We are concerned with the magnitude of the output $\bar{z}$ of the system

$$\frac{d}{dt}(p - \bar{p}) = -\left(\int \xi x x^\top\right)^{-1} \xi x x^\top (p - \bar{p}) + \dot{p}$$
$$\bar{z} = x^\top (p - \bar{p}).$$

As long as $|\bar{z}|^2 > \frac{1}{3}|z|^2$, $\dot{h}_{i_{\min}}$ remains positive, the minimum value attained by the Hessian is increased, and J stays convex! This needs to happen for all directions v_i, of course.

We are finally in a position to justify the fastidious distinction between p and p_*'s estimate $\hat{p}$. For if $\hat{p}(t)$ were the output of some successful tuner, $\dot{\hat{p}}$ would become small. Thus if we were to set $p(t) = \hat{p}(t)$, the tuner's success would render$\bar{z}$ small and thus possibly undermine convexity — the basis on which its success rests.

References

[1] Felipe M Pait. *On the Design of Direct Adaptive Controllers.* Submitted.

2 The Tuner

The method just described requires that we set the tuning error equal to z, that is to say, z must be known in sign as well as magnitude. If only the latter were known, we might choose as our tuning error

$$e_T = (\hat{p} - p)^\top x x^\top (\hat{p} - p) - z^2. \tag{3}$$

Intuition motivates the definition of e_T above: if the sign of z is unknown, then at each instant we might equally believe that $y = x^\top p + |z|$ or that $y = x^\top p - |z|$. Unable to minimize $x^\top \hat{p} - y$ as before, we content ourselves with trying to keep the product $(x^\top \hat{p} - x^\top p - |z|)(x^\top \hat{p} - x^\top p + |z|) = e_T$ small. Notice that e_T is a function of $\hat{p}$ as well as of time.

Now consider the cost function

$$J(t) = \frac{1}{4}(\hat{p}^\top M \hat{p})^2 + \frac{1}{4}\int_0^t d\tau\, e_T(t,\tau)^2.$$

The $\hat{p}$ that minimizes J must satisfy

$$\frac{\partial J}{\partial \hat{p}}(t) = \hat{p}^\top M \hat{p}\hat{p}^\top M + \int_0^t d\tau\, e_T(\hat{p},\tau) x^\top (\hat{p} - p(\tau)) x^\top = 0.$$

A sufficient condition for this solution to be the unique minimum is that the Hessian

$$\frac{\partial^2 J}{\partial \hat{p}\,\partial \hat{p}^\top}(t) = \hat{p}^\top M \hat{p} M + 2M\hat{p}\hat{p}^\top M + \int_0^t d\tau\, \left(2(x^\top(\tau)(\hat{p} - p(\tau))^2 + e_T\right) x x^\top(\tau) > 0.$$

Given an arbitrary vector v_i, let

$$h_i = v_i^\top M v_i \hat{p}^\top M \hat{p} + 2(v_i^\top M \hat{p})^2 + \int_0^t d\tau\, \xi_i(\tau) \left(2(x^\top(\tau)(\hat{p} - p(\tau))^2 + e_T\right)$$

where $\xi_i = (v_i^\top x)^2$. The minimum of h_i occurs when

$$\frac{\partial h_i}{\partial \hat{p}^\top}(t) = 2v_i^\top M v_i M \hat{p} + 4v_i^\top M \hat{p} M v_i + 6\int_0^t d\tau\, \xi_i(\tau) x x^\top (\hat{p} - p) = 0,$$

namely, when

$$\hat{p} = \left(2v_i^\top M v_i M + 4v_i^\top M M v_i \int_0^t + d\tau\, \xi_i x x^\top\right)^{-1} \int_0^t d\tau\, \xi_i x x^\top p := \bar{p},$$

so

$$h_{i_{\min}} = 3\int \xi_i((x^\top p)^2 - z^2) - 3\left(\int \xi p^\top x x^\top\right)\left(\int \xi x x^\top\right)^{-1}\left(\int \xi p^\top x x^\top\right).$$

We are, of course, assuming that $\int \xi x x^\top$ is invertible. Let $\bar{p} = \left(\int \xi x x^\top\right)^{-1}\left(\int \xi p^\top x x^\top\right)$. The patient reader will not complain overmuch if we continue the orgy of derivatives and compute

$$\dot{h}_{i_{\min}} = 3\xi\left(x^\top(p - \bar{p})\right)^2 - z^2.$$

so that

$$\hat{p} = \left(M + \int_0^t d\tau \, x x^\top \right)^{-1} \int_0^t d\tau \, x y \tag{2}$$

is the solution to the least-squares estimation problem. If $M > 0$ this solution is unique, and because the Hessian

$$\frac{\partial^2 J}{\partial \hat{p} \, \partial \hat{p}^\top}(t) = M + \int_0^t d\tau \, x x^\top > 0,$$

$J(t)$ is a convex function of $\hat{p}$ and the solution is a global minimum.

Equation (2) can be used to construct a recursive solution $\hat{p}$ to the least-squares problem. However, the important thing is that the solution to the least-squares problem exists and is unique. Once this condition is satisfied, a number of techniques can be used. If y were available, perhaps the simplest way of constructing a recursive estimate $\hat{p}$ of p_* would be to choose the tuning error $\hat{z} = x^\top \hat{p} - y$ and set

$$\dot{\hat{p}} = -x\hat{z}.$$

In what follows we assume that y is a noisy measurement of the scalar product of data x with an unknown parameter vector p_*, namely, $y = x^\top p_* + w$. Then $x^\top(\hat{p} - p_*) = \hat{z} + w$, so the derivative of

$$V = \frac{1}{2}(\hat{p} - p_*)^\top (\hat{p} - p_*),$$

the squared parameter error, is

$$\begin{aligned} \dot{V} &= (\hat{p} - p_*)^\top \dot{\hat{p}} = -(\hat{p} - p_*)^\top x \hat{z} \\ &= -\hat{z}^2 - \hat{z}w. \end{aligned}$$

Using the inequality $|\hat{z}w| \le \frac{1}{2}\hat{z}^2 + \frac{1}{2}w^2$ and integrating results in

$$V(t) \le V(0) - \frac{1}{2}\int_0^t \hat{z}^2 + \frac{1}{2}\int_0^t w^2.$$

Because $V \ge 0$, this shows that the energy of the signal $\hat{z}$ is limited by that of w plus a finite constant; in particular if $w \in \mathcal{L}^2$ then $\hat{z} \in \mathcal{L}^2$ as well, and $\hat{p}$ is bounded. If we now choose the adjustable parameters p equal to p_*'s estimate $\hat{p}$, which of course we have the freedom to do, then $\hat{z} = z$ and

$$z = x^\top p - y \in \mathcal{L}^2 \text{ and } p \in \mathcal{L}^\infty.$$

If p are parameters of a direct adaptive feedback controller, then $p = \hat{p}$ can be considered an expression of the certainty-equivalence principle. The distinction between the adjustable parameters p and the parameter estimate $\hat{p}$ may appear recherché, but it plays a crucial role in the solution of the more complex problem treated in §2 of this paper.

Tuning via Measurements of the Squared Error

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Abstract

Tuning when the estimation error is known in magnitude only.

Key Words: Adaptive Control; Parameter Estimation; Direct Control.

1 Introduction

Given data $x : [0, \bar{t}) \to \mathbb{R}^n$, we wish to adjust a parameter vector $p(t)$ so as to minimize

$$z(t) = x^\top p - y \tag{1}$$

as best we can in some norm sense. If $y : [0, \bar{t}) \to \mathbb{R}$ (or equivalently z) were available, we might choose as our cost function the integral of z^2 and minimize it using standard least-squares algorithms. We consider the case when neither y nor z are available; rather, at each instant we are able to choose $p(t)$ and measure $z^2(t)$, that is to say, z's magnitude but not its sign. The garden-variety least-squares algorithm depends on explicit knowledge of y, and its recursive version, as other recursive gradient-type methods, relies on $z(t)$ to specify the direction of the adjustment. In this note we propose a recursive algorithm to tune estimates of $p(t)$. One application we have in mind is direct adaptive control [1].

Running the risk of boring the reader, we shall briefly review some standard least-squares parameter estimation concepts. With M symmetric, positive-semidefinite, define

$$J(t) = \hat{p}^\top M \hat{p} + \int_0^t d\tau\, (x^\top \hat{p} - y)^2.$$

$J(t)$ represents a cost to be minimized at a given instant t by choosing an appropriate $\hat{p}$, which must satisfy

$$\frac{\partial J}{\partial \hat{p}}(t) = 2\hat{p}^\top M + 2 \int_0^t d\tau\, (x^\top \hat{p} - y) x^\top = 0,$$

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