A$ defined by*

$$[\tau, \theta](f) = L_\tau(L_\eta(f)) - L_\eta(L_\tau(f)). \tag{6}$$ {eBrackett}

Given two smooth functions f_1 and f_2, by definition, as $\tau(x)$ is a tangent vector, one may write

$$L_\tau(f_1 + f_2) = L_\tau(f_1) + L_\tau(f_2) \tag{7a}$$ {eLinearTau}

$$L_\tau(f_1 f_2) = f_2 L_\tau(f_1) + f_1 L_\tau(f_2) \tag{7b}$$ {eProductTau}

[7] It is an easy exercise to show that $g_*(x) : T_x U \to T_{g(x)} S$ is a well defined linear map.

The next result shows that the previous definition of Lie-Bracket defines a field.

{pLieWell}

Proposition 9 *The definition of Lie-bracket stated in* (11) *is well posed, that is, the Lie-Bracket of two vector fields is a vector field. Furthermore, if* $\tau = \sum_{i=1}^{\infty} \tau_i(x) \frac{\partial}{\partial x_i}|_x$ *and* $\theta = \sum_{i=1}^{\infty} \theta_i(x) \frac{\partial}{\partial x_i}|_x$*, then* $[\tau, \theta](x) = \sum_{i=1}^{\infty} \alpha_i(x) \frac{\partial}{\partial x_i}|_x$*, where* $\alpha_i = \tau(\eta(x_i)) - \eta(\tau(x_i))$.

Proof. See appendix D □

2.5 The cotangent bundle $T^* I\!R^A$ and one-forms

The set of continuous linear functionals $\gamma : I\!R^A \to I\!R$ is denoted by $(I\!R^A)^*$. Let $x = (x_j, j \in I\!N^*) \in I\!R^A$. Let $dx_i : I\!R^A \to I\!R$ stand for the coordinate function $dx_i(x) = x_i$. The notation dx_i is used because dx_i coincides with the differential of x_i.

{pFunctionals}

Proposition 10 *An element* $\gamma \in (I\!R^A)^*$ *is of the form* $\gamma = \sum_{i=1}^{k} \alpha_i dx_i$ *for convenient* $\alpha_i \in I\!R, i = 1, \ldots k$.

Proof. See appendix B. □

Let $(\tilde{x}_1, \ldots, \tilde{x}_k)$ be canonical coordinates of $I\!R^k$. Let $\tilde{\omega} = \sum_{i=1}^{k} \alpha_i d\tilde{x}_i$ be a linear functional of $(R^k)^*$. Let $x \in I\!R^A$. Denote by $\pi_k^* : (R^k)^* \to (I\!R^A)^*$ the linear functional defined by $\pi_k^*(\tilde{\omega})(x) = \tilde{\omega}(\pi_k(x))$. Then it is clear that $\pi_k^* \sum_{i=1}^{k} \alpha_i d\tilde{x}_i = \sum_{i=1}^{k} \alpha_i dx_i$. The last proposition says that an element $\gamma \in (I\!R^A)^*$ is of the form $\pi_k^* \tilde{\omega}$, where $\tilde{\omega}$ is a linear functional belonging to $(I\!R^k)^*$. In particular, one may identify $(I\!R^A)^*$ with the subspace of $I\!R^A$ formed by the vectors $x = (x_i, i \in I\!N^*)$ for which only a finite number of components x_i are nonzero. It can be shown that $(I\!R^A)^*$ is an inductive limit (Bernšteĭn & Rosenfel′d 1973). One will adopt a topology of $(I\!R^A)^*$ for which the corresponding basis are the sets of the form $\mathcal{B} = \{(x_i, i \in I\!N^*) \in I\!R^A \mid |x_i - \delta_i| < \epsilon_i, i \in I\!N^*, \epsilon_i > 0\}$. Note that, when one regards $(I\!R^A)^*$ as a subset of $I\!R^A$, this topology does not coincide with the subset topology. For instance, the open set $U = \{x \in I\!R^A || x_i| < 1, i \in I\!N^*\}$ is an open set of $(I\!R^A)^*$ that is not an open set of $I\!R^A$.

Definition 12 *The cotangent bundle* T^*U *on* U *is the triple* $(U \times (I\!R^A)^*, U, \pi)$*, where* $\pi : U \times (I\!R^A)^* \to U$ *is the canonical projection. A smooth section of* T^*U *is a map* $\omega : U \to T^*U$ *such that* $x \mapsto (x, \bar{\omega}(x))$ *where* $\bar{\omega}(x) = (\omega_1(x), \omega_2(x), \ldots) \in (I\!R^A)^*$ *and each* $\omega_i : U \to I\!R$ *is a smooth function,* $i \in I\!N^*$*. The set of smooth sections of* T^*U *is denoted by* $\Omega(U)$.

The map $x \mapsto (x, dx_i)$, where dx_i is the differential of the coordinate function x_i (which coincides with the coordinate function itself), is an example of a smooth section of T^*U that is denoted by $dx_i|_x$.

Clearly $dx_i|x$ maps x into $(x, (\omega_1(x), \omega_2(x), \ldots))$, where $\omega_j = 0$ if $j \neq i$, and $\omega_i = 1$. Hence, abusing notation, given a section $x \mapsto (x, (\omega_1(x), \omega_2(x), \ldots))$ of T^*U, one may write

$$\omega(x) = \sum_{k=1}^{\infty} \omega_i(x) dx_i|_x.$$

It is important to be pointed out that, at every fixed $x \in U$, then $\omega_i(x) \neq 0$ only for i belonging to a finite subset $F \subset I\!N^*$.

Definition 13 *Let $U \subset I\!R^A$ be an open set, and let $\tau : U \to TU$ be a field such that $x \mapsto (x, \bar{\tau}(x))$, and a section $\omega : U \to T^*U$ such that $x \mapsto (x, \bar{\omega}(x))$ of T^*U. Then define the function $\langle \omega, \tau \rangle : U \to I\!R$ (also denoted by $\omega(\tau)$) by $x \mapsto \bar{\omega}(x)(\tau(x))$. Since $\bar{\omega}(x) \in (I\!R^A)^*$ and $\bar{\tau}(x) \in I\!R^A$, then $\langle \omega, \tau \rangle(x)$ is always well defined by a finite sum $\sum_{i=0}^{k_x} \omega_i(x)\tau_i(x)$, where k_x may depend on x.*

{d13}

Definition 14 *A 1-form on U is a section ω on T^*U such that the function $\langle \omega, \tau \rangle : U \to I\!R$ is smooth for every field τ defined on U.*

Let $(\tilde{x}_1, \ldots, \tilde{x}_k)$ be canonical coordinates of $I\!R^k$ and $\{x_i, i \in I\!N^*\}$ the canonical coordinates of $I\!R^A$. Let V be an open set of $I\!R^k$ and let $U = \pi_k^{-1}(V)$. Let $\tilde{\omega}(\tilde{x}) = \sum_{i=1}^k \tilde{\omega}_i(\tilde{x}) d\tilde{x}_i|_{\tilde{x}}$ be a one-form on V. Let $(\pi_k)^*\tilde{\omega}$ stand for the one-form ω on U defined by $\omega(x) = \sum_{i=1}^k (\tilde{\omega}_i \circ \pi_k) dx_i|_x$. Later, such notation will be generalized and it will be redefined in a more intrinsic manner.

{tFormsAreFinite}

Theorem 2 *Let U be an open subset of $I\!R^A$. The following affirmations holds:*

1. *A one-form ω on U is a smooth section of T^*U.*
2. *A smooth section of T^*U is a one-form if and only if, for every $x \in U$, there exists an open neighborhood V_x of x, and $k \in I\!N^*$ such that, for all $x \in V_x$ one has $\omega(x) = \sum_{i=1}^k \alpha_i(x) dx_i|_x$, where $\alpha_i : V_x \to I\!R, i = 1, \ldots, k$ are smooth functions. defined on $\tilde{V} \subset I\!R^k$, where $\tilde{V} = \pi_k(V_x)$, and $k^* \in I\!N^*$ big enough, such that $\omega|_{V_x} = (\pi_{k^*})^*\tilde{\omega}$.*

Proof. The proof of this theorem is deferred to Appendix C. □

The proof of the following proposition is straightforward, and is left to the reader.

Proposition 11 *(The differential of a smooth function is a one-form) Let $\phi : U \to I\!R$ be a smooth function. Let $d\phi(x) : I\!R^A \to I\!R$ be the differential of ϕ (see Def. 5). The map $d\phi : U \to U \times (I\!R^A)^*$ defined by $x \mapsto (x, d\phi(x))$ is a one form. Furthermore, if $\tau : U \to U \times (I\!R^A)$ is a field, then $\tau(f) = \langle d\phi, \tau \rangle$.*

Proof. By Def. 5, if one may locally write $\phi = \tilde{\phi} \circ \pi_k$, then $d\phi(x) = \sum_{i=1}^k \frac{\partial \tilde{\phi}}{\partial x_k}|_{\pi_k(x)} dx_i|_x$. In particular, it follows that $d\phi$ is a one form. The other affirmation follows from (1). □

The differential $d\phi$ of a smooth function is also a smooth section of T^*U.

2.6 The Cotangent Space T_x^*U

{ssCotangent}

Fix some $x \in U \subset \mathbb{R}^A$. The set of points of $T^*U = U \times (\mathbb{R}^A)^*$ of the form $(x, \bar{\omega})$, with $\bar{\omega} \in (\mathbb{R}^A)^*$, is denoted by T_x^*U. Now the $\mathbb{R}$-vector space T_x^*U is generated by the basis $\{dx_i|_x : i \in \mathbb{N}^*\}$.

Remember that $(\mathbb{R}^A)^*$ may be identified with the subspace of $\mathbb{R}^A$ formed by the elements $\nu = (\nu_\alpha, \alpha \in A)$ such that $\nu_\alpha \neq 0$ for $\alpha \in F$, where $F \subset A$ is finite. The "natural" topology $\mathcal{T}$ of $(\mathbb{R}^A)^*$ is the one induced by an injective limit (Bernšteĭn & Rosenfel′d 1973). The basis of this topology contains the sets of the form $\mathcal{B} = \{(\nu_\alpha, \alpha \in A) \in \mathbb{R}^A \mid |\nu_\alpha - \bar{\nu}_\alpha| < \epsilon_\alpha, \alpha \in A\}$, where $\bar{\nu} = (\bar{\nu}_\alpha, \alpha \in A) \in (\mathbb{R}^A)^*$ and $\epsilon_\alpha > 0, \alpha \in A$. This topology is stronger than the Fréchet topology, since an infinite product of open intervals of $\mathbb{R}$ can be an open set of $\mathcal{T}$.

The map $\beta : T_x^*\mathbb{R}^A \to (\mathbb{R}^A)^*$ by $\sum_{i=1}^k \alpha_i dx_i \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_k, 0, 0, \ldots)$ is easily seen to be an isomorphism. In particular one may endow $T_x^*\mathbb{R}^A$ with the topology induced by $\mathcal{T}$.

The next definition establishes the concept of cotangent maps. This concept will be generalized to $\mathbb{R}^A$-manifolds in section 4.3. For the moment, only the following definition is needed.

{dCoTangentMapRA}

Definition 15 *Let $g : U \subset \mathbb{R}^A \to V \subset \mathbb{R}^B$ be a smooth map. The cotangent map $g^*(x) : T_{g(x)}^*V \to T_xU$ is the $\mathbb{R}$-linear map*[8] *defined by the following rule: one maps $\omega \in T_{g(x)}^*V$ to $g^*\omega \in T_x^*U$, where $\langle g^*(x)\omega, \tau\rangle = \langle \omega, g^*\tau\rangle$ for every $\tau \in T_xU$.*

{pGEstrela1}

Proposition 12 *Given a one-form ω on V, then define the section $g^*\omega$ of T^*U by the rule $\langle g^*(x)\omega(x), \tau(x)\rangle = \langle \omega(g(x)), g^*(x)\tau(x)\rangle$, where $\tau(x)$ is a field on U. Then $g^*\omega$ is a one-form.*

Proof. It is clear that $g^*\omega$ is a section of T^*U. To show that $g^*\omega$ is a one-form, it suffices to show that $\langle g^*\omega, \tau\rangle$ is a smooth function for every field τ on U. One locally has $\omega = \sum_{j=1}^k \alpha_j(y) dy_j|_y$. Note that $\langle g^*\omega, \tau\rangle(x) = \langle \omega(g(x)), g_*(x)\tau(x)\rangle$. From (5a)-(5b), it follows that $g_*(x)\tau(x) = \sum_{j=1}^\infty \tau(g_j(x)) \frac{\partial}{\partial y_j}|_{g(x)}$, where $g_j = y_j \circ g$. Hence $\langle g^*\omega, \tau\rangle(x)$ is locally given by $\sum_{j=1}^k \alpha_j(g_j(x))\tau(g_j(x))$, which depends smoothly on x. $\square$

2.7 p-forms, wedge product, and exterior differentiation on $\mathbb{R}^A$

One has shown that a one form ω on $\mathbb{R}^A$ is locally the pull-back $\pi_k^*\tilde{\omega}$, where $\tilde{\omega}$ is a one-form on $\mathbb{R}^k$. (see Theorem 2). One can define a p-form on $\mathbb{R}^A$ by generalizing this property[9].

---

[8] It is an easy exercise to show that $g^*(x) : T_{g(x)}^*V \to T_xU$ is a well defined linear map.

[9] A more intrinsic way for defining a p-form can be found in (Bernšteĭn & Rosenfel′d 1973).

{def16}

Definition 16 *Let $V \subset I\!R^B$ and $U \subset I\!R^A$ be open sets. Let $y \in U$ and define*

$$\Gamma^p_U(y) = \underbrace{T_yU \times T_yU \times \ldots \times T_yU}_{p \text{ times}}$$

Consider the fiber bundle $\Gamma^p_U = \bigcup_{y\in U} \Gamma^p_U(y)$. Let $\omega : \Gamma^p_U \to I\!R$ be a map. Let $g : U \subset I\!R^B \to V \subset I\!R^A$ be a smooth map. Let $y = g(x)$ Define the pull back $g^*|_x\omega : \Gamma^p_V(x) \to I\!R$ by the rule

$$g^*|_x\omega(\tau_1, \ldots, \tau_p) = \omega(g_*\tau_1, \ldots, g_*\tau_p), \textit{for every } \{\tau_1, \ldots, \tau_p\} \textit{ in } \Gamma^p_U(y)$$

Define the pull-back $g^*\omega : \Gamma^p_V \to I\!R$, pointwise, by the same rule above.

We are ready to define a p-form.

{dPForms}

Definition 17 *(p-forms on $U \subset I\!R^A$)*

- *A p-form ω on U is a map $\omega : \Gamma^p_U \to I\!R$ such that, around every $y \in U$ there exists an open neighborhood W_y of y such that $\omega|_{\widetilde{W}} = (\pi_k)^*\tilde{\omega}$, where $\tilde{\omega}$ is a p-form on V, $V = \pi_k(W_y) \subset I\!R^k$, and $\widetilde{W} = \Gamma^p_{W_y}$.*
- *The set of p-forms on U will be denoted by $\Lambda_p(U)$, where $\Lambda_0(U)$ denotes the set of smooth functions $f : Y \to I\!R$.*
- *The exterior derivative*[10] *is the map $d : \Lambda_p(U) \to \Lambda_{p+1}(U)$ defined in the following way. For $p = 0$, the map $df = d(f)$ is the differential of the function f. For $p > 0$, as one locally has $\omega|_{\widetilde{W}} = \pi^*_k\tilde{\omega}$, then one may locally define $d(\omega)|_{\tilde{W}} = \pi^*_k(d\tilde{\omega})$, where $\widetilde{W}$ and $\tilde{\omega}$ are defined above.*
- *The wedge product "$\wedge$" is defined*[11] *in the following way. Let $\omega_x : \Gamma^m_U(x) \to I\!R$ and $\eta_x : \Gamma^p_U(x) \to I\!R$ be two maps. Let $\mathcal{T} = (\tau_1, \ldots, \tau_{p+m}) \in \Gamma^{p+m}_U(y)$. then*

$$\omega_x \wedge \eta_x(\mathcal{T}) = \sum_{\sigma\in Sh_{p,m}} \operatorname{sgn}(\sigma)\, \omega_x(\mathcal{T}^\sigma_{1,p})\eta_x(\mathcal{T}^\sigma_{p+1,p+m}) \tag{8}$$ {eRuleWedge}

  where $\mathcal{T}^\sigma_{1,p} = (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(p)})$, $\mathcal{T}^\sigma_{p+1,p+m} = (\tau_{\sigma(p+1)}, \ldots, \tau_{\sigma(p+m)})$, $Sh_{p,m}$ denotes the (p,m)-shuffles, *that is, the permutations σ of the set $\lfloor p+m\rceil$ such that $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+m)$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation.*

  Consider now a p-form ω and a m-form η defined on U. Then, $\omega \wedge \eta : \Gamma^{p+m}_U \to I\!R$ is the map that is pointwise defined by $\omega(x) \wedge \eta(x)$.

---

[10]It will be shown that this definition does not depend on the chosen representation $\pi^*_k\tilde{\omega}$ (see (15)).

[11]This definition mimics the standard definition of wedge product that appears in finite dimensional exterior algebra (see (Warner 1971)).

The following Proposition is instrumental in the sequel. From this result, one may show that the wedge product commutes with g^* (that is, $g^*(\omega \wedge \eta) = (g^*\omega) \wedge (g^*\eta)$ in the general situation

{pWedgePi}

Proposition 13 *Consider the notation of Prop. 17. Given a p -form ω, and and m-form η on U, assume that $\omega = \pi_k^* \tilde{\omega}$ and $\eta = \pi_k^* \tilde{\eta}$, where $\tilde{\omega}$ and $\tilde{\eta}$ are convenient forms on $V = \pi_k(U) \subset I\!R^k$. Then,*

$$\pi_k^*(\tilde{\eta} \wedge \tilde{\tau}) = \pi_k^* \tilde{\eta} \wedge \pi_k^* \tilde{\tau}. \tag{9}$$ {eComuta}

Proof. Note that

$$\begin{aligned}
\omega \wedge \eta(\mathcal{T}) &= \sum_{\sigma \in Sh_{p,m}}^{\text{sgn}} (\sigma)\, \omega(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(k)}) \eta(\tau_{\sigma(k+1)}, \ldots, \tau_{\sigma(k+m)}) \\
&= \sum_{\sigma \in Sh_{p,m}}^{\text{sgn}} (\sigma)\, \tilde{\omega}((\pi_k)_* \tau_{\sigma(1)}, \ldots, (\pi_k)_* \tau_{\sigma(k)}) \eta((\pi_k)_* \tau_{\sigma(k+1)}, \ldots, (\pi_k)_* \tau_{\sigma(k+m)}) \\
&= \tilde{\eta} \wedge \tilde{\tau}((\pi_k)_* \tau_1, \ldots, (\pi_k)_* \tau_{p+m}) = (\pi_k)_* (\tilde{\eta} \wedge \tilde{\tau})
\end{aligned}$$

In particular, (9) follows. □

{dMultiindex}

Definition 18 *A vector $I = (i_1, \ldots, i_p)$, where $i_j \in \{1, \ldots, k\}, j = 1, \ldots, p$, and $i_1 < i_2 < \ldots < i_p$ is called a p-multiindex of class k. One may define the* class *of I, denoted by $|I|$ and given by $|I| = \max_{i \in \lfloor p \rceil} \{i_1, i_2, \ldots, i_p\}$. The set of p-multiindeces I such that $|I| \leq k$ is denoted by $H_p(k)$. By definition, $H_p(k)$ is the set of p-multiindeces of class that is not greater than k.*

It will be shown that one may compute exterior derivatives and wedge products of p-forms on $I\!R^A$ in the same way that one computes those objects in finite dimensional differential geometry. All the properties of finite dimensional geometry are transfered via the pull-back π_k^* in the expected way.

For this consider that the canonical coordinates of $I\!R^k$ are $\{\tilde{x}_1, \ldots, \tilde{x}_k\}$. Let $V I\!R^k$ be an open subset. Let $H = (h_1, \ldots, h_p)$ be a p-multiindex. One consider the standard notation $d\tilde{x}_H = d\tilde{x}_{h_1} \wedge d\tilde{x}_{h_2} \wedge \ldots \wedge d\tilde{x}_{h_p}$ of finite dimensional geometry. Given tangent vectors $\{\tilde{X}_1, \ldots \tilde{X}_p\} \subset T_{\tilde{x}} V$, recall that[12], one has $d\tilde{x}_H(\tilde{X}_1, \ldots \tilde{X}_p) = \det \langle d\tilde{x}_{h_i}, \tilde{X}_j \rangle$, where $\det \langle d\tilde{x}_{h_i}, \tilde{X}_j \rangle$ denotes the determinant of the $p \times p$ matrix whose (i,j)-element is $\langle d\tilde{x}_{h_i}, \tilde{X}_j \rangle$.

It is well known that a p-form on $V \subset I\!R^k$ is given by

$$\tilde{\omega} = \sum_{H \in H_p(k)} \tilde{\alpha}_H(\tilde{x}) d\tilde{x}_H. \tag{10}$$ {eOmegaTil}

Let U be an open set of $I\!R^A$. Let $V = \pi_k(U)$. Define the p-form on $U \subset I\!R^A$ given by

$$dx_H = \pi_k^*(d\tilde{x}_H)$$

[12]Other suitable definitions of the action of $d\tilde{x}_H$ on tangent vectors may be found in (Warner 1971)

Given a set of p fields $\{X_1, \dots X_p\}$ on U, one has

$$\begin{aligned} dx_H(X_1, \dots, X_p) &= d\tilde{x}_H((\pi_k)_* X_1, \dots, (\pi_k)_* X_p) \\ &= \det\langle d\tilde{x}_{h_i}, (\pi_k)_* X_j \rangle \\ &= \det\langle d\tilde{x}_{h_i}, (\pi_k)_* X_j \rangle \\ &= \det\{(\pi_k)_* X_j(\tilde{x}_{h_i})\} \\ &= \det\{X_j(\tilde{x}_{h_i} \circ \pi_k)\} \\ &= \det\{\langle dx_{h_i}, X_j\rangle\}\,. \end{aligned}$$

This last equality does not depend on the chosen projection π_k. In other words, the notation dx_H makes sense without the need of specifying the corresponding projection that was originally used to define dx_H. In particular, it follows easily that, all p-forms on $I\!R^A$ may be locally written in the form.

$$\omega|_{\tilde{W}} = \sum_{H \in H_p(k)} \alpha_H dx_H = \sum_{H \in H_p(k)} \pi_k^*(\tilde{\alpha}_H d\tilde{x}_H) = \pi_k^* \tilde{\omega} \tag{11}$$ {eFORMRA}

where $\alpha_H = \tilde{\alpha}_H \circ \pi_k$ and $\tilde{\omega}$ is given by (10). Conversely, given a form (11), it is clear that, for any $m \geq k$, one may write[13]

$$\omega|_{\tilde{W}} = \pi_m^* \left(\sum_{H \in H_p(m)} \tilde{\alpha}_H d\tilde{x}_H \right) \tag{12}$$ {eFORMRB}

where the functions $\tilde{\alpha}_H = 0$ coincides with the ones of (11) for $|H| \leq k$ and $\tilde{\alpha}_H = 0$ for $|H|$ such that $k < |H| \leq m$. From this arguments, one have shown the following Proposition.

{pKbigEnough}

Proposition 14 *Given a p-form ω_1 and a q-form ω_2 on $I\!R^A$, choosing k^* big enough, one may locally write $\omega_1 = \pi_k^* \tilde{\omega}_1 = \sum_{H \in H_p(k)} \alpha_H dx_H$ and $\omega_2 = \pi_k^* \tilde{\omega}_2 = \sum_{J \in H_q(k)} \beta_J dx_J$. It follows from Proposition 13, that the wedge product $\omega_1 \wedge \omega_2$ of two forms is locally the pull-back $(\pi_{k^*})^*(\tilde{\omega}_1 \wedge \tilde{\omega}_2)$ of a form $\tilde{\omega}_1 \wedge \tilde{\omega}_2$, defined on an open set of $I\!R^k$. In particular, the definition of wedge product given in Def. 17 well posed in the sense that it is a $p+q$-form in the sense of the same Def. 17.*

Now, one computes the expression of the wedge product:

$$\begin{aligned} \omega_1 \wedge \omega_2 &= \pi_k^*(\tilde{\omega}_1) \wedge \pi_k^*(\tilde{\omega}_2) \\ &= \pi_k^*(\tilde{\omega}_1 \wedge \tilde{\omega}_2) \\ &= \sum_{H \in H_p(k)} \sum_{J \in H_q(k)} \alpha_H \beta_J dx_H \wedge dx_J. \end{aligned} \tag{13}$$ {eWedge}

[13] From now on, for simplicity of notation, one will not mention the restriction to $\tilde{W} = \Gamma_W^p$ in the local expressions of the forms (see Def. 16).

Note that the last expression do not depend on the chosen local projection π_k that was used to represent ω locally. Now given a p-form $\tilde{\omega}$ on $V \subset I\!R^k$ given by (10). Then

$$d\tilde{\omega} = \sum_{H \in H_p(k)} d\tilde{\alpha}_H \wedge (\tilde{x}) d\tilde{x}_H. \tag{14}$$ {eDOmegaTil}

Now by definition, note that, given a p-form (11), by (14) one may locally write

$$d\omega = \pi_k^* d\tilde{\omega} = \pi_k^* \left(\sum_{H \in H_p(k)} d\tilde{\alpha}_H(\tilde{x}) \wedge d\tilde{x}_H \right).$$

From (13), it follows that

$$d\omega = \sum_{H \in H_p(k)} \pi_k^* \left(d\tilde{\alpha}_H(\tilde{x}) \right) \wedge \left(\pi_k^* d\tilde{x}_H \right)$$

In particular, given a p-form ω given by (11), then

$$d\omega = \sum_{H \in H_p(k)} d\alpha_H(x) \wedge dx_H \tag{15}$$ {eDOmegaRA}

It is clear from the last formula that the concept of exterior differentiation, that is stated in Definition 17, does not depend on the chosen local projection π_k that was used to represent ω locally. Furthermore, given $\omega = \pi_k^* \tilde{\omega}$, as $d\omega = \pi_k^* d\tilde{\omega}$, then $d^2(\omega) = \pi_k^* d^2 \tilde{\omega} = 0$.

3 Internal product, and the Lie derivative of p-forms on $I\!R^A$

{s3}

Definition 19 *Let $X, Y_1, \ldots, Y_{p-1}$ be fields on an open set $U \subset I\!R^A$. Let ω be a p-form on U. Define the* interior product $i(X) : \Lambda_p(U) \to \Lambda_{p-1}(U)$ *by the $p-1$ form defined by $\bar{\omega}(Y_1, \ldots, Y_{p-1}) = \omega(X, Y_1, \ldots, Y_{p-1})$. The Lie-derivative of ω along X is the p-form $L_X\omega$ defined by $L_X\omega = i(X) \circ d(\omega) + d \circ i(X)(\omega)$.*

One must show that the last definition is well posed in the sense that both objects $\theta_1 = i(X)\omega$ and $\theta_2 = L_X\omega$ locally coincide respectively with $(\pi_k)^*\tilde{\theta}_1$ and $(\pi_k)^*\tilde{\theta}_2$, where $\tilde{\theta}_i$ are forms defined on some open subset of $I\!R^k$. For this, let ω be locally given by (11). Let $Y = (Y_1, \ldots, Y_p)\Gamma_U^p(x)$ and let $(\pi_k)_*Y$ stands for $((\pi_k)_*Y_1, \ldots, (\pi_k)_*Y_p)$. Then, if $X \in T_xU$ one may write

$$\begin{aligned} i(X)(\omega)(Y) &= \sum_{H \in H_p(k)} \pi_k^* \left(\tilde{\alpha}_H d\tilde{x}_H \right) (X, Y) \\ &= \sum_{H \in H_p(k)} \left((\tilde{\alpha}_H \circ \pi_k) d\tilde{x}_H \right) \left((\pi_k)_* X, (\pi_k)_* Y \right) \end{aligned}$$

Let $H = (h_1, \dots, h_p)$ be a multiindex, and let $\widetilde{X}, \widetilde{Y}_1, \dots, \widetilde{Y}_p \in T_{\tilde{x}}V$, where $V = \pi_k(U) \subset I\!R^k$ is an open set. Let $\widetilde{Y}$ stands for $(\widetilde{Y}_1, \dots, \widetilde{Y}_p)$. Then,

$$\begin{aligned} i(\widetilde{X})d\tilde{x}_H(\widetilde{Y}) &= i(\widetilde{X})\left(d\tilde{x}_{h_1} \wedge \dots \wedge d\tilde{x}_{h_p}\right)(\widetilde{Y}) \\ &= \sum_{i=1}^{p}(-1)^{i+1}\langle d\tilde{x}_{h_i}, \widetilde{X}\rangle \left(d\tilde{x}_{h_1} \wedge \dots \widehat{dx_{h_i}} \wedge \dots \wedge d\tilde{x}_{h_p}\right)(\widetilde{Y}), \end{aligned}$$

where the notation $\widehat{dx_{h_i}}$ means that the differential dx_{h_i} is omitted. Now take $\widetilde{X} = (\pi_k)_*X$ and $\widetilde{Y}_i = \pi_k)_*Y_i$ for $i = 1, \dots, p$. Since $\langle d\tilde{x}_{h_i}, \pi_k^*Y_j\rangle = \langle dx_{h_i}, Y_j\rangle$, one gets

$$i(X)(a_H dx_H) = a_H \sum_{i=1}^{p}(-1)^{i+1}\langle dx_{h_i}, X\rangle \; (dx_{h_1} \wedge \dots \wedge \widehat{dx_{h_i}} \wedge \dots \wedge dx_{h_p}) \quad (16) \qquad \{eIXRA\}$$

It is now clear that, for $\bar{k}$ big enough[14], then $i(X)\omega = \pi_{\bar{k}}^*\tilde{\theta}_1$, for a convenient one-form θ_1 locally defined on $I\!R^{\bar{k}}$. Now, as $d : \Lambda_k(S) \to \Lambda_{k+1}(S)$ is well posed, it is clear that the operator L_X is well posed.

For a function $\phi : U \subset I\!R^A \to I\!R$ and a field X on U, remember that, by definition $L_X\phi = X(\phi) = \langle d\phi, X\rangle$. Then $L_X(d\phi) = d(i(X)d\phi) + i(X)(d^2\phi) = d\langle d\phi, X\rangle = dL_X\phi$.

Let ω be a p-form ω given by (11). Then $d\omega$ is given by (15), and $i(X)(\omega)$ can be easily determined by (16). These expressions are the same that are found in finite dimensional geometry. As $L_X(\omega)$ is defined by $i(X)\,(d\omega) + d\,(i(X)\omega)$, then it is not difficult to show from (15) and (16), that the expression of $L_X(a_H dx_H)$ for $H = (h_1, \dots h_p) \in H_k(p)$ also coincides with the finite dimensional formula (Warner 1971, Dieudonneé 1974):

$$L_X(a_H dx_H) = (L_X a_H)dx_H + \sum_{i=1}^{p} a_H dx_{h_1} \wedge \dots \wedge L_X(dx_{h_i}) \wedge \dots dx_{h_p}$$

For one-forms, the last expression implies that

$$L_X(\sum_{i=1}^{k} a_i dx_i) = \sum_{i=1}^{k} L_X(a_i)dx_i + a_i L_X(dx_i)$$

Proposition 15 *Let $X, Y_0, Y_1, \dots, Y_p$ be fields on an open subset $U \subset I\!R^A$. Let ω be a p-form on U. Then*

$$\begin{aligned} dL_X\omega &= L_X(d\omega) && \text{(17a)} \\ Y_0(\omega(Y_1, \dots, Y_p)) &= (L_{Y_0}\omega)(Y_1, \dots, Y_p) + \\ & \quad \sum_{i=1}^{p} \omega(Y_1, \dots, Y_{i-1}, [Y_0, Y_i], Y_{i+1}, \dots, Y_p) && \text{(17b)} \end{aligned}$$

[14] Given $H = (h_1, \dots h_p) \in H_k(p)$, let l_H^i be the minimal index of the function $\langle dx_{h_i}, X\rangle$. Let $l_H = \max\{l_H^1, \dots, l_H^p\}$, and let $L = \max_{H \in H_k(p)} l_H$. Then one may take $\bar{k} = \max\{L, k\}$.

$$d\omega(Y_0,\ldots,Y_p) = \sum_{i=0}^{p}(-1)^i Y_i\left(\omega(Y_1,\ldots,\widehat{Y_i},\ldots,Y_p)\right) + \sum_{i<j}(-1)^{i+j}\omega([Y_i,Y_j],Y_0,\ldots,\widehat{Y_i},\ldots,\widehat{Y_j},\ldots,Y_p) \quad (17c)$$

Proof. The first formula follows from the fact that $L_X(\cdot) = d\circ i(X) + i(X)\circ d$. Hence $d\circ L_X = d^2\circ i(X) + d\circ i(X)\circ d = L_X\circ d$. The second and third formulas follows from the definition of the operator $L_X(\cdot)$, from the formulae (15), and (16) (that coincide with the corresponding finite dimensional formulae) and from the same arguments that are used to establish similar formulae in finite dimensional geometry (see (Warner 1971, Dieudonneé 1974)). $\square$

In particular, for one-forms ω and fields X, Y on an open set $U \subset \mathbb{R}^A$, one may write

$$\begin{aligned} L_X\langle\omega, Y\rangle &= \langle L_X\omega, Y\rangle + \langle\omega, L_X Y\rangle \\ d\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \end{aligned}$$

The last expression is known as *Cartan's Magic Formula*.

Some important properties are collected in the following result:

{pCollectRA}

Proposition 16 *Let θ be a one-form, ω be a p-form and η be a q-form , all of them defined on an open set $V \subset \mathbb{R}^A$. Let $g : U \subset \mathbb{R}^A \to V$ be a smooth map.*

1. *One has*[15] $g^*(\omega\wedge\eta) = (g^*\omega)\wedge(g^*\eta)$.
2. *The map $g^*\omega$ is a p-form.*
3. $dg^*\theta = g^*(d\theta)$.
4. $d(g^*\omega) = g^*(d\omega)$.

Proof. **1**. It follows easily from the third item of Def. 17 (see the proof of Prop. 13).

2. By Proposition 12, this holds for $p = 1$. By definition ω is locally given by $(\pi_k)^*\tilde\omega$, where $\tilde\omega$ is a p-form on some open set of $\mathbb{R}^k$. Let $\tilde\omega = \sum_{I\in H_p(k)}\tilde a_I dx_I$, where $dx_I = d\tilde x_{i_1}\wedge\ldots d\tilde x_{i_p}$. Then, by the linearity of g^* and $(\pi_k)^*$, and by 1, one may write

$$\begin{aligned} g^*(\pi_k)^*\tilde a_I d\tilde x_I &= \tilde a_I\circ\pi_k\circ g\,(g^*(\pi_k)^*)\,d\tilde x_I \\ &= \tilde a_I\circ\pi_k\circ g(\pi_k\circ g)^* d\tilde x_I \\ &= \tilde a_I\circ\pi_k\circ g(\pi_k\circ g)^* d\tilde x_{i_1}\wedge\ldots\wedge d\tilde x_{i_p} \\ &= \tilde a_I\circ\pi_k\circ g((\pi_k\circ g)^* d\tilde x_{i_1}\wedge\ldots\wedge(\pi_k\circ g)^* d\tilde x_{i_p} \end{aligned}$$

[15] For the moment one is not claiming that g^* maps forms to forms. Remember that the wedge product was defined for arbitrary maps $\omega:\Gamma_U^p\to\mathbb{R}$.

Now, by Proposition 24, $(\pi_k \circ g)^* d\tilde{x}_{i_j} = d(\tilde{x}_{i_j} \circ \pi_k \circ g) = dg_{i_j}$ where $g_{i_j} = \tilde{x}_{i_j} \circ \pi_k \circ g$ is the i_j-th component of g. In particular, the array of equations above means that

$$g^*(\pi_k)^* \tilde{a}_I d\tilde{x}_I = \tilde{a}_I \circ \pi_k \circ g(dg_{i_1} \wedge \ldots \wedge dgi_p)$$

Now, let k_j^* and K_I be the respectively minimal index of g_j and $\tilde{a}_I \circ \pi_k \circ g$ at x. Let $l = \max\{k, k_1^*, \ldots, k_p^*, (K_I I \in H_p(k))\}$. Then it is clear that one may locally write $g_j = \pi_l \circ \tilde{g}_j$ where $\tilde{g}_j$ is a function defined on some open set of $I\!R^l$. So

$$\begin{aligned}(dg_{i_1} \wedge \ldots \wedge dgi_p) &= (\pi_l)^* d\tilde{g}_1 \wedge \ldots \wedge (\pi_l)^* d\tilde{g}_p && (18)\\ &= (\pi_l)^*(d\tilde{g}_{i_1} \wedge \ldots \wedge d\tilde{g}i_p) && (19)\end{aligned}$$

As $(d\tilde{g}_{i_1} \wedge \ldots \wedge d\tilde{g}i_p)$ is a p-form on $I\!R^l$, this shows the second claim.

3. From Theorem 2, one may locally write $\theta = \sum_{i=1}^k \alpha_i dx_i$, where the α_i depends only on $x_1, \ldots, x_k$. From the proof of 2, one has $g^*\theta = \sum_{i=1}^k (\alpha_i \circ g) dg_i = \sum_{i=1}^p (\pi_l)^* (\beta_i) d\tilde{g}_i$, $\tilde{g}_i \circ \pi_l = g_i$, where $\beta_i \circ \pi_l = (\alpha_i \circ g)$, $l = \max\{k, k_1^*, \ldots, k_p^*, K_1, \ldots, K_p\}$, and k_i^* and K_i are respectively the minimal indeces at x of g_i and $\alpha_i \circ g$. Hence $d(g^*\theta) = \sum_{i=1}^p (\pi_l)^* d\beta_i \wedge d\tilde{g}_i = \sum_{i=1}^p d(\beta_i \circ \pi_l) \wedge d(\tilde{g}_i \circ \pi_l) = \sum_{i=1}^p d(\alpha_i \circ g) \wedge d(g_i)$. Without loss of generality, assume that[16] $l = k$.

Now, let $\tilde{\alpha}_i$ be such that $\alpha_i = \tilde{\alpha}_i \circ \pi_l$). Then $d\theta = g^*(\sum_{i=1}^l d(\pi_l)^* \tilde{\alpha}_i d\tilde{x}_i) = (\sum_{i=1}^l (\pi_l)^* d(\tilde{\alpha}_i d\tilde{x}_i) = (\sum_{i=1}^l (\pi_l)^* d\tilde{\alpha}_i \wedge d\tilde{x}_i) = (\sum_{i=1}^l ((\pi_l)^* d\tilde{\alpha}_i) \wedge ((\pi_l)^* d\tilde{x}_i))$. So,

$$\begin{aligned} g^* d\theta &= \sum_{i=1}^l g^*(\pi_l)^* d\tilde{\alpha}_i \wedge g^*(\pi_l)^* d\tilde{x}_i) \\ &= \sum_{i=1}^l (\pi_l \circ g)^* d\tilde{\alpha}_i \wedge d\tilde{x}_i) \\ &= \sum_{i=1}^l d(\tilde{\alpha}_i \circ \pi_l \circ g) \wedge d(\tilde{x}_i \circ \pi_l \circ g) \\ &= \sum_{i=1}^l d(\alpha_i \circ g) \wedge dg_i \end{aligned}$$

4. By the Proof of 2, it follows that $d(g^*\omega) = d\pi_l^* \sum_{I \in H_p(k)} (\tilde{\alpha}_I \circ \pi_l \circ g) \wedge d\tilde{g}_I = \pi_l^* \sum_{I \in H_p(k)} d(\tilde{\alpha}_I \circ \pi_l \circ g) \wedge d\tilde{g}_i = \sum_{I \in H_p(k)} d(\alpha_I \circ \circ g) \wedge dg_i$.

Now, $g^*(d\omega) = g^*(\pi_l)^* \sum_{I \in H_p(k)} d\tilde{\alpha}_I \wedge d\tilde{x}_I = (\pi_l \circ g)^* \sum_{I \in H_p(k)} d\tilde{\alpha}_I \wedge d\tilde{x}_I = \sum_{I \in H_p(k)} d(\tilde{\alpha}_I \circ \pi_l \circ g) \wedge (\pi_l \circ g)^* d\tilde{x}_I$. Now, recall that $\tilde{\alpha}_I \circ \pi_l = \alpha_I$ and

$$\begin{aligned} (\pi_l \circ g)^* d\tilde{x}_I &= (\pi_l \circ g)^*(d\tilde{x}_{i_1} \wedge \ldots \wedge d\tilde{x}_{i_p}) \\ &= (d(\tilde{x}_{i_1} \circ \pi_l \circ g) \wedge \ldots \wedge d(\tilde{x}_{i_p} \circ \pi_l \circ g)) \\ &= dg_I \end{aligned}$$

[16] By construction, $l \geq k$. If $k < l$, one may write $\theta = \sum_{i=1}^l \alpha_i dx_i$, where $\alpha_i = 0$ for $i \geq k$.

□

4 $I\!R^A$-manifolds

{sRAManifolds}

The definition of a $I\!R^A$-manifold is similar to the definition of a finite-dimensional manifold.

Definition 20 *A smooth $I\!R^A$-manifold is a Hausdorff topological space S and a family of pairs $\{(U_i, \phi_i), i \in \Lambda\}$, where U_i is an open subset of S and $\phi_i : U_i \to V_i \subset I\!R^A$ is a homeomorphism*[17] *such that:*

1. $\bigcup_{i\in\Lambda} U_i = S$;
2. *If $U_i \bigcap U_j$ for some pair $i, j \in \Lambda$, then the mapping $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \bigcap U_j) \to \phi_j(U_i \bigcap U_j)$ is a smooth map between open sets of $I\!R^A$.*

As in the case of finite-dimensional manifolds, the family $\{(U_i, \phi_i), i \in \Lambda\}$ is called *atlas* and each map $\phi_i : U_i \to V_i$ is called a *(local) chart*. Given a set of canonical coordinates $\{y_j, i \in I\!N^*\}$ of $I\!R^A$, the set $\{x_j, j \in I\!N^*\}$ of functions $x_j : U_i \to I\!R$ defined by $x_j = y_j \circ \phi_i$ is called set of *local coordinate functions*. Let $\xi \in S$. A local chart (U, ϕ) with $\xi \in U$ is called a *local chart around ξ*.

An *atlas* is *maximal* if one may not add any pair (U, ϕ) to this atlas in a way that property 2 still holds. As in the finite dimensional case, using Zorn's Lemma, one may show the existence a maximal Atlas and, without loss of generality, one may assume that a given atlas is maximal (Warner 1971). In this entire section, S will be a given $I\!R^A$-manifold.

4.1 Smooth functions on $I\!R^A$-manifolds

Let $U \subset S$ be an open set. A function $f : U \to I\!R$ is smooth if for every local chart $\phi : W \to Z$ with $W \subset U$, the map $f \circ \phi^{-1} : \phi(W) \subset I\!R^A \to I\!R$ is smooth. Let $\{x_j, j \in I\!N^*\}$ be the set of local coordinate functions of the local chart ϕ. Let $\xi \in W \subset U$ and let $x = \phi(\xi)$. By the definition of smooth function on an open set of $I\!R^A$, then $f \circ \phi^{-1}$ may be locally expressed in the form $f \circ \phi^{-1} = \tilde{f} \circ \pi_k$. Abusing notation, one may locally write $f \circ \phi^{-1} = \tilde{f}(x_1, \ldots, x_k)$, that is called *expression of f in local coordinates.*

It is clear from the definition of a $I\!R^A$-manifold that there exists a subfamily $\{(U_i, \phi_i), i \in \Gamma\}$ of the maximal atlas such that $\bigcup_{i\in\Gamma} U_i = U$. Then if one wants to prove that a given $f : U \to I\!R$ is smooth, it suffices to show that $f \circ {\phi_i}^{-1}$ is smooth for all $i \in \Gamma$.

Definition 21 *A map $g : R \to S$ between $I\!R^A$-manifolds is smooth if, for every local chart (U, ϕ) of R and (V, ψ) of S, the map $\tilde{g} = \psi \circ g \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth. the map $\tilde{g}$ is called local expression of g in coordinates.*

[17] That is, it is a continuous bijection with continuous inverse.

4.2 Tangent bundle, tangent maps, and fields on $I\!R^A$-manifolds

{sTTF}

One may define derivations acting on germs of smooth functions on open sets of $I\!R^A$-manifolds in the same way that one has defined such objects on open sets of $I\!R^A$ (see Section 2.3). Hence, given a $I\!R^A$-manifold S, then $T_\xi S$ will denote the set of tangent vectors $v_\xi : \mathcal{C}^\infty\{\xi\} \to I\!R$, where $\mathcal{C}^\infty\{\xi\}$ denote the set of functions that are defined on some open neighborhood of $\xi \in S$.

{d22T}

Definition 22 *Let $g : R \to S$ be a smooth map. The tangent map $g_*(x) : T_x R \to T_{g(x)} S$ is the $I\!R$-linear map defined by $g_*(x)(v_x)(\lambda) = v_x(\lambda \circ g)$.*

The following proposition has three important meanings, namely: the first one is the chain rule; the second one assures that the tangent map of a diffeomorphism is an isomorphism of tangent spaces; the last one, shows that one may canonically endow the tangent space with the Fréchet topology.

{pChainRule}

Proposition 17 *Let X, Y and Z be $I\!R^A$-manifolds. The following properties of tangent maps hold for smooth maps $g : Y \to Z$ and $h : X \to Z$.*

1. *(Chain Rule) $(g \circ h)_*(x) = g_*(h(x)) \circ h_*(x)$.*
2. *(Diffeomorphisms induce Isomorphisms of Tangent Spaces) If $Z = X$, g is a diffeomorphism, and $g = h^{-1}$, then $g_*(x)$ is an isomorphism between the $I\!R$ linear vector spaces $T_x Y$ and $T_{g(x)} Z$ with inverse $h_*(g(x))$.*
3. *A Local Chart ϕ of X induce an isomorphism $\phi_*(x) : T_x X \to T_x I\!R^A$. In particular one may endow $T_x X$ with the Fréchet topology, and the induced topology does not depend on the chosen chart.*

Proof. See appendix E. □

{dField}

Definition 23 *(Tangent bundle and sections on $I\!R^A$-manifolds) Let S be a $I\!R^A$-manifold. Define $TS = \bigcup_{x \in S} T_x S$. The canonical projection $\pi : TS \to S$ is the map $v_x \mapsto x$ for every $x \in S$ and $v_x \in T_x S$. The bundle (S, TS, π) is called* tangent bundle *of S. A section τ of the tangent bundle is a map $\tau : S \to TS$ such that $\pi \circ \tau$ is the identity map. In other words, $\tau(x)$ is a tangent vector v_x at $x \in S$, that is, $\tau(x) \in T_x S$.*

{p11}

Proposition 18 *Let S be a $I\!R^A$ manifold and let $\tau : S \to TS$ be a section. Let (U, ϕ) be a coordinate system, and let $V = \phi(U)$. Let $\{y_i : i \in I\!N^*\}$ be the canonical coordinate functions of $I\!R^A$, and let $\frac{\partial}{\partial y_i}|_y$ be the fields on $I\!R^A$ defined by $\frac{\partial}{\partial y_i}|_y(y_j) = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. Let*

$$\frac{\partial}{\partial x_i}|_x = (\phi^{-1})_*|_{\phi(x)}(\frac{\partial}{\partial y_i}|_{\phi(x)}) \tag{20}$$ {eDdx}

Then

1. $\frac{\partial}{\partial x_i}|_x$ *is a section on* U *such that* $\frac{\partial}{\partial x_i}|_x(x_j) = \delta_{ij}$.

2. *Every section* $\tau : S \to TS$ *may be locally represented in the form*

$$(\tau|U)(x) = \sum_{i=1}^{\infty} \tau_i(x) \frac{\partial}{\partial x_i}|_x \tag{21}$$ {eUnique}

where the functions $\tau_i : U \to I\!R$ *are given by* $\tau_i = \tau(x_i)$.

3. *Let* $\tilde{\tau} : V \to TV$ *be defined by*

$$\tilde{\tau}(y) = \sum_{i=1}^{\infty} \tilde{\tau}_i(y) \frac{\partial}{\partial y_i}|_y$$

where $\tilde{\tau}_i = \tau_i \circ \phi^{-1}$. *Then* $(\tau|U)(x) = (\phi^{-1})_*(\phi(x))\tilde{\tau}(\phi(x))$. *In particular, on* U *one may write for* $\lambda : U \to I\!R$, $\tau(\lambda)|_x = (\phi^{-1})_*(\phi(x))\tilde{\tau}(\phi(x))(\lambda) = \tilde{\tau}(\phi(x))(\lambda \circ \phi^{-1}) = \sum_{i=0}^{k} \tilde{\tau}_i(y) \frac{\partial \tilde{\lambda}}{\partial y_i}|_{y=\phi^{-1}(x)}$, *where* $\tilde{\lambda} = \lambda \circ \phi^{-1}$.

Proof. The fact that $\frac{\partial}{\partial x_i}|_x$ is a section on U follows easily from the definition of tangent maps. Remember that $x_j = y_j \circ \phi$. Then $\frac{\partial}{\partial x_i}|_x(x_j) = (\phi^{-1})_*|_{\phi(x))}(\frac{\partial}{\partial y_i}|_{\phi(x)})(x_j) = \frac{\partial}{\partial y_i}|_{\phi(x)}(x_j \circ \phi^{-1}) = \frac{\partial}{\partial y_i}|_{\phi(x)}(y_j) = \delta_{ij}$. By proposition 35 of Appendix F, one may define the section $\tilde{\tau} : V \to TV$ on $V = \phi(U) \subset I\!R^A$, by $\tilde{\tau}(y) = \phi_*(x)\tau(x)$, where $x = \phi^{-1}(y)$. By Prop. 7, the section $\tilde{\tau}$ may be represented by

$$\tilde{\tau} = \sum_{i=0}^{\infty} \tilde{\tau}_i \frac{\partial}{\partial y_i}|_y$$

where $\tau_i, i \in I\!N$ are convenient functions (not necessarily smooth). By construction it is clear that

$$\tau(x) = (\phi^{-1})_*(\phi(x))\tilde{\tau}(\phi(x)), x \in U$$

Hence, if $\lambda : U \to I\!R$ is a function, it follows that

$$\begin{aligned}
\tau(\lambda) &= (\phi^{-1})_*(\phi(x))\tilde{\tau}(\phi(x))(\lambda) \\
&= \tilde{\tau}(\phi(x))(\lambda \circ \phi^{-1}) \\
&= \sum_{i=0}^{\infty} (\tilde{\tau}_i \circ \phi(x)) \frac{\partial}{\partial y_i}|_{\phi(x)}(\lambda \circ \phi^{-1}) \\
&= \sum_{i=0}^{\infty} (\phi_*^{-1})(\phi(x)) \left\{ (\tilde{\tau}_i \circ \phi(x)) \frac{\partial}{\partial y_i}|_{\phi(x)}(\lambda) \right\} \\
&= \sum_{i=1}^{\infty} (\tilde{\tau}_i \circ \phi(x)) \frac{\partial}{\partial x_i}|_x(\lambda)
\end{aligned}$$

Hence one may take $\tau_i = \tilde{\tau}_i \circ \phi(x)$. By 1, one must have $\tau(x_j) = \tau_j$. Note that the infinite sum always makes sense since, at a point $x \in U$, only a finite number of summands are nonzero. $\square$

{p12s}

Proposition 19 *One call* (21) *by* local expression of τ in coordinates. *One says that the local expression is smooth if the functions τ_i are smooth on U. Let (U, ϕ_1) an (U, ϕ_2) be two local charts of a $I\!R^A$-manifold S and let τ be a section of TS. Then the local expression of τ in the coordinates ϕ_1 is smooth if and only if the local expression of τ in the coordinates ϕ_2 is smooth.*

Proof. Straightforward from Proposition 18 and Proposition 35 of Appendix F. □

The Propositions 18 and 19 allows one to state the following definition

{d18}

Definition 24 *A section $\tau :\to TS$ is a field if, for every local chart (U, ϕ) of S, with $V = \phi(U) \subset I\!R^A$, the local expression of τ in local coordinates is smooth.*

Among other important things, the next proposition shows that, when particularized to a map between open sets of $I\!R^A$, the tangent map is a generalization of the Jacobian matrix.

{pJacobian}

Proposition 20 *Let $g : U \subset I\!R^A \to V \subset I\!R^B$. Let $\{x_j, j \in I\!N^*\}$ and $\{y_i, i \in I\!N^*\}$ stand for the canonical coordinates respectively of $I\!R^A$ and $I\!R^B$. Denote the component function $y_i \circ g$ by g_i. Then*

1. $g_*(x)(\frac{\partial}{\partial x_j}|_x) = \sum_{i=1}^{\infty} \frac{\partial g_i}{\partial x_j}|_x \frac{\partial}{\partial y_i}$.
2. $g_*(x)(\sum_{j=1}^{\infty} \alpha_j \frac{\partial}{\partial x_j}|_x) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{k^*_{g_i}} \alpha_j \frac{\partial g_i}{\partial x_j}|_x \right) \frac{\partial}{\partial y_i}$*, where $k^*_{g_i}$ is the maximal index of g_i at x.*
3. *Fix a point $x \in I\!R^A$. The map $g_*(x) : T_xU \to T_yV$ is continuous*[18].

Proof. Note that $g_*(x)(\frac{\partial}{\partial x_j}|_x)(y_i) = (\frac{\partial}{\partial x_j}|_x)(y_i \circ g) = \frac{\partial g_i}{\partial x_j}$, showing 1. Note that 2 is straightforward from 1. Now, to show 3, it suffices to see that the ith component function of $g_*(x)$ is the linear map that associates $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ to $\sum_{j=1}^{k^*_{g_i}} \alpha_j \frac{\partial g_i}{\partial x_j}|_x$. As it depend only on a finite number $k^*_{g_i}$ of coordinates, since these functions are linear, from Proposition 10 it follows that they are continuous. Now, the result follows from Proposition 1. □

4.3 Cotangent bundle, pull-backs, one-forms and differentials on $I\!R^A$-manifolds

{sCTPFD}

From Proposition 17, given an $I\!R^A$ manifold S, and $x \in S$, one may identify T_xS with $T_{\phi(x)}V$ for a given local chart $\phi : U \to V$ defined around x. As $T_{\phi(x)}V$ may be identified with $I\!R^A$ (with the Fréchet topology), one may endow the $I\!R$-linear space T_xS with the topology induced by those identifications.

[18] With the Fréchet topology constructed in Part 3 of Prop. 17

Proposition 21 *Let X and Y be $I\!R^A$-manifolds and let $g : X \to Y$ be a smooth map. Fix some $x \in X$ and let $y = g(x)$. Then the $I\!R$-linear map $g_*(x) : T_xX \to T_yY$ is continuous.*

Proof. Let (U, ϕ) and (V, ψ) be respectively local charts around x and y. Let $\tilde{x} = \phi(x)$, $\tilde{y} = \psi(y)$, $\tilde{U} = \phi(U)$, $\tilde{V} = \psi(V)$. Without loss of generality, assume that $U = g^{-1}(V)$ (otherwise, one may restrict to $g^{-1}(V)$). Remember that W is an open set of T_yY if and only $W = (\psi^{-1})_*(\tilde{y})(\tilde{W})$, where $\tilde{W}$ is an open set of T_xU. Hence, $\tilde{W}$ is open. An analogous remark may be stated for the open sets of T_xY.

Let $\tilde{g} = \psi \circ g \circ \phi^{-1}$ be the expression of g in local coordinates. Then $\tilde{g}$ is differentiable. In particular, from Proposition 20, then $\tilde{g}_*(\tilde{x}) : T_{\tilde{x}}\tilde{U} \to T_{\tilde{y}}\tilde{V}$ is continuous. Since $g_*(x) = \psi_*(g(x))\tilde{g}_*(x)\phi_*^{-1}(\tilde{x})$, given an open set W of T_yY, it follows that $(g_*(x))^{-1}(W)$ is an open set of T_xX. □

Definition 25 *(Cotangent bundle, one-forms and pull-backs)* {d25}

- *(Cotangent Bundle) Let T_x^*S denote the space of continuous linear maps $\omega_x : T_xS \to I\!R$. Let $T^*S = \bigcup_{x \in s} T_x^*S$. Let $\pi : TS \to S$ be the canonical projection that maps $\omega_x \mapsto x$. The Cotangent Bundle is the triple (S, T^*S, π).*

- *(One-form) A section of the cotangent bundle of S is a map $\omega : S \to TS$ such that $\pi \circ \omega$ is the identity map on S. In other words, $\omega(x) \in T_x^*S$. A one form ω is a section of the cotangent bundle such that, for every field $\tau : S \to TS$ the map $\langle \omega, \tau \rangle : S \to I\!R$ defined by $\langle \omega, \tau \rangle(x) = \omega(x)(\tau(x))$ is a smooth function.*

- *(Cotangent Map and Pull-Back) Let X and Y be $I\!R^A$-manifolds (or open subsets of $I\!R^A$) and let $\phi : X \to Y$ be a smooth map. Let $x \in X$, $y = \phi(x)$, and let $\theta_y \in T_y^*Y$. Let $\tau_x \in T_xX$. The cotangent map $\phi^*(x) : T_y^*Y \to T_x^*X$ is defined[19] by $\langle \phi^*(x)(\theta_y), \tau_x \rangle = \langle \theta_y, \phi_*(x)\tau_x \rangle$. Let τ be a field on X. Given a section θ of $T * Y$, one may define a section $\phi^*(\theta)$ of T^*X called pull-back of θ, by the rule $\phi^*(\theta)(\tau)(x) = \langle \phi^*(x)\theta(x), \tau(x) \rangle$.*

The next proposition shows that a one-form on a $I\!R^A$-manifold is necessarily the pull-back by a local chart of a one-form on an open set of $I\!R^A$. Moreover, the pull-back of a one form is a one-form.

Proposition 22 *Let X, Y be $I\!R^A$ manifolds. Let $g : X \to Y$ be a smooth map, $y = g(x)$ and let (U, ϕ) and (V, ψ) be local charts respectively around x and y, with $V = \phi(U)$.* {p13}

1. *Let $\tilde{x} = \phi(x)$ and $\tilde{U} = \phi(U)$. The map $\phi^*(x) : T_{\tilde{x}}^*\tilde{U} \to T_x^*X$ is an isomorphism of $I\!R$-vector spaces. In particular, every section θ of T^*U is of the form $\theta = \phi^*\tilde{\theta}$, where $\tilde{\theta}$ is a section of $T^*\tilde{U}$.*

[19] Clearly, as $\phi^*(x)(\theta_y) = \theta_y \circ \phi_*(x)$, if θ_y is continuous, so is $\phi^*(x)(\theta_y)$.

2. *A one form θ on U is of the form $\theta = \phi^*\tilde{\theta}$, where $\tilde{\theta}$ is a one-form of $T^*\tilde{U}$.*

3. *(Dual of chain-rule) Let X, Y, Z be $I\!R^A$ manifolds and let $g : X \to Y$ and $f : Y \to Z$ be smooth maps. Then $(f \circ g)^* = g^* f^*$.*

4. *If ω is a one-form on Y, then $g^*\omega$ is a one-form on X.*

Proof. See appendix G. □

{pPull}

Proposition 23 *Let S be an $I\!R^A$ manifold and let (U, ϕ) be a local coordinate system. Let $(x_i, i \in I\!N^*)$ be the corresponding local coordinate functions. Let $\pi_k^\phi : U \to I\!R^k$ be the map defined by*

$$\pi_k^\phi(\xi) = (x_1(\xi), x_2(\xi), \ldots, x_k(\xi)), \xi \in S$$

Note that $\phi_k^\phi = \pi_k \circ \phi$. A section ω of T^*S is a one form if and only if, around every point $\xi \in U$, there exists a local chart (U, ϕ) around ξ and a one form $\tilde{\omega}$ on $I\!R^k$, such that $\omega|_U = (\pi_k^\phi)^*\tilde{\omega}$.

Proof. From the part 3 of the last Proposition, there exists a local chart (U, ϕ) such that $\omega|_U = \phi^*\hat{\omega}$, where $\hat{\omega}$ is a one form on an open subset of $I\!R^A$. Now, from Part 2 of theorem 2, $\hat{\omega} = \pi_k^*\tilde{\omega}$, where $\tilde{\omega}$ is a one-form on an open set of $I\!R^k$. Hence, using Proposition 22, $\omega|_U = (\phi^*\pi_k^*\tilde{\omega}) = (\pi_k \circ \phi)^*\tilde{\omega} = (\pi_k^\phi)^*\tilde{\omega}$. □

Now we are ready to define the notion of differential of a function on a $I\!R^A$-manifold.

{d26}

Definition 26 *One may define the differential df of a function $f : S \to I\!R$ as the one-form such that $df(\tau) = \tau(f)$ for every field τ on S.*

{p15}

Proposition 24 *Let $\phi : X \to Y$ be a smooth map between $I\!R^A$ manifolds. Let $\lambda : Y \to I\!R$ be a function. Then $\phi^*(d\lambda) = d(\lambda \circ \phi)$.*

Proof. We shall show that the action of both 1-forms on every field τ on X coincides. In fact, $\langle \phi^*(d\lambda), \tau \rangle = \langle d\lambda, \phi_*\tau \rangle = (\phi_*\tau)(\lambda) = \tau(\lambda \circ \phi) = \langle d(\lambda \circ \phi), \tau \rangle$ □

{pComputational}

Proposition 25 *(Computational Issues) Let S be a $I\!R^A$-manifold. Let $\xi \in S$ and let (U_ξ, ϕ) be a local chart of S around ξ. Let $(x_i, i \in I\!N^*)$ be the corresponding coordinate functions. Let $dx_i|_x$ be the differential of $x_i : U \to I\!R$. Then:*

1. *$\langle dx_i|_x, \frac{\partial}{\partial x_j}|_x \rangle = \delta_{ij}$, where δ_{ij} was defined in Prop. 18.*

2. *If $\{y_i, i \in I\!N^*\}$ are the canonical coordinates of $I\!R^A$, then*

$$dx_i|_x = \phi^*(x) dy_i|_{\phi(x)}.$$

3. *If $\{z_i, i = 1, \ldots, k\}$ are the canonical coordinates of $I\!R^k$, then $dx_i|_x = (\pi_k^\phi)^*(z)dz_i|_z$, $i = 1, \ldots, k$, where $z = \pi_k^\phi(x)$.*

4. *If ω is a one-form on S, then around every $\xi \in U$ there exists an open neighborhood V_ξ of ξ, and some $k \in I\!N$, such that $\omega|_{V_\xi}(x) = \sum_{i=1}^k \omega_i(x)dx_i|_x$ where $\omega_i : V_\xi \to I\!R$ is the smooth function given by $\omega_i(x) = \langle \omega(x), \frac{\partial}{\partial x_i}|_x \rangle$, and the minimal index of the functions ω_1 at ξ are less than or equal to k. Furthermore, $\omega = (phi)^*\hat{\omega}$ where $\hat{\omega}$ is the one-form on $\phi(V_\xi) \subset I\!R^A$ given by $\hat{\omega}(y) = \sum_{i=1}^k \tilde{\alpha}_i(y)dy_i|_y$, and $\tilde{\alpha}_i(y) = \omega_i \circ \phi^{-1}(y)$.*

5. *Let k be the minimal index of f at ξ and let $V_\xi \subset U_\xi$ be a minimal neighborhood. Then $df(\nu) = \sum_{i=1}^k \frac{\partial}{\partial x_i}|_\nu(f)dx_i|_\nu$, for all $\nu \in V_\xi$. Furthermore, if $\tilde{f} = f \circ \phi^{-1}$, then $\frac{\partial}{\partial x_i}|_\nu(f) = \frac{\partial \tilde{f}}{\partial y_i}|_{\phi(\nu)}$.*

Proof. See appendix H. □

Remark 7 *One may endow TS and T^*S with structures of a $I\!R^A$ manifold in a very similar way that one can do in the finite dimensional case (Warner 1971). In this context, one may define the canonical projection $\pi : TS \to S$ in the same way one has defined above, and a section is a map $\tau : S \to TS$ such that $\pi \circ \tau$ is the identity map. Then a field may be defined as a smooth section of TS, which gives a intrinsic definition of a field. However, a one-form is not only a smooth section of T^*S, as one have noted in the discussions above, and intrinsic definition of a one-form ω must assure that $\langle \omega, \tau \rangle$ is a smooth function for every field τ.*

4.4 Independent functions are part of a local coordinate system

It is known in finite dimensional geometry that a set of functions with pointwise independent differentials is part of a local coordinate system. The theorem that allows one to prove this statement is the inverse function theorem. The next lemma shows that this result holds in our infinite dimensional setting, at least for a finite number of functions. However it does not hold for an infinite number of functions (Zharinov 1992). As a consequence of this result, one shows that, if the differential of a function u is generated by a finite set $d\theta$ of pointwise independent differentials, then u is locally a function of θ.

{lLemma2}

Lemma 1 *Let S be a $I\!R^A$ manifold, and V be an open neighborhood of $\xi \in S$. Let $\theta = \{\theta_1, \ldots, \theta_k\}$ be a set of smooth functions $\theta_i : V \to I\!R$, $i = 1, \ldots, k$, and assume that set of one-forms $d\theta = \{d\theta_1, \ldots, d\theta_k\}$ is pointwise independent at every point of V.*

1. *Let $\nu \in V$. Then there exists an open neighborhood U of ν and a set of functions z such that (θ, z) is a local coordinate system defined in U.*

In particular the map $\delta : U \to \delta(U) = \widehat{V} \subset I\!R^k$ defined by $\delta(\zeta) = (\theta_1(\zeta), \ldots, \theta_k(\zeta))$ is open and surjective.

2. *Let $u : V \to I\!R$ be a smooth function such that $du \in$ span $\{d\theta_1, \ldots, d\theta_k\}$ in V. Let $\nu \in V$. Then, there exist a smooth map $\mu : \widehat{\delta}(U) \to I\!R$ such that $u|U = \mu \circ \delta$. In particular, the mapping μ may be identified with the expression of u in the local coordinates (θ, z).*

3. *Let (θ, w) be a local coordinate system around ξ and let $\eta = \{\eta_1, \ldots, \eta_k\}$ be a set of smooth functions such that span $\{d\theta\}$ = span $\{d\eta\}$ on an open neighborhood containing ξ and the set $d\eta$ is linearly independent at ξ. Then (η, w) is also a local coordinate system on some open neighborhood of ξ.*

Proof. See Appendix I. □

Proposition 26 *Let S be a $I\!R^A$-manifold. Let $u : S \to I\!R$ be a function. Let (U, ϕ) be a local coordinate system, and let $x = (\theta, w)$, be the corresponding coordinate function*[20] *around ξ, where $\theta = \{\theta_i, i \in B\}$ and $w = \{w_j, j \in C\}$. Assume that one locally has span $\{d\mu\} \subset$ span $\{d\theta\}$. Hence, there exists an open neighborhood of ξ such that the local expression of u in coordinates given by $\mu = u \circ \phi^{-1}$ in coordinates is of the form $\mu(\theta_1, \ldots, \theta_k)$.*

Proof. Abusing notation, let $\{\theta, w\}$ be coordinates of $I\!R^A$. By Part 5 of Proposition 25, one gets $\frac{\partial \mu}{\partial w_j} = 0, j \in C$. If the minimal index of u at ξ is k, on a minimal neighborhood, if the cardinal of B is greater than k, one gets $\frac{\partial \mu}{\partial \theta_i} = 0$ for $i \geq k$. The desired result then follows. □

4.5 Integral curves on $I\!R^A$-Manifolds

A smooth curve on an $I\!R^A$-manifold S is a smooth map $\sigma : (a, b) \to S$, where $(a, b) \subset I\!R$. Let f be a field on S. One may define on $I\!R$ the field $\frac{d}{ds}$ given by the standard operation of derivation of smooth functions.

Definition 27 *A smooth curve $\sigma : (a, b)(a, b) \to S$ is an integral curve of a field f if*

$$\sigma_*(t)\frac{d}{ds}|_t = f(\sigma(t)). \tag{22}$$ {eSolution}

Given a local coordinate system (U, ϕ) with local coordinate functions $(x_i, i \in I\!N^*)$, one may abuse notation, letting $x_i(t)$ stands for $x_i \circ \sigma(t)$, that is, one shall denote $\phi \circ \sigma(t)$ by $(x_1(t), x_2(t), \ldots)$. Let $\tilde{U} = \phi(U)$. Assume that

$$f(x) = \sum_{i=1}^{\infty} f_i(x) \frac{\partial}{\partial x_i}|_x$$

[20] Here one does not assume that θ is a finite set.

where $f_i = f(x_i)$. Then, when one seeks an integral curve $\sigma(t) \in U$, the equation (22) is locally equivalent to the infinite dimensional differential equation[21]

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), \ldots, x_{\alpha_i}), i \in I\!N^* \tag{23}$$ {eSolutionL}

An integral curve σ is also called a *solution* of f. One may take a closed interval $[a, b]$, instead of (a, b). In this case $\sigma(a)$ is called the *initial condition of* σ.

Contrarily to the finite dimensional case, a version of flow-box theorem is not available in the context of $I\!R^A$ manifolds. In fact, for some fields there exists no integral curves (Zharinov 1992). It is easy to construct an example for which there exist infinitely many solutions with the same initial condition. In fact, on $I\!R^A$, define the field $f = \sum_{i=1}^{\infty} x_{i+1} \frac{\partial}{\partial x_i}|_x$, where $x = (x_1, x_2, \ldots)$. Now, Borel's theorem (Borel 1895) says that, for every $\bar{x} \in I\!R^A$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$, there exists a smooth function $g : (-1, 1)$ with $g^{(k-1)}(0) = \bar{x}_k, k \in I\!N^*$. This assures the existence of a solution $\sigma(t) = (g(t), g^{(1)}(t), g^{(2)}(t), \ldots)$ for any initial condition $\bar{x} \in I\!R^A$. Remember that the smooth function[22] $h : I\!R \to I\!R$ defined by

$$h(t) = \begin{cases} 0, t \leq 0 \\ \exp(-1/t), t > 0 \end{cases} \tag{24}$$

is such that $h^{(k)}(0) = 0$ for $k \in I\!N^*$. Hence, the curve $\sigma(t) = (g(t)+h(t), g^{(1)}(t)+h^{(1)}(t), g^{(2)}(t) + h^{(2)}(t), \ldots)$ is also a solution with initial condition $\bar{x} \in I\!R^A$. So uniqueness of solutions is not expected to be a general property in our infinite dimensional context.

4.6 Submersions, immersions, immersed manifolds and embeddings

Let A, B be two countable sets. Let $X = I\!R^A$ and $Y = I\!R^B$. Let $\pi : I\!R^A \times I\!R^B \to I\!R^B$ be such that $\pi(x, y) = y$. Let $\iota : I\!R^A \to I\!R^A \times I\!R^B$ be such that $\iota(x, y) = (x, 0)$. Let S and T be two $I\!R^A$-manifolds. A map $\Sigma : S \to T$ is said to be a *submersion* (respectively, an *immersion*) if, for every $\xi \in S$, there exists local charts (U, ϕ) and (V, ψ), with $\xi \in U$ and $\phi(\xi) \in V$, with coordinate functions such that the expression of Σ in local coordinates, given by $\psi \circ \Sigma \circ \phi^{-1}$, coincides with π (respectively, coincides with ι). In the finite dimension case, one may show that a map Σ is a local submersion around some $\xi \in S$ (respectively a local immersion) if and only if $\Sigma_*(\xi)$ is an surjective linear map (respectively, an injective linear map). This nice feature does not hold in our infinite dimensional setting (Zharinov 1992). However, if Σ is a submersion (respectively an immersion), Prop. 27 shows that Σ_* is pointwise surjective (respectively pointwise injective).

Let S be an $I\!R^A$-manifold. A subset $\tilde{\Delta} \subset \Delta$ is said to be an immersed manifold if there exists a $I\!R^A$-manifold Δ and an injective immersion $\Sigma : \Delta \to S$

[21] In fact, $\sigma_*(\frac{d}{ds}|_t)(x_i) = \frac{d}{ds}|_t(x_i \circ \sigma) = \dot{x}_i(t) = f(x_i)|_{\sigma(t)} = f_i|_{\sigma t}$.
[22] This function is used in the construction of partitions of unity (Warner 1971).

with $\tilde{\Delta} = \Sigma(\Delta)$. Note that one may endow $\tilde{\Delta}$ with two different topologies. The first one is the *subset topology*, for which the open sets of $\tilde{\Delta}$ are of the form $U \bigcap \tilde{\Delta}$, where U is an open set of S. The second one is the induced topology, for which the open sets of $\tilde{\Delta}$ are of the form $\Sigma(V)$, where V is an open set of Δ. Note that $\tilde{\Delta}$ has a structure of $I\!R^A$-manifold when one chooses the induced topology. In fact, giving an atlas $\{(U_\lambda, \phi_\lambda), \lambda \in \Lambda\}$ of Δ, as the map $\Sigma : \Delta \to \tilde{\Delta}$ is a bijection, then one may define the atlas $\{(\tilde{U}_\lambda, \tilde{\phi}_\lambda), \lambda \in \Lambda\}$ of $\tilde{\Delta}$, where $\tilde{U}_\lambda = \Sigma(U_\lambda)$ and $\tilde{\phi}_\lambda = \phi_\lambda \circ \Sigma^{-1}$. A simple exercise shows that this defines a structure of $I\!R^A-$manifold for $\tilde{\Delta}$. Note that, as Σ is an immersion, there exists convenient local charts such that the local expression of Σ in these coordinates reads $x \mapsto (x, 0)$. So the immersed manifold is locally "a slice" of the manifold S, as one have seen in the finite dimensional case.

The immersion Σ is said to be an embedding if the induced topology coincides with the subset topology. In this case there is no loss of generality of thinking that Σ is the insertion map (since it is a bijection onto its image), $\tilde{\Delta} = \Delta$ and the topology is the subset topology. As in the finite dimensional case, an embedding cannot exhibit the same known problem of the "*figure eight*", that is, an embedded manifold cannot approximate to itself indefinitely, because it is formed by a disjoint union of slices (Warner 1971).

In finite dimensional geometry, the converse of the next proposition holds. However this is not true in general in our infinite dimensional context. For instance, the tangent map ϕ_* of a map $phi : S \to T$ may be injective at a given point $\xi \in S$, but the map ϕ is not an immersion around this point (see an example of this fact (Zharinov 1992)).

{p16}

Proposition 27 *The tangent map of an immersion (respectively submersion) is pointwise injective (respectively surjective). The cotangent map of an immersion (respectively submersion) is surjective (respectively injective).*

Proof. We show only the claims for injections. The proof of the given affirmations for submersion is similar, and is left to the reader. Locally an injection $\iota : \Gamma \to S$ is of the form $x \mapsto (x, 0)$. Given a function $\lambda : S \to \Gamma$, namely $\lambda(x, z)$, note that $\lambda \circ \iota = \lambda(x, 0)$. In particular, if $\lambda = x_i$, abusing notation, one has $\lambda \circ \iota = x_i$. Now, if $\iota_*(\tau)(\lambda) = 0$ for all λ, this means that, $\tau(\lambda \circ \iota) = 0$ for all λ. In particular $\tau(x_i) = 0$ for all i implies that $\tau = 0$. Hence $\ker \iota_* = \{0\}$ and so ι_* is injective. Now, as $\iota^*(d\lambda) = d(\lambda \circ \iota)$, by choosing $\lambda = x_i$ one gets $\iota^*(d\lambda) = dx_i$. In particular, the cotangent space $T_x X$ is generated by the image of $i^*(x)$ for all $x \in X$. $\square$

{lSubmersion}

Lemma 2 *Let S be a $I\!R^A$ manifold an let $\phi : S \to I\!R^B$ be a submersion, where B is a (finite or infinite) countable set. Then around all $\xi \in S$ there exists a local coordinate system (U, ψ) such that the coordinate functions are of the form (ϕ, z). When written in these coordinates, the map ϕ reads $(h, z) \mapsto h$.*

Proof. Looking $I\!R^B$ as a $I\!R^B$-manifold, the assumption means that, for all $\xi \in S$, there exists local charts (U, α) around ξ and (V, β) around $\phi(\xi)$ such that the map $\beta \circ \phi \circ \alpha^{-1}$ reads $(x, z) \mapsto x$. Note that the map $(x, z) \mapsto (\beta^{-1}(x), z)$ is a local diffeomorfism with inverse $(w, z) \mapsto (\beta(w), z)$. Hence $(\beta^{-1}(x), z)$ is also a set of coordinate functions. Now, for all $\nu \in S$ with $\alpha(\nu) = (x, z)$, one has $\beta^{-1}(x) = \beta^{-1}\left(\beta \circ \phi \circ \alpha^{-1}(x, z)\right) = \phi \circ \alpha^{-1}(x, z) = \phi \circ \alpha^{-1} \circ \alpha(\nu) = \phi(\nu)$. In particular, $\nu \mapsto (\phi(\nu), z(\nu))$ is the local chart with the claimed properties. $\square$

4.7 Distributions and Codistributions

Let S be a $\mathbb{R}^A$ manifold. Let $x \in S$ and let $\mathcal{G}_x$ stands for the set of all subspaces of the $\mathbb{R}$-linear space $T_x S$. Define $\mathcal{G} = \bigcup_{x \in S} \mathcal{G}_x$. A distribution $\mathcal{D} : S \to \mathcal{G}$ is a map $x \mapsto \mathcal{D}(x)$ such that, for all $x \in S$, $\mathcal{D}(x)$ is a subspace of $T_x S$. In finite dimensional theory, one may endow $\mathcal{G}$ with a structure of manifold (called the Grassmanian of S). In this context one may define a smooth codistribution as being a smooth section of $\mathcal{G}$. In our infinite dimensional setting, this construction is much more involved, and it is not necessary, at least for our purposes. Hence, we shall consider another definition of smooth distribution, that is also considered in finite dimensional geometry.

Definition 28 *(Distributions) The set $\mathcal{F}(S)$ of fields on S has a structure of $C^\infty(S)$-module induced by the operation of (pointwise) sum of fields, and (pointwise) multiplication of a field by a function (evaluated on the working point). A smooth distribution on S is a submodule D of $\mathcal{F}(S)$. Note that, given a distribution D, one may define a $\mathcal{D}(x) = span\,\{\tau(x) \,|\, \tau \in D\}$. A point x of S is a* regular *point of D, if there exists an open neigborhood V_x of x such that the dimension of $\mathcal{D}(\nu)$ is finite and constant for all $\nu \in V_x$. A distribution is involutive if for all τ_1 and τ_2 in D, then $[\tau_1, \tau_2] \in D$.*

Unfortunately, the Frobenius theorem does not hold for finite dimensional distributions in our infinite dimensional setting. As a matter of fact, it holds when its codimension is finite. It is much easier to considere the Cartan's version of the Frobenius theorem, which is related to codistributions (see section 4.9).

Definition 29 *(Codistributions) The set $\Lambda_1(S)$ of one-forms on S has a structure of $C^\infty(S)$-module induced by the operation of (pointwise) sum of one-forms, and (pointwise) multiplication of a one-form by a function (evaluated on the working point). A (smooth)* codistribution *on S is a submodule Γ of $\Lambda_1(S)$. Note that, given a distribution Γ, one may define a map $\mathcal{C}(\Gamma)$, that associates to every point $x \in S$ the subspace of $T_x^* S$ given by $\mathcal{C}(\Gamma)(x) = span\,\{\omega(x) \,|\, \omega \in \Gamma\}$. A point x of S is a regular point of Γ, if there exists an open neighborhood V_x of x such that the dimension of $\mathcal{C}(\Gamma)(\nu)$ is finite and constant for all $\nu \in V_x$. If all $x \in S$ is a regular point of Γ, we say that Γ is a* nonsingular *codistribution.*

Remark 8 *By simplicity, one may abuse notation, letting $\Gamma(x)$ stands for $\mathcal{C}(\Gamma)(x)$.*

Given a codistribution Γ defined on S, let $V \subset S$ be an open set. Then $\Gamma|_V$ stands for the $C^\infty(V)$-submodule defined by $\Gamma|_V = \mathrm{span}_{C^\infty(V)}\,\{\omega|_V \;:\; \omega \in \Gamma\}$.

Proposition 28 *Let Γ be a smooth codistribution on S. Define the function $\dim\Gamma : S \to I\!R$ by $x \mapsto \dim\Gamma(x)$. Assume*[23] *that $\dim\Gamma(x) \leq k^*$ for some $k^* \in I\!N$ for all $x \in S$.*

1. *The function $\dim(\Gamma) : S \to I\!R$ is upper semi-continuous. In other words, around every point $\nu \in S$, there exist an open set $V_\nu \subset s$ such that $(\dim(\Gamma)(\nu)) \leq (\dim(\Gamma)(x))$ for every $x \in V_\nu$.*
2. *The set of regular points of Γ is open and dense on S.*
3. *Let $x \in S$ be a regular point of a codistribution Γ, then there exists an open neighborhood V_x of x and a set of pointwise independent one-forms $\omega = \{\omega_1, \ldots, \omega_r\} \subset \Gamma$ such that $\Gamma|_{V_x} = span_{C^\infty(V_x)}\{\omega_1|_{V_x}, \ldots, \omega_r|_{V_x}\}$. The set ω is called a* local basis *of Γ.*
4. *Let $\Gamma_i, i = 1, 2$ be two codistributions. They are said to be locally coincident, if for all $\xi \in S$, one has $\Gamma_1|_{V_\xi} = \Gamma_2|_{V_\xi}$, where V_ξ is an open neighborhood of ξ. Assume that Γ_1 is nonsingular. Then the codistributions Γ_1 and Γ_2 are locally coincident if and only if $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$.*

Proof. 1. Let $s = \dim\Gamma(\nu)$. Then $\Gamma(\nu)$ is a s-dimensional subspace of $T^*_\nu S$. In particular, there exists a set of one-forms $\{\omega_1, \ldots \omega_s\}$ in Γ, such that $\{\omega_1(\nu), \ldots, \omega_s(\nu)\}$ is a basis of $\Gamma(\nu)$. By Part 4 of Proposition 25, choosing a local coordinate system, on some open neighborhood V of ν, one may write

$$\omega_i(x) = \sum_{i=1}^{k} \alpha_{ij}(x) dx_i|_x$$

where $\alpha_{ij} : V \to I\!R$ are smooth functions. The $k \times s$ matrix $\alpha(x)$ such that $\{\alpha\}_{ij} = \alpha_{ij}$ depends smoothly[24] on x. As $\alpha(\nu)$ has rank s, it admits a nonzero minor determinant. Since the (minor) determinant is a continuous function, and so is α, this minor will be nonzero on some open neighborhood W_ν of ν. In particular $\dim\Gamma(x) \geq s$ for all $x \in W_\nu$.

2. The fact that the set of nonsingular points is open is straightforward from the definition. To show that this set is dense, choose any open set V. As $\dim\Gamma(x) \leq k^*, x \in S$ there must exist $\nu \in V$ such that $\dim\Gamma(\nu) \geq \dim\Gamma(x)$ for all $x \in V$. By 1, it follows that ν must be a regular point of Γ.

3. In the proof of 1, take the constructed set $\omega = \{\omega_1, \ldots, \omega_r\}$, where $r = \dim\Gamma(x)$. This set is pointwise independent on some open neighborhood V_x of $x \in S$. By dimensional arguments, it is clear that $\Gamma(\xi) = \text{span}\{\omega_1(\xi), \ldots, \omega_r(\xi)\}$ for all $\xi \in V_x$. This shows only the pointwise coincidence of the vector spaces $\Gamma(\xi)$ and $\text{span}\{\omega_1(\xi), \omega_r(\xi)\}$. To show the statement of 2 one must show that,

[23] For instance, this is true if Γ is finitely generated.
[24] One may regard $\alpha : V \to I\!R^{k\times s}$ as a map that associates $x \in V$ to a $k \times s$ matrix.

for all $\omega \in \Gamma|_{V_x}$, $\omega(\xi) = \sum_{i=1}^{r} \delta_i(\xi)\omega_i(\xi)$ for all $\xi \in V_x$, where $\alpha_i, i = 1, \dots r$, are smooth functions. One has already shown that

$$\omega(\xi) = \sum_{i=1}^{r} \delta_i(\xi)\omega_i(\xi). \tag{25}$$ {eOmegaXi}

It suffices to show the smoothness of the δ_i. From the same arguments of the proof of 1, after choosing a local coordinate system

$$\omega_i(\xi) = \sum_{i=1}^{k} \alpha_{ij}(\xi) dx_i|_x, i = 1, \dots, r \tag{26}$$ {eAAAAA}

and define the smooth map $\alpha : V_x \to I\!R^{k\times r}$. As rank $\alpha(x) = r$, one may complete the real matrix $\alpha(\xi)$ with $k-r$ constant rows $\beta_l, l = 1, \dots, k-r$, constructing the $k \times k$ invertible matrix $\beta(\xi)$. By continuity, $\beta(\xi)$ will be invertible for ξ in some open neighborhood $W_x \subset V_x$ of x. So define

$$\omega_i(\xi) = \sum_{i=1}^{k} \beta_{ij}(\xi) dx_i|_\xi, i = 1, \dots, k$$

By Proposition 25, one may locally write

$$\omega(\xi) = \sum_{i=1}^{k^*} \gamma_i(\xi) dx_i|_\xi$$

for convenient smooth functions γ_i. Substituting (26) on (25), one shows that one may take $k^* = k$. Note that this substitution is equivalent to the following matrix equation

$$\begin{pmatrix} \gamma_1(\xi) \\ \gamma_2(\xi) \\ \vdots \\ \gamma_k(\xi) \end{pmatrix} = \beta(\xi) \begin{pmatrix} \delta_1(\xi) \\ \delta_2(\xi) \\ \vdots \\ \delta_r(\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So one may locally write

$$\begin{pmatrix} \delta_1(\xi) \\ \delta_2(\xi) \\ \vdots \\ \delta_r(\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \beta^{-1}(\xi) \begin{pmatrix} \gamma_1(\xi) \\ \gamma_2(\xi) \\ \vdots \\ \gamma_k(\xi) \end{pmatrix}$$

Showing the smoothness of the functions δ_i.

Another proof can be given, without the need of completing the matrix α into an invertible matrix β. The idea is to use a pseudo-inverse[25] of α in order to compute the δ_i from the knowledge of the γ_j.

4. Straightforward consequence of the proof of 3. The details are left to the reader. $\square$

Remark 9 *Note that proofs of Parts 1 and 2 Proposition 28 also shows that $\nu \in S$ is a regular point of Γ if and only if $\dim \Gamma$ is locally maximal around ν.*

4.8 Lie-Brackets, Lie-derivatives and Exterior Calculus on $\mathbb{R}^A$-manifolds

In this section one will generalize for $\mathbb{R}^A$-manifolds the concepts of Lie-Brackets, Lie-derivatives and all the results of exterior calculus, that have been established for open sets of $\mathbb{R}^A$. The main idea is to transfer directly all the results by using the tangent mappings of the local charts (for fields), or by the pull-backs of the forms. One may define Lie-Brackets on $\mathbb{R}^A$-manifolds exactly as it was stated in definition 11, only replacing $\mathbb{R}^A$ by $\mathbb{R}^A$-manifold. The proof that such definition is well posed is identical to the one of Proposition 9.

Lemma 3 *Let $\phi : S \to \tilde{S}$ be a smooth mapping between $\mathbb{R}^A$-manifolds. Let τ, θ be fields on S and $\tilde{\tau}, \tilde{\theta}$ be fields on $\tilde{S}$. The fields $\{\tilde{\tau}, \tau\}$ are said to be ϕ-related if $\phi_* \tau = \tilde{\tau} \circ \phi$. Assume that $\{\tilde{\tau}, \tau\}$ and $\{\tilde{\theta}, \theta\}$ are ϕ-related. Then $\{[\tilde{\tau}, \tilde{\theta}], [\tau, \theta]\}$ are also ϕ-related.*

Proof. The proof is identical to the same result of finite dimensional geometry (see (Warner 1971)). $\square$

The consequence of the last Lemma is that, to compute the expression of the Lie-Bracket in coordinates, it suffices to apply the expression that is developed in section 2.4. This follows from the fact that a field τ on S is locally given by $\tau \circ \phi = \phi^{-1} * \tilde{\tau}$, where (U, ϕ) is a local chart of S, $\tilde{\tau}$ is a field on $V \subset \mathbb{R}^A$, and $V = \phi(U)$.

One has shown that a one form ω on a $\mathbb{R}^A$-manifold is locally the pull-back $\phi^* \tilde{\omega}$ where ϕ is a local chart and $\tilde{\omega}$ is a one-form on an open set of $\mathbb{R}^A$. One can define a p-form on S by generalizing this property. The next definition generalizes the Def. 16 for $\mathbb{R}^A$-manifolds.

Definition 30 *Let Y be a $\mathbb{R}^A$-manifold. Let*

$$\Gamma_Y^p(y) = \underbrace{T_y Y \times T_y Y \times \ldots \times T_y Y}_{p \ times}$$

[25] For the definition and properties of the pseudo-inverse see for instance (Strang 1988).

and define $\Gamma^k_Y = \bigcup_{y\in Y}\Gamma_k(y)$. Let $\omega : \Gamma_k Y \to I\!R$ be a map. Let $g : X \to Y$ be a smooth map between $I\!R^A$-manifolds. Let $y = g(x)$. Define the pull-back $g^*|_x\omega : \Gamma^k_X(x) \to I\!R$ by the rule

$$g^*_x\omega(\tau_1,\ldots,\tau_p) = \omega(g_*\tau_1,\ldots,g_*\tau_p), \text{for every } \{\tau_1,\ldots,\tau_p\} \text{ in } \Gamma^p_X(x)$$

Define the pull-back $g^*\omega : \Gamma^k_X \to I\!R$, pointwise, by the same rule above.

Let (U,ϕ) be a local chart on Y. Let $V = \phi(U) \subset I\!R^A$. By definition, if U is an open subset of Y, then Γ^k_U is a subset of Γ^k_Y. We are ready to define a p-form.

{dPFormsS}

Definition 31 *(p-forms on $I\!R^A$-manifolds)*

- *A p-form ω on Y is a map $\omega : \Gamma^p_Y \to I\!R$ such that, around every $y \in Y$ there exists a local chart (U,ϕ) around y such that $\omega|_{\tilde U} = \phi^*\tilde\omega$, where $\tilde\omega$ is a one-form on $V = \phi(U) \subset I\!R^A$, and $\tilde U = \Gamma^p_U$.*

- *The set of p-forms on Y will be denoted by $\Lambda_p(Y)$, where $\Gamma_0(Y)$ denotes the set of smooth functions $f : Y \to I\!R$.*

- *The wedge product "$\wedge$" is defined[26] in the following way. Let $\omega_x : \Gamma^p_U(x) \to I\!R$ and $\eta_x : \Gamma^m_U(x) \to I\!R$ be two maps. Then $\omega \wedge \eta$ is defined by the same rule* (8) *that was used to define the wedge product on $I\!R^A$..*

- *The exterior derivative is the map $d : \Lambda_p(Y) \to \Lambda_{p+1}(Y)$ defined in the following way. For $p = 0$, the map $df = d(f)$ is the differential of the function f. For $p > 0$, then[27] $d(\omega)|_{\tilde U} = \phi^*(d\tilde\omega)$, where $\tilde U$ and $\tilde\omega$ are defined above.*

Some important properties of forms on a $I\!R^A$-manifold are collected in the following result:

{pCollectS}

Proposition 29 *Let θ be a one-form, ω be a p-form and η be a q-form , all of them defined on an open set V of a $I\!R^A$-manifold. Let $g : U \to V$ be a smooth map between open subsets of $I\!R^A$-manifolds.*

1. *The definition of exterior differential stated in Def. 31 des not depend on the chosen local chart that is used to represent the p-form.*
2. *One has[28] $g^*(\omega\wedge\eta) = (g^*\omega)\wedge(g^*\eta)$.*
3. *The map $g^*\omega$ is a p-form.*
4. *$dg^*\theta = g^*(d\theta)$.*

[26] This definition mimics the standard definition of wedge product that appears in finite dimensional exterior algebra (see (Warner 1971)).

[27] It is not difficult to show that the definition does not depend on the chosen local chart (U,ϕ).

[28] For the moment one is not claiming that g^* maps forms to forms. Remember that the wedge product was defined for arbitrary maps $\omega : \Gamma^p_U \to I\!R$.

5. $d(g^*\omega) = g^*(d\omega)$.

Proof.
1. Since $\phi_2^{-1} \circ \phi_2$ is the identity map, and $(\phi_2 \circ \phi_2^{-1})^* = (\phi_2^{-1})^* \phi_2^*$, then it follows that $\phi_2^{-1})^* = (\phi_2^*)^{-1}$. Assume that (U_1, ϕ_1) and (U_2, ϕ_2) are two local charts of a $I\!R^A$-manifold S, with $W = U_1 \cap U_2 \neq \emptyset$. Let ω be a p-form on W such that $\omega = \phi_1^* \tilde{\omega}_1$ and $\omega = \phi_2^* \tilde{\omega}_2$, where ω_1 and ω_2 are p-forms defined respectively on the open subsets respectively, $\phi_1(U_1)$ and $\phi_2(U_2)$. It will be shown that $\phi_1^* d\omega_1 = \phi_2^* d\omega_2$ on W. In fact, note that $(\phi_2^{-1})^* \phi_1^* \omega_1 = \omega_2$. Then, $(phi_1 \circ \phi_2^{-1})^* \omega_1 = \omega_2$. By proposition part 3 of 16, it follows that $d(phi_1 \circ \phi_2^{-1})^* \omega_1 = (phi_1 \circ \phi_2^{-1})^* d\omega_1 = d\omega_2$. Then, this implies that $\phi_2^*(phi_1 \circ \phi_2^{-1})^* d\omega_1 = \phi_2^* \phi_2^{-1})^* \phi_1^* d\omega_1 = (\phi_1) * d\omega_1 = (\phi_2)^* d\omega_2$.
2. It follows easily from the third item of Def. 31 (see the proof of Prop. 13).
3. Let (U, ϕ) be a local coordinate system. As in Prop. 23, one lets π_k^ϕ stands for $\pi_k \circ \phi$. By Def. 31, ω is locally given by $\phi^* \hat{\omega}$, $\tilde{\omega}$ is a p-form on some the open set $\phi(U) \subset I\!R^A$. By Def. 17, $\hat{\omega} = (\pi_k)^* \tilde{\omega}$, where $\tilde{\omega} = \sum_{i=1}^k \tilde{\alpha}_i d\tilde{x}_i$ is a form on an open set of $I\!R^k$. Then $\omega = (\pi_k^\phi)^* \sum_{i=1}^k \tilde{\alpha}_i d\tilde{x}_i$. The rest of the proof is similar to the Proof of Prop. 16 with π_k (or π_l) replaced by π_k^ϕ (or respectively, replaced by ϕ_l^ϕ). The details are left to the reader.

□

It is clear now that one may compute exterior derivatives and wedge products of p-forms on $I\!R^A$ manifolds in the same way that one computes exterior derivatives of forms on $I\!R^A$. Given a local chart (U, ϕ), all the properties of finite dimensional geometry may be obtained via the pull-back ϕ^* in the expected way. Now let $I = (i_1, \ldots, i_p) \in H_p(k)$ be a multiindex. Given a local chart (U, ϕ) with coordinate functions $\{x_i, i \in I\!N^*\}$, one may locally define on U the p-forms $\phi^*(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}) = (\pi_k^\phi)^*(d\tilde{x}_{i_1} \wedge d\tilde{x}_{i_2} \wedge \ldots \wedge d\tilde{x}_{i_p})$, where $\tilde{x}_i \circ \phi = x_i$. Abusing notation, one lets $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$ stands for $\phi^*(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p})$. Now, if ω is locally written by $\sum_{I \in H_p(k)} \alpha_I(x) dx_I$ then $d\omega$ is locally written by

$$d\omega = \sum_{I \in H_p(k)} d\alpha_I \wedge dx_I.$$

If one locally has $\omega_1 = \sum_{I \in H_p^k}^{l_1} \alpha_I dx_I$ and $\omega_2 = \sum_{J \in I_q(l)} \beta_J dx_J$, then

$$\omega_1 \wedge \omega_2 = \sum_{I \in H_p(k)} \sum_{J \in I_q(l)} \alpha_I \beta_J dx_I \wedge dx_J. \tag{27}$$ {eWedgeI}

Other properties may be easily obtained from the properties of p-forms on $I\!R^A$. For instance, all the definitions and properties of Section 3 may be easily generalized for $I\!R^A$-manifolds.

4.9 The Frobenius theorem on $I\!R^A$-manifolds {sFrob}

As stated above, the Frobenius theorem for fields is a delicate matter in our infinite dimensional setting. However, since 1-forms are pull-backs of one-forms that are defined on finite dimensional linear spaces, the Cartan's version of Frobenius theorem is easy to obtain from the same theorem for finite dimensional manifolds.

Definition 32 *Let $\Gamma = span\,\{\omega_1, \ldots, \omega_k\}$ be a codistribution on a $I\!R^A$ manifold S.*

- *One says that Γ is (locally) integrable around $\xi \in S$ if there exists a local coordinate system (U, ϕ) around ξ with local coordinate functions $\{x_i, i \in I\!N^*\}$ such that $\Gamma|_U = span\,\{dx_1, \ldots, dx_r\}\,|_U$.*
- *Let $\tilde{\Gamma} \subset \Lambda^2(S) = \{\omega \in \Lambda^2(S) \mid \omega = \sum_{i=1}^{s} \theta_i \wedge \eta_i, \theta_i \in \Lambda_1(S), \eta_i \in \Gamma\}$. One says that Γ is involutive if $d(\Gamma) \subset \tilde{\Gamma})$, or equivalently, $d(\omega_i) = \sum_{j=1}^{k} \theta_j^i \wedge \omega_j$, $\theta_j^i \in \Lambda^1(S)$, $i, j \in \lfloor k \rceil$.*

The next theorem states the Cartan local integrability criterium for $I\!R^A$ manifolds.

{t3}

Theorem 3 *A codistribution $\Gamma = span\,\{\omega_1, \ldots, \omega_k\}$ on S is locally integrable around ξ if and only if ξ is a regular point of Γ and Γ is locally involutive around ξ.*

Proof. See Appendix J. □

5 Cartan fields and Diffieties

An ordinary diffiety is an $I\!R^A$-manifold for which there exists a field $\frac{d}{dt}$, called Cartan field.

5.1 Lie Bäcklund maps between diffieties

Let S_1 and S_2 be two diffieties with Cartan fields respectively given by ∂_1 and ∂_2. A smooth map $\phi : S_1 \to S_2$ is Lie-Bäcklund if $\phi_* \partial_1 = \partial_2 \circ \phi$.

Let ϕ be a Lie-Bäcklund immersion. In adequate local coordinates z_1 for S_1 and $(\tilde{z}_1, \tilde{z}_2)$ for S_2, one has $z_1 \mapsto (z_1, 0)$, that is, $\tilde{z}_1 \circ \phi = z_1$ and $\tilde{z}_2 \circ \phi = 0$. Then $\phi_* \partial_1(\tilde{z}_1) = \partial_1(\tilde{z}_1 \circ \phi) = \partial_1(z_1) = \partial_2(\tilde{z}_2) \circ \phi$. Furthermore,

$$\begin{aligned} 0 &= \partial_1(\tilde{z}_2 \circ \phi) \\ &= \phi_* \partial_1(\tilde{z}_2) \\ &= \partial_2(\tilde{z}_2) \circ \phi. \end{aligned}$$

This means that, one has $\partial_2(\tilde{z}_2)|_{\mathrm{im}\phi} = 0$. The following result may be easily proved from the remarks above.

{p19}

Proposition 30 *Let $\phi : S_1 \to S_2$ be an immersion between two diffieties S_1 and S_2. Let z_1 and $(\tilde{z}_1, \tilde{z}_2)$ be suitable local coordinats respectively of S_1 and S_2 such that the local expression of ϕ reads $z_1 \mapsto (z_1, 0)$. Then ϕ is a Lie-Backlund immersion if and only if $\partial_2(\tilde{z}_2) = 0$ and $\partial_2(\tilde{z}_1) \circ \phi = \partial_1(z_1)$. Let $\partial_1 = \sum_{i=1}^{\infty} \alpha_i(z_1) \frac{\partial}{\partial z_{1_i}}$. Then,*

$$\partial_2|_{(z_1,z_2)} = \sum_{i=1}^{\infty} \beta_i(z_1, z_2) \frac{\partial}{\partial \tilde{z}_{1_i}} + \sum_{j=1}^{\infty} \gamma_i(z_1, z_2) \frac{\partial}{\partial \tilde{z}_{2_i}},$$

where $\beta_i(z_1, 0) = \alpha_i(z_1)$ and $\gamma_j(z_1, 0) = 0$. In particular,

$$\partial_2|_{(z_1,0)} = \sum_{i=1}^{\infty} \alpha_i(z_1) \frac{\partial}{\partial \tilde{z}_{1_i}}.$$

Roughly speaking, the proposition above says that the Cartan field ∂_2 of S_2, when restricted to the immersed manifold $\Delta = \mathrm{im}\phi$, may be identified with ∂_1. This may be understood in a more intrinsic way, by noting that ϕ_* is an injective linear map at every point ξ of S_1. Hence $\phi_*(\xi)$ is an isomorphism onto its image. So, as $\phi_* \partial_1 = \partial_2 \circ \phi$, one may identify ∂_1 at $\xi \in S_1$ with ∂_2 at $\phi(\xi)$ via this isomorphism.

In a similar way one may prove that.

{p20}

Proposition 31 *Let $\phi : S_1 \to S_2$ be an submersion between two diffieties S_1 and S_2. Let (z_1, z_2) and $\tilde{z}_2$ be suitable local coordinates, respectively of S_1 and S_2 such that the local expression of ϕ reads $(z_1, z_2) \mapsto z_2$. Then ϕ is a Lie-Backlund submersion if and only if $\partial_2(\tilde{z}_2) \circ \phi = \partial_1(z_2)$. In particular, if $\partial_2 = \sum_{j=1}^{\infty} \beta_j(\tilde{z}_2) \frac{\partial}{\partial \tilde{z}_{2_j}}$ then $\partial_1 = \sum_{i=1}^{\infty} \alpha_i(z_1, z_2) \frac{\partial}{\partial z_{1_i}} + \sum_{j=1}^{\infty} \beta_j(z_2) \frac{\partial}{\partial z_{2_j}}$, where the smooth functions $\alpha_i, i \in \mathbb{N}^*$ are arbitrary.*

5.2 Time notion and systems

A system is a diffiety S with Cartan field $\frac{d}{dt}$ for which one can define a *global notion of time*. In other words, for each point ξ of S, one may associate the time $\tau(\xi)$, where $\tau : S \to \mathbb{R}$ is a smooth function. For each $t \in \tau(S)$ the fiber $\tau^{-1}(t)$ corresponds to the set of all points that exists at time t. Furthermore, the derivative of time is identically equal to one, that is $\frac{d}{dt}(\tau)|_\xi = 1$ for every $\xi \in S$.

The next definition is an intrinsic definition of a system (see (Fliess, Lévine, Martin & Rouchon 1997, Fliess, Lévine, Martin & Rouchon 1999)).

{d26a}

Definition 33 *The field $\mathbb{R}$ of real numbers can be regarded as an ordinary diffiety with the Cartan field $\frac{\partial}{\partial s}$ defined by the standard operation of diferentiation $\frac{\partial}{\partial s}|_t(\phi) = \frac{\partial \phi}{\partial s}(t)$. A system S is a triple $(S, \tau, \mathbb{R})$, where S is an ordinary diffiety with Cartan field $\frac{d}{dt}$ and $\tau : S \to \mathbb{R}$ is a Lie-Bäcklund submersion. The function τ is called* time notion.

Now, as $\tau : S \to I\!R$ is a submersion, by Lemma 2, there exists local coordinates functions of the form $\{t, z\}$, where $t = \tau$ and the function τ, when written in these coordinates reads $(t, z) \mapsto t$. Let $\frac{d}{dt} = \alpha_0 \frac{\partial}{\partial t} + \sum_{i=1}^{\infty} \alpha_i \frac{\partial}{\partial z_i}$. Let $\mathcal{R} = \tau(S)$. Note that τ is an open map, and so $\mathcal{R}$ is an open subset of $I\!R$. As $\tau = t$ is Lie-Bäcklund, this means that for every function $\phi : \mathcal{R} \to I\!R$ one has $\frac{\partial}{\partial s}\phi(s) = \tau_*(\frac{d}{dt})(\phi) = \frac{d}{dt}(\phi \circ \tau)$. Letting ϕ be the identity map on the last equation, one obtains $\alpha_0 = \frac{d}{dt}(\tau) = 1$. The remarks above can be stated as an alternate definition of a system. It is easy to note that both definitions are equivalent.

Definition 34 *A system S is an ordinary diffiety with Cartan field $\frac{d}{dt}$ and a function $\tau : S \to I\!R$ called time notion, such that:* {d27}

- *Around each point $\xi \in S$, there exists local coordinates (t, z) such that $\tau(t, z) = t$.*
- *In these coordinates one may write $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{\infty} \alpha_i(t, z) \frac{\partial}{\partial z_i}$. In other words $\frac{d}{dt}(t) = 1$.*

Note that a system $(S, \tau, I\!R)$ is a fiber-bundle. Remember that S is a trivial fiber bundle if $S = I\!R \times S_1$ and τ is the projection in the first factor. Assume that S_1 is a diffiety with Cartan field $\frac{d}{dt}_1$. Let $\pi_2 : S \to S_1$ be the projection on the second factor. Assume that

$$\frac{d}{dt}_1 \circ \pi_2 = (\pi_2)_* \frac{d}{dt} \qquad (28)$$ {eTimeInvariant}

Then the Cartan field is said to be time-invariant. In this case it is easy to see that, for convenient coordinates z of S_1 and (t, z) of S, one gets $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{\infty} \alpha_i(z) \frac{\partial}{\partial z_i}$.

Note that, in general, the fiber bundle S is not necessarily trivial. When S is trivial, and (28) hold, then the system is called *time-invariant*. This means that the notion of time is exogenous to the system, and furthermore the Cartan field does not depend on time. For a time-invariant system every fiber $\tau^{-1}(t)$ does exist for every time t. Furthermore, the solution of the field $\frac{\partial}{\partial t}$, that is, the time-translation, has a canonical meaning.

5.3 Lie Bäcklund maps between systems, immersed systems and subsystems

Let $(S_1, \tau_1, I\!R)$ and $(S_2, \tau_2, I\!R)$ be two systems. A Lie-Bäcklund map $\phi : S_1 \to S_2$ is a Lie-Bäcklund map such that $\tau_1 = \tau_2 \circ \phi$. In other words, the notions of time of both systems is compatible with the map ϕ. One says that S_1 is immersed on S_2 if ϕ is a Lie-Bäcklund immersion.

Proposition 32 *The map $\phi : S_1 \to S_2$ is a Lie-Bäcklund immersion between systems S_1 and S_2 if and only if:* {p29}

- *Around every point $s_1 \in S_1$, and around $\phi(s_1) \in S_2$), there exists local coordinates (t, z_1), respectively of S_1 and (t, z_1, z_2) of*[29] *S_2, such that the map ϕ reads $(t, z_1) \mapsto (t, z_1, 0)$.*

- *If τ_1 and τ_2 are the time notions respectively of S_1 and S_2, then in these coordinates one may write $t = \tau_1(t, z_1)$ and $t = \tau_2(t, z_1, z_2)$.*

- *In these coordinates, the Cartan field ∂_1 of S_1 is given by $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{\infty} \alpha_i(t, z_1) \frac{\partial}{\partial z_{1_i}}$, whereas the Cartan field ∂_2 of S_2 reads*

$$\partial_2|_{(z_1,z_2)} = \sum_{i=1}^{\infty} \beta_i(t, z_1, z_2) \frac{\partial}{\partial \tilde{z}_{1_i}} + \sum_{j=1}^{\infty} \gamma_i(t, z_1, z_2) \frac{\partial}{\partial \tilde{z}_{2_i}},$$

  where $\beta_i(t, z_1, 0) = \alpha_i(t, z_1)$ and $\gamma_j(t, z_1, 0) = 0$. In particular,

$$\partial_2|_{(t,z_1,0)} = \sum_{i=1}^{\infty} \alpha_i(t, z_1) \frac{\partial}{\partial \tilde{z}_{1_i}}$$

Proof. See Appendix K. □

The following definition is very important in the study of implicit systems.

Definition 35 *(Subsystem) A system S_2 is said to be a subsystem S_1 if and only if there exists a Lie-Bäcklund submersion $\phi : S_1 \to S_2$ between S_1 and S_2. If there exists a Lie-Bäcklund submmersion $\phi : U \subset S_1 \to S_2$, where U is an open subset, then S_2 is said to be a local subsystem of S_1.*

A similar result may be stated for subsystems. The proof of this result is similar to the last one, and is left to the reader.

Proposition 33 *The map $\phi : S_1 \to S_2$ is an Lie-Bäcklund submersion between systems S_1 and S_2 if and only if:*

- *Around every point $s_1 \in S_1$ and $\phi(s_1) \in S_2$), there exists local coordinates (t, z_1, z_2) of S_1 and (t, z_1) of*[30] *S_2 such that the map ϕ reads $(t, z_1, z_2) \mapsto (t, z_1)$.*

- *In these coordinates, one may write $t = \tau_1(t, z_1, z_2)$ and $t = \tau_2(t, z_1)$.*

- *In these coordinates, the Cartan field ∂_1 of S_1 is given by $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{\infty} \beta_j(t, z_1, z_2) \frac{\partial}{\partial z_{2_j}} + \sum_{i=1}^{\infty} \alpha_i(t, z_1) \frac{\partial}{\partial z_{1_i}}$, whereas the Cartan field ∂_2 of S_2 reads $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{\infty} \alpha_i(t, z_1) \frac{\partial}{\partial z_{1_i}}$.*

[29] For convenience, we abuse notation letting (t, z_1) stands for a set of coordinate functions of both S_1 and S_2.

[30] For convenience, we abuse notation letting z_1 stands for a set of coordinate functions of both S_1 and S_2.

References

Abraham, R. & Marsden, J. E. (1988). *Manifolds, tensor analysis and applications*, second edn, Springer-Verlag, New York.

Alekseevskij, D. V., Vinogradov, A. M. & Lychagin, V. V. (1991). *Ideas and concepts from differential geometry*, Vol. 28 of *Encycl. Math. Sci.*, Springer, Berlin. Translated from the Russian.

Anderson, R. L. & Ibragimov, N. H. (1979). *Lie-Bäcklund Transformations in Applications*, SIAM, Philadelphia.

Bernšteĭn, I. N. & Rosenfel′d, B. I. (1973). Homogeneous spaces of infinite-dimensional Lie algebras and the characteristic classes of foliations, *Uspehi Mat. Nauk* **28**(4(172)): 103–138. There exists an english translation.

Borel, E. (1895). Sur quelques points de la théorie des fonctions, *Annales Scientifiques de l'École Normale Supérieure. Troisième Série* **12**: 9–55.

Dieudonneé, J. (1974). *Treatise on Analysis*, Vol. IV, Academic Press, London.

Fliess, M., Lévine, J., Martin, P. & Rouchon, P. (1997). On a new differential geometric setting in nonlinear control, *J. Math. Sci. (New York)* **83**(4): 524–530. Dynamical systems, 3.

Fliess, M., Lévine, J., Martin, P. & Rouchon, P. (1999). A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems, *IEEE Trans. Automat. Control* **44**(5): 922–937.

Golubitsky, M. & Guillemin, V. (1973). *Stable mappings an their singularities*, Springer, New York.

Ibragimov, N. H. (1985). *Transformation groups apllied to mathematical physics*, Reidel, Dordrecht.

Krasil'shchik, I. S., Lychagin, V. V. & Vinogradov, A. M. (1986). *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon and Breach, New York.

Lang, S. (1995). *Differential and Riemmanian manifolds*, Springer-Verlag, New York.

Olver, P. J. (1993). *Applications of Lie groups to differential equations*, 2nd edn, Springer, New York.

Pereira da Silva, P. S. e. a. (2008). An infinite dimensional differential-geometric approach for nonlinear systems : Part II - system theory. . submitted, available in www.lac.usp.br/∼paulo/.

Saunders, D. J. (1989). *The geometry of jet bundles*, Vol. 142 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge.

Strang, G. (1988). *Linear algebra and its applications*, 3rd edn, Brooks Cole.

Tsujishita, T. (1990). Formal geometry of systems of differential equations, *Sugaku Expos.* **3**: 25–73.

Vinogradov, A. M. (1984). Local symmetries and conservation laws, *Acta Appl. Math.* **2**: 21–78.

Warner, F. W. (1971). *Foundations of differentiable manifolds and Lie Groups*, Scott, Foresman and Company, Glenview, Illinois.

Zharinov, V. V. (1992). *Geometrical Aspects of Partial Differentials Equations*, World Scientific, Singapore.

A Uniqueness of the differential in $I\!R^A$

{aA}

It will be shown that the definition of differential stated in definition 5 is independent on the chosen local finite representation of the function. Before showing this fact, consider the following lemma of finite dimensional calculus.

Lemma 4 *Let $k \geq p$ and let $\pi_k^p : I\!R^k \to I\!R^p$ be defined by $\pi_k^p(x_1, \ldots, x_k) = (x_1, \ldots x_p)$. Let $U \subset I\!R^k$ and $V \subset I\!R^p$ be two open sets such that $\pi_k^p(U) = V$. Let $\tilde{f}_1 : U \subset I\!R^k \to I\!R^s$ and $\tilde{f}_2 : V \subset I\!R^p \to I\!R^s$ be two smooth functions such that $\tilde{f}_1 = \tilde{f}_2 \circ \pi_k^p$. Then, for all $y \in U$ and $z \in I\!R^k$, one may write*[31] $d\tilde{f}_1(y)(z) = d\tilde{f}_2|_{\pi_k^p(y)} \circ \pi_k^p(z)$.

The proof of the last lemma is straightforward and is left to the reader. One may denote the statement of the Lemma using the following notation, which means an equality of linear transformations for every fixed y:

$$d\tilde{f}_1(y) = d\tilde{f}_2(\pi_k^p(y)) \circ \pi_k^p$$

Now, assume that a function admits two different local representations $f = \tilde{f}_1 \circ \pi_1 = \tilde{f}_2 \circ \pi_2$, where $\pi_1(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_k)$ and $\pi_1(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_p)$. Without loss of generality, assume that this local representations are defined on the same open set W and $k \geq p$. Then, $\pi_2 = \pi_k^p \circ \pi_1$, where $\pi_k^p(x_1, \ldots, x_k) = (x_1, \ldots x_p)$. Note that $f = \tilde{f}_1 \circ \pi_1 = \tilde{f}_2 \circ \pi_2 = \tilde{f}_2 \circ \pi_k^p \circ \pi_1$. As π_1 is surjective, then $\tilde{f}_1 = \tilde{f}_2 \circ \pi_k^p$. Let $y = \pi_1(x)$. Note now that $d\tilde{f}_2(\pi_2(x)) \circ \pi_2 = d\tilde{f}_2(\pi_k^p \circ \pi_1(x)) \circ (\pi_k^p \circ \pi_1) = [(d\tilde{f}_2(\pi_k^p(y)) \circ \pi_k^p] \circ \pi_1 = d\tilde{f}_1(y) \circ \pi_1 = d\tilde{f}_1(\pi_1(x)) \circ \pi_1$. This shows the claimed uniqueness.

B Proof of proposition 10

{sFunctional}

Choose an arbitrary $k \in I\!N^*$. One may write $I\!R^A \cong I\!R^k \times I\!R^B$, where $B = \{j \in I\!N^* | j > k\}$. Hence, a point of $I\!R^A$ will be denoted by (x, z), where

[31] Here $df_1(y)(z)$ means the linear mapping $z \mapsto [\lim_{t \to 0} \frac{f_1(y+tz) - f_1(y)}{t}]$

$x = (x_1, \ldots, x_k) \in I\!R^k$ and $z \in I\!R^B$. Let $X \subset I\!R^A$ be the k-dimensional subspace $X = \{(x,z) \in I\!R^A \mid z = 0\}$. It is clear that $X \cong I\!R^k$. Hence, as $\gamma : I\!R^A \to I\!R$ is $I\!R$-linear, it follows that $\gamma|_X(x,0) = \sum_{i=0}^k \alpha_i x_i$, where $\alpha_i \in I\!R$. Now fix $\epsilon \in I\!R$, with $\epsilon > 0$. As γ is continuous, then $\mathcal{A} = \gamma^{-1}((-\epsilon,\epsilon))$ is an open subset of $I\!R^A$. Note that $\mathcal{A}$ contains a basic open set containing the origin $(0,0)$. From the definition of the basis of the Fréchet topology (see section 2.1), choosing k big enough, this basic open set must contain every point $\xi \in I\!R^A$ of the form $\xi = (0,z)$, where $z \in I\!R^B$ is arbitrary. Let $Z \subset I\!R^A$ be the subspace defined by $Z = \{\xi = (x,z) \in I\!R^A | x = 0\}$. One will show now that $\gamma(\xi) = 0$ for every $\xi \in Z$. In fact, by linearity, one has $\gamma((0,tz)) = t\gamma((0,z))$ for every $t \in I\!R$. Hence, if $\gamma((0,z)) = a \neq 0$, then one may choose $t = 2\epsilon/|a|$, obtaining $\gamma((0,tz)) = 2\epsilon \notin (-\epsilon,\epsilon)$. This is a contradiction, and shows that $\gamma(\xi) = 0$. Hence

$$\gamma((x,z)) = \gamma((x,0)) + \gamma((0,z)) = \gamma((x,0)) = \sum_{i=1}^k \alpha_i x_i$$

In particular, $\gamma = \sum_{i=}^k \alpha_i dx_i$.

C Proof of Theorem 2

{aFormsAreFinite}

Proof. Remember that a section ω of T^*U must be of the form

$$\omega(x) = \sum_{i=0}^{\infty} \alpha_i(x) dx_i|_x.$$

Now take $\tau = \frac{\partial}{\partial x_i}$. Then $\langle \omega, \tau \rangle = \alpha_i(x)$. In particular, if ω is a one-form, then $\alpha_i(x)$ must be a smooth function, showing 1.

Now if $\omega(x) = \sum_{i=1}^k \alpha_i(x) dx_i|_x$, with α_i being smooth, it is clear that ω is a smooth section, showing the sufficiency of 2.

To show the necessity of 2, assume, as an absurd, that $\omega : U \to T^*U$ is a one-form and, for every $l \in I\!N^*$ and for every open neighborhood $V_\xi \subset U$ of ξ, there exists $\bar{x} \in V_\xi$ and $k \geq l$ such that

$$\omega(x) = \sum_{k=1}^{\infty} \omega_i(x) dx_i|_x, \text{where } \omega_k(\bar{x}) \neq 0. \tag{29}$$ {eAbsurd}

Let l_i and U_ξ^i be respectively the minimal index at ξ, and a minimal neighborhood at ξ of $\omega_i : U \to I\!R, i \in I\!N^*$. Given $k \in I\!N^*$, let $W_\xi^k = U_\xi^1 \bigcap U_\xi^2 \cdots \bigcap U_\xi^k$ and define $l_0^* = 0$ and $l_k^* = \max\{l_1, l_2, \ldots, l_k, l_{k-1}^* + 1, k\}$, for $k = 1,2,3,\ldots$. By construction $l_k^* > l_{k-1}^*$ and $l_k^* \geq k$. Let $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$. define the field τ on U by

$$\tau(x) = \sum_{i=1}^{\infty} (x_{l_i^*+1} - \xi_{l_i^*+1}) \frac{\partial}{\partial x_i}|_x$$

By construction, $\tau_i(x) = (x_{l_i^*+1} - \xi_{l_i^*+1}), i \in I\!N^*$. In particular, $\tau_i(\xi) = 0, i \in I\!N^*$.

Now assume that $V_\xi \subset U$ is a minimal neighborhood of the smooth function $\langle \omega, \tau \rangle : U \to I\!R$. Suppose also that l is the minimal index the function $\langle \omega, \tau \rangle$ at ξ. Let Z be a basic open set such that $Z \subset V_\xi \bigcap W_\xi^k$. By absurd, there exists $\bar{x} \in Z$ and $k \geq l$ such that (29) holds.

Now note that, inside Z, $\omega_i, i = 1, \ldots, k$ does not depend on x_j for $j > l_k^*$. Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots)$ and let $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$. Define the map $\Psi : I\!R \to I\!R^A$ by

$$\Psi(t) = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{l_k^*}, \xi_{l_k^*+1} + t, \xi_{l_k^*+2}, \xi_{l_k^*+3}, \ldots)$$

As Z is a basic open set, it is easy to show that $\Psi(0) \in Z$. As Ψ is continuous, then $\Psi(t) \in Z$ for $|t| < \epsilon$, for ϵ small enough. By construction it is easy to see that

$$\tau(\Psi(t)) = \sum_{i=1}^{k-1} (\bar{x}_{l_i^*+1} - \xi_{l_i^*+1}) \frac{\partial}{\partial x_i} + t \frac{\partial}{\partial x_k}$$

Hence

$$\langle \omega, \tau \rangle (\Psi(t)) = \sum_{i=1}^{k-1} (\bar{x}_{l_i^*+1} - \xi_{l_i^*+1}) \omega_i(\Psi(t)) + t\omega_k(\psi(t))$$

By construction, ω_i depend only on the first l_i coordinates and $l_k^* \geq l_i, i = 1, \ldots k$ on Z. It follows that $\omega_i(\Psi(t)) = \omega_i(\bar{x}), i = 1, \ldots k$. In particular, $h = \omega_k(\Psi(t)) \neq 0$. One concludes that

$$\langle \omega, \tau \rangle (\Psi(t)) - \langle \omega, \tau \rangle (\Psi(0)) = th$$

By Proposition 6, it follows that the minimal index of $\langle \omega, \tau \rangle$ is greater than $l_k^* + 1 > k \geq l$. This is an absurd.

One has already shown that, for every $\xi \in U$ there exists $k \in I\!N^*$ and an open neighborhood V_ξ of ξ such that $\omega(x) = \sum_{i=1}^k \omega_i(x) dx_i$ for $x \in V_\xi$. Without loss of generality, assume that V_ξ is a minimal neighborhood of ω_i at ξ, and the minimal index of ω_i is $l_i^*, i = 1, \ldots k$. Let $k^* = \max\{k, l_1^*, l_2^*, \ldots, l_k^*\}$. It is clear that $\omega_i|_{V_\xi} = \tilde{\omega}_i \circ \pi_{k^*}|_{V_\xi}$ for convenient smooth functions $\tilde{\omega}_i$. In particular, on V_x one may write $\omega(x) = \sum_{i=1}^k (\tilde{\omega}_i \circ \pi_{k^*}) dx_i$. Hence $\omega|_{V_\xi} = (\pi_{k^*})^* \tilde{\omega}$, for a convenient $\tilde{\omega}$.

□

D Proof of Proposition 9

{aLieWell}

It is easy to see from (7a) that $[\tau, \theta](f_1 + f_2) = [\tau, \theta](f_1) + [\tau, \theta](f_2)$. Now,

$$\begin{aligned} [\tau, \theta](f_1 f_2) &= L_\tau(L_\theta(f_1 f_2)) - L_\theta(L_\tau(f_1 f_2)) \\ &= L_\tau[f_2 L_\theta(f_1) + f_1 L_\theta(f_2)] - L_\theta[f_2 L_\tau(f_1) + f_1 L_\tau(f_2)] \\ &= \{f_1 L_\tau(L_\theta(f_2)) + (L_\theta f_2)(L_\tau f_1) + \end{aligned}$$

$$
\begin{aligned}
& f_2 L_\tau(L_\theta(f_1)) + (L_\theta f_1)(L_\tau f_2)\} - \\
& \{f_1 L_\theta(L_\tau(f_2)) + (L_\tau f_2)(L_\theta f_1) + \\
& f_2 L_\theta(L_\tau(f_1)) + (L_\tau f_1)(L_\theta f_2)\} \\
= \quad & f_1[\tau,\theta](f_2) + f_2[\tau,\theta](f_1).
\end{aligned}
$$

In particular, $[\tau,\theta](x)$ is a tangent vector for all $x \in \mathbb{R}^A$. To show that $[\tau,\theta]$ is a field (that is, a smooth section of $T\mathbb{R}^A$), one may compute the expression $[\tau,\theta]$ in coordinates by taking $f = x_i$, obtaining $[\tau,\theta]_i = \alpha_i = \tau(\eta(x_i)) - \eta(\tau(x_i))$. In particular, $[\tau,\theta](x) = \sum_{i=1}^{\infty} \alpha_i(x) \frac{\partial}{\partial x_i}|_x$, where α_i is a smooth function.

E Proof of Proposition 17

{aChainRule}

Let $v_x \in T_x X$. Let $y = g(x)$ and $z = h(y)$. Let $\lambda : V_z \to \mathbb{R}$ be a smooth function, where V_z is an open neighborhood of z. By definition $(h \circ g)_*(v_x)(\lambda) = v_x(\lambda \circ h \circ g) = (g_*(x)v_x)(\lambda \circ h)$. Since $\tau_y = g_*(x)(v_x) \in T_y Y$, then $\tau_y(\lambda \circ h) = h_*(y)(\tau_y)(\lambda) = (h_*(g(x)) \circ g_*(x)(v_x))(\lambda)$. This shows the chain rule. Now, let X be a $\mathbb{R}^A$-manifold and let $\Psi : X \to X$ be the identity map. Then, by definition, it is clear that $\Psi_*(x)$ is the identity map on the linear space $T_x X$. If $g = h^{-1}$, then $\Psi = h \circ g$ is the identity map on X, and so $h_*(g(x))$ is the left-inverse of $g_*(x)$. As $\Phi = g \circ h$ is the identity map on Y, one concludes easily that $h_*(g(x))$ is also the right-inverse of $g_*(x)$. This shows 2.

The proof of 3 is identical to the proof of 2. By section 2.3, one may induce a topology on $T_x X$. It is easy to show that the topology induced on $T_x X$ does not depend on the chosen local chart ϕ. In fact, if ψ is another chart defined around x, one may write $\phi_*(x) = (\phi \circ \psi^{-1})_*|_{\psi(x)} \psi_*(x)$. Now, from 2, it follows that $(\phi \circ \psi^{-1})_*|_{\psi(x)}$ is an isomorphism, and so the uniqueness of the induced topology follows.

F Smoothness of vector fields trasformed by diffeomorphisms

{aTransfer}

The following results are instrumental for the definition of fields on $\mathbb{R}^A$-manifolds. They are useful in order to show that the notion of smoothness that is introduced in the Def. 24 does not depend on the chosen local chart.

{pTransfer}

Proposition 34 *(Diffeomorphisms induce a transformation of fields) If $\phi : U \subset \mathbb{R}^A \to V \subset \mathbb{R}^B$ is a diffeomorphism of open sets, then given a field $\tau : U \to TU$, the map $\tilde{\tau} : V \to TV$ is a field, where $\tilde{\tau}(y) = \phi_*(x)\tau(x)$, with $x = \phi^{-1}(y)$).*

Proof. Note that $\tilde{\tau}$ is well defined as a section from V to TV. Let $\{y_j, j \in \mathbb{N}^*\}$ be canonical coordinates of $V \subset \mathbb{R}^A$. To show that $\tilde{\tau}$ is a field, it suffices to show that $\tilde{\tau}(y) = \sum_{j=1}^{\infty} \tau_j(y) \frac{\partial}{\partial y_j}$, where $\tau_j : V \to \mathbb{R}$ is a smooth function. Denote the functions $y_j \circ \phi$ by ϕ_j. Now remember that, $\tau_j = \tilde{\tau}(y_j) =$

$\phi_*(x)\tau(x)(y_j)|_{x=\phi^{-1}(y)} = \tau(x)(y_j \circ \phi)|_{x=\phi^{-1}(y)} = \tau(x)(\phi_j)|_{x=\phi^{-1}(y)} = \tau(\phi_j) \circ \phi^{-1}(y)$. As ϕ and τ are smooth, $\tau(\phi) : U \to I\!R$ is smooth, and so is $\tau(\phi_j) \circ \phi^{-1}$. $\square$

{p10}

Proposition 35 *Let (U, ϕ_1) and (U, ϕ_2) be local charts of the $I\!R^A$-manifold S. Let $\tau : S \to TS$ be a section of TS. Take some $x \in U$ and let $y_i = \phi_i(x)$ and $V_i = \phi_i(U) \subset I\!R^A$, $i = 1, 2$.*

1. *The map $\tilde{\tau}_i : V_i \to TV_i$ defined by $\tilde{\tau}_i(y_i) = (\phi_i)_*(x)\tau(x)$, where $x = \phi^{-1}(y_i)$ is a section of TV_i for $i = 1, 2$.*
2. *The section $\tilde{\tau}_1$ is smooth (that is, $\tilde{\tau}_1$ is a field) if and only if $\tilde{\tau}_2$ is smooth.*

Proof. The fact that $\tilde{\tau}_i$ is a section of TV_i is a consequence of the fact that $(\phi_i)_*$ maps a a tangent vector of T_xS onto a tangent vector of $T_\phi(x)V$. To show 2, note that $\tilde{\tau}_2(y_2) = (\phi_2)_*(x)\tau(x) = (\phi_2 \circ \phi_1^{-1} \circ \phi_1)_*(x)\tau(x) = (\phi_2 \circ \phi_1^{-1})_*(y_1)(\phi_1)_*(x)\tau(x) = (\phi_2 \circ \phi_1^{-1})_*(y_1)\tilde{\tau}_1(y_1)$, with $y_1 = \phi_1 \circ \phi_2^{-1}(y_2)$. By Prop. 34, as $\phi_1 \circ \phi_2^{-1} : U_2 \to U_1$ is a local diffeomorphism, then if τ_1 is smooth, it follows that τ_2 is also smooth. By similar arguments, if τ_2 is smooth, then τ_1 is smooth. $\square$

G Proof of Proposition 22

{a13}

Let τ_x be a tangent vector in T_xX, and let $\theta_{\tilde{x}} \in T^*_{\tilde{x}}\tilde{U}$. Remember that $\phi^*(x)(\theta_{\tilde{x}})(\tau_x) = \theta_{\tilde{x}}(\phi_*(x)\tau_x)$. As ϕ_* is an isomorphism, it is clear that $\phi^*(x)(\theta_{\tilde{x}})(\tau_x) = 0$ for all $\tau_x \in T_xX$ if and only if $\theta_{\tilde{x}} = 0$. Hence $\ker \phi^*(x) = \{0\}$. Now, given a continuous linear function $h : T_xX \to I\!R$, as $\phi_*(x)$ is continuous with continuous inverse, then $\tilde{h} = h \circ (\phi_*)^{-1} \in T_{\tilde{x}}\tilde{U}$, and $\phi^*(x)\tilde{h} = h$. This shows 1. Now, to show 2, remember that a smooth vector field on U is of the form $\tau(x) = \phi_*^{-1}(\phi(x))(\tilde{\tau}(\phi(x))$, where $\tilde{\tau} : \tilde{U} \to T\tilde{U}$ is a field. Now let $\theta = \phi^*\tilde{\theta}$ be a one-form on U. Then $\langle\theta(x), \tau\rangle(x) = \langle\theta(x), \tau(x)\rangle = \langle\phi^*(x)\tilde{\theta}(\phi(x)), \phi_*^{-1}(\phi(x))(\tilde{\tau}(\phi(x))\rangle = \langle\tilde{\theta}(x), \phi_*(x)\phi_*^{-1}(\phi(x))(\tilde{\tau}) \circ \phi\rangle = \langle\tilde{\theta}(\phi(x)), \tilde{\tau}(\phi(x))\rangle = \langle\tilde{\theta}, \tilde{\tau}\rangle \circ \phi(x)$. In particular, this function is smooth if and only if the function $\langle\tilde{\theta}, \tilde{\tau}\rangle$ is smooth on $\phi(U)$. Hence, by Def. 14, $\tilde{\theta}$ must be a one form on $\phi(U)$. Now given a one-form ω on Z and a field τ on X, then $\langle(f \circ g)^*\omega, \tau\rangle = \langle\omega, (f \circ g)_*\tau\rangle = \langle\omega, f_*(g_*\tau)\rangle = \langle f^*\omega, g_*\tau\rangle = \langle g^*f^*\omega, \tau\rangle$. To show (4), note first that $g^*\omega$ is a section of TX^*. Hence, it suffices to show that $\langle g^*\omega, \tau\rangle$ is smooth for every field $\tau : Y \to TY$. For this, let (V, Ψ) be a local chart around $y = g(x)$. From Part 3 of Proposition 18, one locally has $\tau = \Psi_*^{-1}(\tilde{x})\tilde{\tau}(\tilde{x})$, where $\tilde{x} = \psi(x)$ and $\tilde{\tau}(\tilde{x}) = \sum_{j=1}^{\infty} \tilde{\tau}_j(\tilde{x})\frac{\partial}{\partial \tilde{x}_j}|_{\tilde{x}}$. From Part 1 of Prop. 17 and Prop. 20, one may write

$$\begin{aligned}
\phi_*(g(x))g_*(x)\tau(x) &= \phi_* g_* \psi_*^{-1}(\tilde{x})\tilde{\tau}(\tilde{x}) \\
&= (\phi \circ g \circ \psi^{-1})_*\tilde{\tau}(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{k^*_{\tilde{g}_i}} \tilde{\tau}_j(\tilde{x})\frac{\partial \tilde{g}_i}{\partial \tilde{x}_j}|_{\tilde{x}}\frac{\partial}{\partial \tilde{y}_i}|_y.
\end{aligned}$$

Now let $\tilde{\omega}(\tilde{y}) = \sum_{i=1}^{k} \alpha_i(\tilde{y}) d\tilde{y}_i$ be such that $\omega = \phi^* \tilde{\omega}$. Then

$$\begin{aligned}
\langle g^*\omega, \tau \rangle &= \langle \omega, g_* \tau \rangle \\
&= \langle \phi^* \tilde{\omega}, g_* \tau \rangle \\
&= \langle \tilde{\omega}, \phi_* g_* \psi_*^{-1} \tilde{\tau} \rangle \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k_{\tilde{g}_i}^*} \alpha_i(\tilde{y}) \tilde{\tau}_j(\tilde{x}) \frac{\partial \tilde{g}_i}{\partial \tilde{x}_j}|_{\tilde{x}}
\end{aligned}$$

where $\tilde{y} = \Psi \circ g \circ \phi^{-1}(\tilde{x})$. From the smoothness of the last equation, the result follows directly by the third part 3 of Def. 25.

H Proof of Proposition 25

{aComputational}

By definition 26, $\langle dx_i|_x, \frac{\partial}{\partial x_j}|_x \rangle = \frac{\partial}{\partial x_j}|_x(x_j)$. By Prop. 18, it follows that $\langle dx_i|_x, \frac{\partial}{\partial x_j}|_x \rangle = \delta_{ij}$. Now, as $x_i = y_i \circ \phi$ and $x_i = z_i \circ \pi_k^\phi$, then 2 and 3 follows from Proposition 24. Now, by proposition 23, on V_ξ one may write $\omega = (\pi_k^\phi)^* \tilde{\omega}$, where $\tilde{\omega}$ is a one-form on V, where $V \subset I\!R^k$ is the open subset of $I\!R^k$ given by $\pi_k^\phi(V_\xi)$. Since $\tilde{\omega}(z) = \sum_{i=1}^k \tilde{\omega}_i(z) dz_i|_z$, then by 3 and Prop. 24, one has $\omega(x) = \sum_{i=1}^k \omega_i(x) dx_i|_x$, where $\omega_i = \tilde{\omega}_i \circ \pi_k^\phi$. By 1, it follows easily that $\omega_i = \langle \omega, \frac{\partial}{\partial x_i}|_x$. Since $(\phi^{-1} \circ \phi)^*$ is the identity map, note from part 3 of Prop. 22, that

$$(\phi^{-1})^* = (\phi^*)^{-1}. \tag{30}$$ {ePhiEst}

Note that $(\phi^{-1})^*\omega = \sum_{i=1}^k \omega_i \circ \phi^{-1} (\phi^{-1})^* \omega dx_i|_x$. By 2, and (30), one may write$(\phi^{-1})^* dx_i|_x = (\phi^{-1})^* \phi^* dy_i|_y = dy_i|_y$. So $(\phi^{-1})^*\omega = \sum_{i=1}^k \alpha_i(y) dy_i|_y$. Hence $\omega(x) = \phi^* \sum_{i=1}^k \alpha_i(y) dy_i|_y$. This shows 4. Note that $\langle df(x), \frac{\partial}{\partial x_i}|_x \rangle = \frac{\partial}{\partial x_i}|_x(f) = (\phi^{-1})_*(\phi(x)) \frac{\partial}{\partial x_i}|_{\phi(x)}(f) = \frac{\partial}{\partial y_i}|_{\phi(x)}(f \circ \phi^{-1}) = \frac{\partial \tilde{f}}{\partial y_i}|_{\phi(x)}$.

I Proof of Lemma 1

{aLemma2}

Let (ϕ, W) be a coordinate system around ξ, where $\phi = \{x_i : i \in A\}$. Without loss of generality, one may assume that $W = V$, otherwise we may restrict V to $V \cap W$. Let $\tilde{V}$ be the open set $\tilde{V} = \phi(V) \subset I\!R^A$. Up to a restriction to the open set $\tilde{V}$, one may write $\tilde{\theta}_i = \theta_i \circ \phi^{-1} = \tilde{\theta}_i(x_1, \ldots, x_{n_i}), i = 1, \ldots, k$, where n_i is the maximal index of θ_i at ξ. Let $n = \max\{n_1, \ldots, n_k\}$ and denote $x = (x_1, \ldots, x_n)$ and $\hat{x} = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots)$. Consider the projection $\pi : \tilde{V} \to V_1 \subset I\!R^n$ defined by $\pi(x, \hat{x}) = x$. The set $\tilde{\theta} = \{\tilde{\theta}_1, \ldots, \tilde{\theta}_k\}$ may be regarded as a set of functions defined on $V_1 \subset I\!R^n$. Note that the independence of $d\theta$ on V implies the independence of $d\tilde{\theta}$ on V_1. So, a convenient application of the (finite dimensional) inverse function theorem, shows that one may choose a set of coordinates $\tilde{x} = \{x_{i_1}, \ldots, x_{i_{n-k}}\}$ and an open neighborhood $U_1 \subset V_1 \subset I\!R^n$ of $\pi \circ \phi(\xi)$, such that the map $H : U_1 \to U_2 \subset I\!R^n$ defined by $H(x) =$

$(\tilde{\theta}(x), \tilde{x})$ is a difeomorphism. Let $\tilde{U} = \pi^{-1}(U_1)$. Then the map $F : \tilde{U} \mapsto \tilde{U}_1$ defined by $F(x, \hat{x}) = (H(x), \hat{x})$ is a difeomorphism with inverse $F^{-1}(\tilde{\theta}, \tilde{x}, \widehat{x}) = (H^{-1}(\tilde{\theta}, \tilde{x}), \widehat{x})$. Let $U = \phi^{-1}(\tilde{U})$. Then, (ψ, U) is a coordinate system of S around ξ, where $\psi = \{\theta, \tilde{x}, \hat{x}\}$, with $\psi = F \circ \phi$. Let $\tilde{U}_1 = \psi(U)$, and define the open surjective map $\pi_1 : \tilde{U}_1 \to \widehat{V} \subset I\!R^k$ by $\pi_1(\theta, \tilde{x}, \hat{x}) = \theta$. Let $\tilde{u} = u \circ \psi^{-1}$. As $du \in \text{span}\,\{d\theta\}$, then $\tilde{u}(\theta, \tilde{x}, \hat{x}) = \mu(\theta)$. Hence, one may regard μ as a function defined in $\widehat{V}$ such that $\mu \circ \pi_1 = \tilde{u}$. Hence, $\mu \circ \pi_1 \circ \psi = u$. One may take $\delta = \pi_1 \circ \psi$, and δ and μ, constructed in this way, have the desired properties. This shows 1 and 2.

To show 3, let $\phi = (\theta, w)$. By part 2, one may construct $\delta : V_\xi \to \hat{V}$ and $\mu : \hat{V} \to W$ such that η is locally given by $\mu(\theta)$. By dimensional arguments, the set $\{d\eta\}$ is pointwise independent. By parts 1 and 2, one may construct a local coordinate system $\phi_1 = (\eta, z)$, and maps $\delta_1 : V_\xi^1 \to \hat{V}_1$ and $\mu_1 : \hat{V}_1 \to W_1$ such that θ is locally given by $\mu_1(\eta)$. Without loss of generality, the two local coordinate systems ϕ and ϕ_1 are defined on the same open neighborhood of V_ξ of ξ. In particular the map μ is the inverse of μ_1, and so μ is a local diffeomorphism. So the map h such that $(\theta, w) \mapsto (\mu(\theta), w)$ is a local diffeomorphism. In particular, the map $h \circ \phi = (\eta, w)$ is a local cooordinate system.

J Proof of Theorem 3

{aT3}

The following result is useful for the Proof of Theorem 3.

{lObs}

Lemma 5 *Assume that $g : X \to Y$ is a smooth map between $I\!R^A$-manifolds such that $g(x)$ is surjective, and $g_*(x)$ is a surjective linear map for every $x \in X$. Let ω_1 and ω_2 be two k-forms on Y. If $g^*\omega_1 = g^*\omega_2$, then $\omega_1 = \omega_2$.*

Proof. As $y = g(x) \in Y$ is arbitrary, it suffices to note that, given any one $\theta_i \in T_yY$, can choose $\tau_i \in T_xX$ in a way that $\theta_i = g_*(x)\tau_i$. Hence, $\omega_1(g_*\tau_1, \ldots, g_*\tau_k) = \omega_2(g_*\tau_1, \ldots, g_*\tau_k)$ implies that $\omega_1(\theta_1, \ldots, \theta_k) = \omega_2(\theta_1, \ldots, \theta_k)$, with $\theta_1, \ldots, \theta_k$ arbitrary vectors in T_yY, with y arbitrary. □

Proof. **(Of theorem 3.)** If Γ is integrable around ξ, our definition implies that ξ is a regular point of Γ. If Γ is integrable, then $\Gamma|_U = \text{span}\,\{dx_1, \ldots, dx_r\}\,|_U$. In particular, as $d(dx_i) = 0$, then $d(\Gamma|_U) \subset \tilde{\Gamma}$. This shows that the statement of the theorem is a necessary condition of integrability. To show that the statement of the theorem gives sufficient conditions, note that, by part 4 of Proposition 25, one may construct a local chart (U, ϕ), with coordinate functions $(x_i : i \in I\!N^*)$, such that, around any $\xi \in S$, for k big enough, $\omega_i = (\pi_k^\phi)^*\tilde{\omega}_i$, where $\tilde{\omega}_i$ are one-forms on the open set $V = \pi_k^\phi(U)$ of $I\!R^k$. Now, from the assumptions of Theorem 3, one will show from the finite dimensional Frobenius theorem that the codistribution $\tilde{\Gamma} = \text{span}\,\{\tilde{\omega}_1, \ldots, \tilde{\omega}_k\}$ is locally integrable around $\tilde{x} \in V$. In fact, by Part 5 of Prop. 29, it follows that

$$d\omega_i = (\pi_k^\phi)^* d\tilde{\omega}_i$$

and $d\omega = \sum_{j=1}^{p_i} \theta_j \wedge \omega_j$. From Part 2 of Prop. 29, one has

$$(\pi_k^\phi)^* d\tilde{\omega}_i = \sum_{j=1}^{p_i} (\pi_k^\phi)^* \left\{ \tilde{\theta}_j^i \wedge \tilde{\omega}_j \right\},$$

As π_k^ϕ is surjective and $(\pi_k^\phi)_*$ is pointwise surjective, by Lemma 5, the assumptions of the finite dimensional Frobenius theorem holds. In particular, there exists a local diffeomorphism θ given by $(\tilde{x}_1, \ldots, \tilde{x}_k) \mapsto (\tilde{z}_1, \ldots, \tilde{z}_k)$, defined around $\tilde{x}$, such that $\tilde{\Gamma} = \operatorname{span}\{d\tilde{z}_1, \ldots, d\tilde{z}_r\}$. Assume that the coordinate functions of the local chart ϕ are given by $(x, \hat{x})$, where $x = (x_1, \ldots, x_k)$, and $\hat{x} = (x_{k+1}, x_{k+2}, x_{k+3}, \ldots)$. Note that the map $(x, \hat{x}) \mapsto (z, \hat{x})$ is a local diffeomorphism. In particular, $\psi = (z, \hat{x})$ is a local chart of S around ξ. It is easy to see that $z_i = \tilde{z}_i \circ \pi_k \circ \psi = \tilde{z}_i \circ \pi_k^\psi$, where $\tilde{z}_i$ is the i-th component of the map θ.

It follows that[32] $\Gamma = \operatorname{span}\left\{(\pi_k^\psi)^* dz_1, \ldots, (\pi_k^\psi)^* dz_r\right\}$. Then

$$\Gamma = \operatorname{span}\left\{(\pi_k^\psi)^* d\tilde{z}_1, \ldots, (\pi_k^\psi)^* d\tilde{z}_r\right\} = \operatorname{span}\left\{(\pi_k \circ \psi)^* dz_1, \ldots, (\pi_k \circ \psi)^* d\tilde{z}_r\right\}.$$

By Proposition 24, one has

$$\Gamma = \operatorname{span}\left\{d(\tilde{z}_1 \circ \pi_k \circ \psi), \ldots, d(\tilde{z}_r \circ \pi_k \circ \psi)\right\} = \operatorname{span}\left\{dz_1, \ldots, dz_k\right\},$$

showing our claim. □

K Proof of Proposition 32

{a29}

(if). Assume that ϕ a Lie-Bäcklund immersion between two systems. Note that, from Definition 34, there exists local coordinates (t, z_1) of S_1 and $(\tilde{t}, y_2)$ of S_2 such that the map τ_1 reads $(t, z_1) \mapsto t$ and the map τ_2 reads $(\tilde{t}, y_2) \mapsto \tilde{t}$. Writing ϕ in these coordinates one gets $(t, z_1) \to ((\tilde{t}, z_1), y_2(t, z_1))$. As $\tau_1 = \tau_2 \circ \phi$, then $t = \tilde{t}$. So the map ϕ in these coordinates reads $(t, z_1) \mapsto (t, y_2(t, z_1))$. So one may abuse notation, taking $\tilde{t} = t$. Now, as ϕ is a immersion, there exists local coordinates w_1 of S_1 and $(\tilde{w}_1, z_2)$ of S_2 such that ϕ reads $w_1 \mapsto (w_1, 0)$. Abusing notation, one may let $\tilde{w}_1$ stand for w_1. After a convenient restriction, the coordinate change map $(t, z_1) \mapsto w_1(t, z_1)$ is a local diffeomorphism. So the map $(t, z_1, z_2) \mapsto (w_1(t, z_1), z_2)$ is also a local diffeomorphism. In particular, (t, z_1, z_2) are local coordinates for S_2 with the claimed properties. From the fact that ϕ is Lie-Bäcklund, the second statement follows from Definition 34 and Proposition 30.
(only if). Straightforward from the Definition 34 and from Proposition 30.

[32] Here one uses the fact that, for nonsingular codistributions, the pointwise definition of codistributions coincides with the definition of a codistribution as a submodule, as shown in Part 4 of Proposition 28.