Optimal Stopping with Continuous Control of Piecewise Deterministic Markov Processes *

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Abstract

In this paper we consider the problem of optimal stopping and continuous control on some local parameters of a piecewise-deterministic Markov processes (PDP's). Optimality equations are obtained in terms of a set of variational inequalities as well as on the first jump time operator of the PDP. It is shown that if the final cost function is absolutely continuous along trajectories then so is the value function of the optimal stopping problem with continuous control. These results unify and generalize previous ones in the current literature.

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1 Introduction

Piecewise-deterministic Markov processes (PDP's) were introduced by Davis [6] as a general family of non-diffusion stochastic models suitable for formulating many optimization problems in several areas of operations research [5],[6]. The motion of a PDP depends on three local characteristics, namely the flow ϕ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from x the motion of the process follows the flow $\phi(t,x)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\phi(t,x))$ or when the flow hits the boundary of the state space. In either case the location of the process at the jump time T_1 is selected by the transition measure $Q(.;\phi(T_1,x))$ and the motion restarts from this new point as before. The theory of PDP's has recently been developed for invariant measures [8], continuous control [5],[7],[18], optimal stopping [3],[5], [11], [13], and impulse control [4],[5],[10], with some specific problems being solved [5]. Related to PDP's are the so-called Markov Decision Drift Processes, which deals with problems involving interventions (similar to impulse control) and continuous control on the rate of jumps and post-jump location measure. Such problems have been analyzed in [17] via time-discretization, and in [19] via Bellman inequalities .

In this paper we consider the optimal stopping with continuous control of PDP's. Optimal stopping problems have been studied in a general framework in [2],[15],[16],[20],[21], and for PDP's in [3],[4], [5],[10],[11],[13]. In [11] the author defines an operator related to the first jump time of the process, which we shall call first jump operator, and shows that the value function of the optimal stopping problem is a fixed point for this operator. The basic assumption in this case is that the final cost function is continuous along trajectories, and it is shown that the value function will also have this property. In [10],[13] the authors adopt some stronger continuity assumptions and boundary conditions to show that the value function of the optimal stopping problem satisfies some variational inequalities, related to integro-differential equations. In [5] Davis assumes that the final value function is bounded and locally Lipschitz along trajectories to show that the variational inequalities is a necessary and sufficient to characterize the value function of the optimal stopping problem, which will also be locally Lipschitz along trajectories.

The main differences between this paper and previous ones [3],[5],[10],[11],[13] are twofold. Firstly we allow for continuous control on the local parameters λ and Q, besides the optimal stopping problem. In the previous papers, only optimal stopping was allowed. In order to do that, we consider a more general family of stochastic processes, not necessarily time-homogeneous Markovian processes. Secondly, we weaken the continuity assumptions of [5],[10],[13] and assume that the final cost function is only absolutely continuous along trajectories, to show that the value function will also have this property. No boundary conditions are considered here. Moreover we show that the value function is the unique fixed point of the operator associated to the first jump operator of the process, as well as the unique solution of a set of variatonal inequalities, defined in terms of the infinitesimal operator of the process. We believe that these results unify and generalize previous ones.

Optimal stopping problems are interesting not only by its own, but also because they are the first step to study the more general problem of impulse control. In this case, control is taken by intervention, that is, the decision to act is taken at discrete times, in response to the random evolution of the system, and the process moved to a new point in the state space. Problems of this kind were first studied by Bensoussan and Lions [1], in the context of diffusion processes, as an application of variational and quasi-variational inequalities. In a companion paper, the authors will apply the results of the present paper to analyze the impulse with continuous control problem of PDP's, under the absolutely continuity along trajectories assumptions adopted here.

This work is organized in the following way. In Section 2 we present some notation and definitions, while in Section 3 we define our optimal stopping with continuous control problem. In Section 4 we present some auxiliary results. The main results are presented in Section 5, with Theorem 5.2 characterizing the value function of our optimal stopping with continuous control problem as the unique fixed point of the first jump operator, or as the unique solution of a set of variational inequalities. It is also show that the absolutely continuous property is preserved from the final cost function to the value function.

2 Notation and Preliminary Definitions

Let E^0 be an open non-empty subset of R^d, ∂E^0 its boundary. We consider $\phi(t,x)$ the flow of a Lipschitz continuous vector field $\mathcal{L}$. Define

$$\partial^{\pm} E^0 = \{z \in \partial E^0; z = \phi(\pm t, x) \text{ for some } x \in E^0\},$$

and $\partial_1 E^0 = \partial^- E^0 - \partial^+ E^0$. Set $E = E^0 \cup \partial_1 E^0, \sigma(E)$ the Borel σ-field of E, $\mathcal{P}(E)$ the set of probability measures on E, $\partial E^* = \partial^+ E^0$, and for each $x \in E$, write

$$t^*(x) = inf\{t > 0; \phi(t,x) \in \partial^* E\},$$

where $inf\{\emptyset\} := \infty$. Define $t^*(z) = 0$ for $z \in \partial E^*$, and for $t < t^*(x)$,

$$\mathcal{I}_t(x) = \begin{cases} [t, \infty) & \text{if } t^*(x) = \infty \\ [t, t^*(x)] & \text{if } t^*(x) < \infty \end{cases}.$$

We set $\mathbb{B}(E)$ and $\mathbb{B}(\partial^* E)$ the set of all bounded Borel measurable functions from E and $\partial^* E$ into R respectively. We define

$$\begin{aligned} \mathbb{B}^c(E) = \{ & f \in \mathbb{B}(E); f(\phi(.,x)) : [0, t^*(x)) \mapsto R \text{ is continuous for each } x \in E, \text{ and} \\ & lim_{t \uparrow t^*(x)} f(\phi(t,x)) \text{ exists}\}. \\ \mathbb{B}^{ac}(E) = \{ & f \in \mathbb{B}^c(E); f(\phi(.,x)) : [0, t^*(x)) \mapsto R \text{ is absolutely continuous for each } x \in E\} \end{aligned}$$

and we define for $f \in \mathbb{B}^c(E)$, $f(z) = lim_{t \uparrow t^*(x)} f(\phi(t,x))$ for all $z = lim_{t \uparrow t^*(x)} \phi(t,x) \in \partial^* E$. We consider the following parameters for our problem:

a) $U(.)$ is a Borel set-valued function such that for each $x \in E \cup \partial^* E$, $U(x) \subset \mathbb{U}$, where $\mathbb{U}$ is a non-empty Borel space, and

a.1) $U(x)$ is compact for each point $x \in E \cup \partial^* E$,

a.2) $U(\phi(.,x)) : [0, t^*(x)) \mapsto \mathbb{U}$ is right continuous for each $x \in E$.

b) $\lambda(.,.) : E \times \mathbb{U} \mapsto R_+$ is a bounded Borel measurable function such that for each $x \in E$

$$\lambda(x,.) : U(x) \mapsto R_+ \text{ is continuous,}$$

c) $Q(.;.,.) : E \cup \partial^* E \times \mathbb{U} \mapsto \mathcal{P}(E)$ are such that for each $g \in \mathbb{B}(E)$, and $x \in E$, $z \in \partial^* E$,

c.1) $Qg(x,.) : U(x) \mapsto R$ is lower semi continuous,

c.2) $Qg(z,.) : U(z) \mapsto R$ is lower semi continuous,

where we use the notation,

$$Qg(x,.) = \int_E g(y) Q(dy; x,.).$$

d) $f(.,.) : E \times \mathbb{U} \mapsto R$ is a bounded Borel measurable function such that for each $x \in E$,

$$f(x,.) : U(x) \mapsto R \text{ is lower semi continuous,}$$

e) $r(.,.) : \partial^* E \times \mathbb{U} \mapsto R$ is a bounded Borel measurable function and for each $z \in \partial^* E$

$$r(z,.) : U(z) \mapsto R \text{ is lower semi continuous.}$$

f) $\varphi \in \mathbb{B}^{ac}(E)$.

Define for each $x \in E$ and $t \in [0, t^*(x))$

$$V_t(x) = \{v(.) : \mathcal{I}_t(x) \mapsto \mathbb{U} \text{ measurable; } v(s) \in U(\phi(s,x)) \text{ for each } s \in \mathcal{I}_t(x)\}$$

and write $V(x) = V_0(x)$. For any $v \in V(x)$,set

$$\begin{aligned}
&1) \lambda^v(\phi(s,x)) = \lambda(\phi(s,x), v(s)),\\
&2)\ Q^v(.;\phi(s,x)) = Q(.;\phi(s,x), v(s)),\\
&3)\ f^v(\phi(s,x)) = f(\phi(s,x), v(s))\\
&4)\ r^v(\phi(t^*(x),x)) = r(\phi(t^*(x),x), v(t^*(x))).
\end{aligned}$$

We also define

$$\begin{aligned}
\mathcal{U} = \{u(.) : E \cup \partial^* E \mapsto \mathbb{U} &\text{ a measurable function such that } u(\phi(t,x)) \in U(\phi(t,x))\\
&\text{for each } x \in E \text{ and } t \in \mathcal{I}_0(x)\}.
\end{aligned}$$

and, as before, set for $u \in \mathcal{U}$, $\lambda^u(x) = \lambda(x, u(x)), Q^u(.;x) = Q(.;x,u(x)), f^u(x) = f(x,u(x))$ and $r^u(z) = r(z, u(z))$ for $z \in \partial^* E$. For any $u \in \mathcal{U}$, we shall define $v(u) \in V(x)$ as

$$v(u)(t) = u(\phi(t,x)), t \in \mathcal{I}_0(x).$$

From the definition of $\mathcal{U}$ above it is clear that indeed $v(u) \in V(x)$.

Remark 2.1. *From the assumptions b), c.1), c.2), d), and e) we have that for $g, h \in \mathbb{B}(E)$,*

$$\begin{aligned}
&f(x,.) - \lambda(x,.) \int_E (h(x) - g(y)) Q(dy; x,.)\\
&\textit{and}\\
&r(z,.) + Qg(z,.)
\end{aligned}$$

are lower semi-continuous in $U(x)$. Therefore from the compactness assumption in a.1) and the results in Proposition D5) in [12], page 183, there exists a measurable selector $u^* \in \mathcal{U}$ such that for each $x \in E$, $z \in \partial^* E$

$$\begin{aligned}
&f^{u^*}(x) - \lambda^{u^*}(x) \int_E (h(x) - g(y)) Q^{u^*}(dy; x)\\
&= min_{a \in U(x)} \{f(x,a) - \lambda(x,a) \int_E (h(x) - g(y)) Q(dy; x, a)\},\\
r^{u^*}(z) + Q^{u^*} g(z) &= min_{a \in U(z)} \{r(z,a) + Qg(z,a)\}.
\end{aligned}$$

3 Optimal Stopping with Continuous Control Problem

3.1 Problem Formulation

Let D denote the space of right-continuous functions $\omega(.)$ on R_+ taking values in E such that the left limit exists for all $t > 0$. Denote by x_t the coordinate function $x_t(\omega) = \omega(t)$ for all $\omega \in \Omega$. Let $\mathcal{F}_t^0 = \sigma\{x_s; 0 \leq s \leq t\}$ and $\mathcal{F}^0 = \bigvee_{t \geq 0} \mathcal{F}_t^0$. For $\omega \in D$, set for $k = 0, 1, 2, \ldots$

$$
\begin{aligned}
&T_0(\omega) = 0,\\
&T_k(\omega) = \begin{cases} inf\{t > T_{k-1}(\omega); x_t(\omega) \neq x_{t^-}(\omega)\} & if\, T_{k-1}(\omega) < \infty \\ \infty & otherwise \end{cases}, k > 0,\\
&T_\infty(\omega) = \lim_{k \to \infty} T_k(\omega)\\
&Z_k(\omega) = \begin{cases} x_{T_k(\omega)}(\omega) & if\, T_k(\omega) < \infty \\ \Delta & if\, T_k(\omega) = \infty \end{cases}
\end{aligned}
$$

where Δ represents a cemetery state. We shall write for $k = 0, 1, \ldots,$

$$\varpi_k(\omega) = (Z_0(\omega), T_1(\omega), Z_1(\omega), \ldots, T_k(\omega), Z_k(\omega)).$$

Let us define $\Omega \subset D$ such that $\omega \in \Omega$ if

$$x_t(\omega) = \phi(t, Z_0(\omega)) 1_{\{0 \leq t < T_1(\omega)\}} + \sum_{k=2}^{\infty} 1_{\{T_{k-1}(\omega) \leq t < T_k(\omega)\}} \phi(t - T_{k-1}(\omega), Z_{k-1}(\omega)).$$

We shall define the set of admissible continuous control strategies Ψ in the following way. A point $\psi = (\zeta_1, \zeta_2, \ldots) \in \Psi$ if it is predictable (see [5], page 264) that is, for every $\omega \in \Omega$,

$$
\begin{aligned}
\psi(\omega, t) =& 1_{\{0 \leq t \leq T_1(\omega)\}} \zeta_1(Z_0(\omega), t) + \sum_{k=2}^{\infty} 1_{\{T_{k-1}(\omega) < t \leq T_k(\omega)\}} \zeta_k \Big(\varpi_{k-1}(\omega), t - T_{k-1}(\omega) \Big) + \\
& 1_{\{T_\infty(\omega) \leq t\}} \hat{\Delta}
\end{aligned}
$$

where ζ_k is a Borel measurable function from $(E \cup \{\Delta\}) \times \prod_{i=1}^{k-1} \Big(R_+ \cup \{\infty\}) \times (E \cup \{\Delta\}) \Big) \times R_+$ to $\mathbb{U} \cup \{\hat{\Delta}\}$ satisfying, for all $s \in \mathcal{I}_0(x)$ and $\omega \in \Omega$ such that $T_{k-1}(\omega) < \infty$,

$$\zeta_k(\varpi_{k-1}(\omega), s) \in U(\phi(s, Z_{k-1}(\omega))),$$

whereas for the case in which $T_{k-1}(\omega) = \infty$, the control variable takes the value of the cemetery state $\hat{\Delta}$. It follows that $\zeta_k(\varpi_{k-1}(\omega), .) \in V(Z_{k-1}^\psi(\omega))$. We define the motion of the process $\{X_t^\psi\}$ associated to the strategy ψ, starting from a point $x \in E$, in the following way. Take a random variable T_1^ψ such that

$$P_x(T_1^\psi > t) = \begin{cases} exp\{-\int_0^t \lambda^{\zeta_1}(\phi(s, x)) ds\} & \text{for } t < t^*(x) \\ 0 & \text{for } t \geq t^*(x) \end{cases}$$

where $\zeta_1(x, .) \in V(x)$. If T_1^ψ generated according to the above probability is equal to infinity, then $X_t^\psi = \phi(t, x)$ for all $t \geq 0$. Otherwise select independently an E-valued random variable having distribution $Q^{\zeta_1}(.;\phi(T_1^\psi,x))$. The trajectory of $\{X_t^\psi\}$ starting at x, for $t \le T_1^\psi$, is given by

$$X_t^\psi = \begin{cases} \phi(t,x) & \text{for } t < T_1^\psi \\ Z_1^\psi & \text{for } t = T_1^\psi \end{cases}.$$

Starting from $X^\psi_{T_1^\psi} = Z_1^\psi$, we now select the next inter-jump time $T_2^\psi - T_1^\psi$ and post-jump location $X^\psi_{T_2^\psi} = Z_2^\psi$ in a similar way, using $\zeta_2(x,T_1^\psi,Z_1^\psi,.)$ (which by definition belongs to $V(Z_1^\psi)$ for T_1^ψ and Z_1^ψ fixed) instead of $\zeta_1(x,.)$. This gives a time-variant piecewise-deterministic trajectory for the process $\{X_t^\psi\}$ with jump times $T_1^\psi, T_2^\psi, \ldots$, and post-jump location $Z_1^\psi, Z_2^\psi, \ldots$. Note that in general we can not guarantee that the process $\{X_t^\psi\}$ is a Markov process, since ζ_k may depend on the whole history of the process up to k^{th}-jump. For the case in which we have, for all $k = 0, 1, \ldots$

$$\zeta_k(\varpi_{k-1}(\omega),.) = u_k(\phi(.,Z_{k-1}(\omega)))$$

for $u_k \in \mathcal{U}$, $k = 0,1,\ldots$,we write $\psi = (u_1,u_2,\ldots)$ and it can be shown by arguments similar to the ones in [5], pages 62-66 that $\{X_t^\psi\}$ is a strong Markov process (although not necessarily homogeneous). The particular case in which $u_k = u$ for all $k = 0,1,\ldots$, for some $u \in \mathcal{U}$, corresponds to the situation of a time homogeneous PDP, to be further detailed in the next subsection.

The procedure above defines a family of measures $\{P_x^\psi; x \in E\}$ on $(\Omega,\mathcal{F}^0)$. The final assumption on ψ to be an admissible control is that for every $x \in E$,

$$E_x^\psi\left(\sum_{k=1}^{\infty} 1_{\{T_k^\psi \le t\}}\right) < \infty \tag{1}$$

for all $t \in R_+$. In particular (1) implies that $T_k^\psi \to \infty$ as $k \to \infty$ almost surely. For any $\mu \in \mathcal{P}(E)$, define P_μ^ψ on $(\Omega,\mathcal{F}^0)$ as

$$P_\mu^\psi(A) = \int_A P_x^\psi(A)\mu(dx).$$

Let $\mathcal{F}_t^{\psi,\mu}$ be the completion of $\mathcal{F}_t^0$ with respect to all P_μ^ψ-null set of $\mathcal{F}^0$, and

$$\mathcal{F}_t^\psi = \bigcap_{\mu \in \mathcal{P}(E)} \mathcal{F}_t^{\psi,\mu}.$$

By the same arguments as in [5], Theorem 25.3, page 63, it follows that $\mathcal{F}_t^\psi$ is right-continuous. Define

$$p^{\psi *}(t) = \sum_{i=1}^{\infty} 1_{\{t \ge T_i^\psi\}} 1_{\{X^\psi_{T_i(\omega)^-} \in \partial^* E\}}$$

which is a measure that counts the number of jumps from the boundary. Associated to an admissible control strategy ψ and an $\mathcal{F}_t^\psi$-stopping time τ, we have the following cost, for $\alpha > 0$:

$$\mathcal{J}^{\psi,\tau}(x) = E_x^\psi\left(\int_0^\tau e^{-\alpha s} f^\psi(X_s^\psi)ds + \int_0^\tau e^{-\alpha s} r^\psi(X_{s^-}^\psi)dp^{\psi *}(s) + e^{-\alpha\tau}\varphi(X_\tau^\psi)\right), \tag{2}$$

where

$$f^\psi(X_s^\psi(\omega)) = f(\phi(s - T_{i-1}^\psi(\omega), Z_{i-1}^\psi(\omega)), \zeta_i(\varpi_{i-1}(\omega), s - T_{i-1}^\psi(\omega))) \text{ for } T_{i-1}^\psi(\omega) \le s < T_i^\psi(\omega)$$
$$r^\psi(X_{s^-}^\psi(\omega)) = r(X^\psi_{T_i(\omega)^-}(\omega), \zeta_i(\varpi_{i-1}(\omega), T_i^\psi(\omega))) \text{ for } s = T_i^\psi(\omega) \text{ and } X^\psi_{T_i(\omega)^-}(\omega) \in \partial^* E.$$

To avoid unbounded lower limit for the value function, we assume that there exists $d_1 \geq 0$ such that for every $\psi \in \Psi$ and $x \in E$,

$$E_x^\psi\left(\int_0^\infty e^{-\alpha s} min(0, r^\psi(X_{s^-}^\psi))dp^{\psi *}(s)\right) \geq -d_1$$

(clearly this is always satisfied for $r \geq 0$). We consider the following optimal stopping with continuous control problem:

$$\rho(x) = \underset{\psi\in\Psi}{inf}\ \underset{\tau\in\mathcal{M}^\psi}{inf}\ \mathcal{J}^{\psi,\tau}(x) \tag{3}$$

where $\mathcal{M}^\psi$ denotes the set of all $\mathcal{F}_t^\psi$-stopping times. It will be convenient to define also the following truncated optimal stopping with continuous control problem. For $n = 0, 1, \ldots$, set

$$\mathcal{J}_n^{\psi,\tau}(x) = E_x^\psi\left(\int_0^{\tau\wedge T_n^\psi} e^{-\alpha s} f^\psi(X_s^\psi)ds + \int_0^{\tau\wedge T_n^\psi} e^{-\alpha s} r^\psi(X_{s^-}^\psi)dp^{\psi *}(s) + e^{-\alpha\tau\wedge T_n^\psi}\varphi(X_{\tau\wedge T_n^\psi}^\psi)\right)$$

$$\rho_n(x) = \underset{\psi\in\Psi}{inf}\ \underset{\tau\in\mathcal{M}^\psi}{inf}\ \mathcal{J}_n^{\psi,\tau}(x). \tag{4}$$

Clearly we have that $\rho(x) \leq \rho_n(x)$ for all $x \in E$.

3.2 Piecewise-Deterministic Markov Processes

As mentioned above, the particular case in which $\psi = (u, u, \ldots)$ for some $u \in \mathcal{U}$ corresponds to the definition of a piecewise-deterministic Markov process (PDP for short), as presented in [5]. We shall assume that (1) is satisfied for every $\psi = (u, u, \ldots)$ and $u \in \mathcal{U}$. For this case we shall just write u instead of ψ for all the definitions of the previous section. As proved in [5], $\{X_t^u\}$ is a homogeneous strong Markov process, characterized by the following parameters: a) the flow $\phi(.,.)$, b) the jump rate $\lambda^u(.)$, c) the transition measure $Q^u(.;.)$. Associated to the extended generator of the PDP we have the infinitesimal operator $\mathcal{A}^u$ defined in the following way. For $h \in \mathbb{B}^{ac}(E)$, $\mathcal{A}^u h$ is defined as

$$\mathcal{A}^u h(x) = \mathcal{X}h(x) + \lambda^u(x)\int_E (h(y) - h(x))Q^u(dy; x),$$

where $\mathcal{X}h(.)$ is such that for any $x \in E$, $t \in [0, t^*(x))$,

$$h(\phi(t,x)) = h(x) + \int_0^t \mathcal{X}h(\phi(s,x))ds.$$

3.3 First Jump Operator

We shall now define the first jump operator and some related operators associated to the optimal stopping with continuous control problem posed in sub-section 3.1. For $h_2 \in \mathbb{B}(E \times R_+)$, $h_1 \in \mathbb{B}^{ac}(E)$, define for each $x \in E$ and $v \in V(x)$,

$$J^v(h_1,h_2)(t,x) = \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds + e^{-\alpha t - \Lambda^v(t,x)}h_1(\phi(t,x))$$

where

$$
\begin{aligned}
Q^v h_2(\phi(s,x))) &= \int_E h_2(y,s) Q(dy;\phi(s,x),v(s)) \\
\Lambda^v(t,x) &= \int_0^t \lambda(\phi(s,x),v(s))ds
\end{aligned}
$$

and

$$
\begin{aligned}
K^v(h_2)(x) &= \int_0^{t^*(x)} e^{-\alpha s-\Lambda^v(s,x)}(f^v(\phi(s,x))+\lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds \\
&+ e^{-\alpha t^*(x)-\Lambda^v(t^*(x),x)}(r^v(\phi(t^*(x),x))+Q^v h_2(\phi(t^*(x),x))) \\
J_s^v(h_1,h_2)(x) &= \inf_{s\le t<t^*(x)} J^v(h_1,h_2)(t,x).
\end{aligned}
$$

Finally set

$$
\begin{aligned}
L(h_1,h_2)(x) &= \inf_{v\in V(x)}\left(\left\{\inf_{0\le t<t^*(x)} J^v(h_1,h_2)(t,x)\right\}\wedge K^v(h_2)(x)\right) \\
&= \inf_{v\in V(x)}\left(J_0^v(h_1,h_2)(x)\wedge K^v(h_2)(x)\right).
\end{aligned} \tag{5}
$$

The operator $L(.,.)(.)$ defined above is the first jump operator. The following result will be useful in the sequel, and the proof can be found in Theorem A2.3 of [5], page 261.

Proposition 3.1. *Let $\tau\in\mathcal{M}^\psi$ for some $\psi\in\Psi$. Then there exists Borel measurable functions s_k, $k=1,2,\dots$ from $(E\cup\{\Delta\})\times\prod_{i=1}^{k-1}\Big(R_+\cup\{\infty\})\times(E\cup\{\Delta\})\Big)$ into $R_+\cup\{\infty\}$ such that*

$$
\begin{aligned}
&\tau 1_{\{\tau\le T_1^\psi\}} = (s_1(Z_0)\wedge T_1^\psi)1_{\{\tau\le T_1^\psi\}} \\
&\tau 1_{\{T_{k-1}^\psi<\tau\le T_k^\psi\}} = (T_{k-1}^\psi + s_k(Z_0,T_1^\psi,Z_1^\psi,\dots,T_{k-1}^\psi,Z_{k-1}^\psi))\wedge T_k^\psi 1_{\{T_{k-1}^\psi<\tau\le T_k^\psi\}}.
\end{aligned}
$$

Consider now $\psi=(\zeta_1,\zeta_2,\dots)\in\Psi$, and $\tau\in\mathcal{M}^\psi$ defined by the functions s_k, $k=1,2,\dots$, as in the Proposition 3.1 above. Given that $Z_0=x, T_1^\psi=s$,we can now construct a new $\psi'=(\zeta_1',\zeta_2',\dots)\in\Psi$ and $\tau'\in\mathcal{M}^{\psi'}$ defined by the functions s_k', $k=1,2,\dots$ as in Proposition 3.1 in the following way. For $\omega\in\Omega$, $t\ge 0$, $k=1,2,\dots$,

$$
\begin{aligned}
&\zeta_k'(\varpi_{k-1}(\omega),t-T_{k-1}(\omega)) = \zeta_{k+1}(x,s,\varpi_{k-1}(\omega),t-T_{k-1}(\omega)) \\
&s_k'(\varpi_{k-1}(\omega)) = s_{k+1}(x,s,\varpi_{k-1}(\omega)).
\end{aligned}
$$

In this way we can define, associated to a $\psi=(\zeta_1,\zeta_2,\dots)\in\Psi$ and $\tau\in\mathcal{M}^\psi$, a truncated cost

$$
\mathcal{J}_{n-1}^{\psi',\tau'}(z\Big|x,s)
$$

given as in (4) with initial condition z, and $\psi'=(\zeta_1',\zeta_2',\dots)\in\Psi$, $\tau'\in\mathcal{M}^{\psi'}$ as defined above for x,s fixed. The following result will be useful in the sequel, and establishes a connection between the operators $J^v(.,.)$, $K^v(.,.)$, and the truncated problem (4).

Proposition 3.2. *For any $\psi = (\zeta_1, \zeta_2, \dots) \in \Psi$, $\tau \in \mathcal{M}^\psi$,*

$$\mathcal{J}_n^{\psi,\tau}(x) = E_x^\psi\Bigg(\Big(\int_0^{s_1(x)} e^{-\alpha s} f^\psi(\phi(s,x))ds + e^{-\alpha s_1(x)}\varphi(\phi(s_1(x),x))\Big)1_{\{T_1^\psi > s_1(x)\}} + \Big(\int_0^{T_1^\psi} e^{-\alpha s} f^\psi(\phi(s,x))ds + \int_0^{T_1^\psi} e^{-\alpha s} r^\psi(\phi(s^-,x))dp^{\psi *}(s) + e^{-\alpha T_1^\psi}\mathcal{J}_{n-1}^{\psi',\tau'}(Z_1^\psi \big| x, T_1^\psi)\Big)1_{\{T_1^\psi \le s_1(x)\}}\Bigg) \tag{6}$$

for $s_1(.)$ as in Proposition 3.1. Moreover,

$$\mathcal{J}_n^{\psi,\tau}(x) = \begin{cases} J^v(\varphi, \mathcal{J}_{n-1}^{\psi',\tau'}(.\big|x,.))(s_1(x),x) & \text{if } s_1(x) < t^*(x) \\ K^v(\mathcal{J}_{n-1}^{\psi',\tau'}(.\big|x,.))(x) & \text{if } s_1(x) \ge t^*(x) \end{cases} \tag{7}$$

where $v(.) = \zeta_1(x,.) \in V(x)$.

Proof. From the representation theorem for $\mathcal{F}_t^\psi$-stopping times in Proposition 3.1, we have that for some measurable function $s_1(.) : E \to [0,\infty]$,

$$\tau \wedge T_1^\psi = s_1(Z_0) \wedge T_1^\psi.$$

Using this property in (4) and noticing that

$$E_x^\psi\Bigg(\Big(\int_{T_1^\psi}^{\tau\wedge T_n^\psi} e^{-\alpha s} f^\psi(X_s^\psi)ds + \int_{T_1^\psi}^{\tau\wedge T_n^\psi} e^{-\alpha s} r^\psi(X_{s^-}^\psi)dp^{\psi *}(s) + e^{-\alpha\tau\wedge T_n^\psi}\varphi(X^\psi_{\tau\wedge T_n^\psi})\Big)1_{\{T_1^\psi \le s_1(x)\}}\Bigg) = E_x^\psi\Big(e^{-\alpha T_1^\psi}\mathcal{J}_{n-1}^{\psi',\tau'}(Z_1^\psi\big|x, T_1^\psi)1_{\{T_1^\psi \le s_1(x)\}}\Big)$$

we deduce (6). By direct calculation we obtain (7). □

4 Auxiliary Results

In this section we shall obtain some continuity results related to the operators presented in subsection 3.3. The proofs of the propositions below are presented in the Appendix.

For $x \in E$, $0 \le t < t^*(x)$, set the operator

$$q_t^x(.,.) : V(x) \times V_t(x) \mapsto V(x)$$

as, for $v \in V(x)$, $w \in V_t(x)$,

$$q_t^x(v,w)(s) = \begin{cases} v(s) & , s < t \\ w(s) & , s \ge t \end{cases}.$$

Clearly $q_t^x(v,w) \in V(x)$ since $q_t^x(v,w)(\phi(.,x)) : \mathcal{I}_0(x) \mapsto \mathbb{U}$ is measurable and for each $s \in \mathcal{I}_0(x)$, $q_t^x(v,w)(s) \in U(\phi(s,x))$. For simplicity we shall drop the superscript x of q_t^x. For $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^c(E)$, and $t \in [0, t^*(x))$, $v \in V(x)$, let us define

$$M^v(h_1,h_2)(t,x) = \inf_{w\in V_t(x)}\Big(\Big\{\inf_{t\le s<t^*(x)} J^{q_t(v,w)}(h_1,h_2)(s,x)\Big\} \wedge K^{q_t(v,w)}(h_2)(x)\Big).$$

which can be easily seen to be a non-decreasing function in t. The following results are proved in the Appendix.

Proposition 4.1. *For $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^c(E)$ and any $v \in V(x)$, $t \in [0, t^*(x))$,*

$$L(h_1, h_2)(\phi(t,x)) = \Big\{ M^v(h_1,h_2)(t,x) - \int_0^t \Big(e^{-\alpha s - \Lambda^v(s,x)} \Big(f^v(\phi(s,x)) + \lambda^v(\phi(s,x)) Q^v h_2(\phi(s,x)) \Big) \Big) ds \Big\} e^{\alpha t + \Lambda^v(t,x)}. \tag{8}$$

Proposition 4.2. *For $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^c(E)$, and $0 \le t < s$, $s \in \mathcal{I}_0(x)$, let $v_t, v_s \in V(x)$ be such that $v_t(\ell) = v_s(\ell)$ for $\ell \in [0,t) \cup \mathcal{I}_s(x)$. Then for any $\nu \in [s, t^*(x))$,*

$$\begin{aligned} \left| J^{v_t}(h_1,h_2)(\nu,x) - J^{v_s}(h_1,h_2)(\nu,x) \right| &\le c(s-t) \\ \left| K^{v_t}(h_2)(x) - K^{v_s}(h_2)(x) \right| &\le c(s-t). \end{aligned}$$

where

$$c = \Big\{ \|f\| + \|\lambda\| \, \|h_2\| \Big\} \Big\{ 2 + \frac{1}{\alpha} \|\lambda\| \Big\} + 2 \|\lambda\| \, max\{ \|r\| + \|h_2\|, \|h_1\| \}.$$

Proposition 4.3. *For $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^c(E)$, consider $0 \le t_i < s_i < t^*(x)$ and an arbitrary $\zeta > 0$. Then we can find $\bar{t}_i$ such that $0 \le t_i \le \bar{t}_i \le s_i < t^*(x)$ and that*

$$0 \le M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x) \le \left| J^v(h_1,h_2)(s_i,x) - J^v(h_1,h_2)(\bar{t}_i,x) \right| + c(s_i - t_i) + \zeta. \tag{9}$$

Proposition 4.4. *For any $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^c(E)$, the function $L(h_1,h_2)(.)$ belongs to $\mathbb{B}^c(E)$.*

Proposition 4.5. *Consider any $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^{ac}(E)$. Then $M^v(h_1,h_2)(.,x)$ is absolutely continuous in $[0, t^*(x))$.*

The next Theorem establishes the connection between the first jump operator and the variational inequalities.

Theorem 4.1. *Consider any $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^{ac}(E)$. Then $h = L(h_1,h_2)$ if and only if $h \in \mathbb{B}^{ac}(E)$ and satisfies, for each $x \in E$,*

a) $h(x) \le h_1(x)$

b) $\mathcal{X}h(x) - \alpha h(x) + min_{a \in U(x)} \{ f(x,a) - \lambda(x,a) \int_E (h(x) - h_2(y)) Q^a(dy;x,a) \} \ge 0$

c) $\Big(h_1(x) - h(x) \Big) \Big(\mathcal{X}h(x) - \alpha h(x) + min_{a \in U(x)} \{ f(x,a) - \lambda(x,a) \int_E (h(x) - h_2(y)) Q(dy;x,a) \} \Big) = 0$

and for $z \in \partial^* E$

d) $h(z) = h_1(z) \wedge min_{a \in U(z)} \{ r(z,a) + Q h_2(z,a) \}.$

Proof. ($\Rightarrow$) Since, from (8) in Proposition 4.1

$$L(h_1,h_2)(\phi(t,x)) = \left\{ M^v(h_1,h_2)(t,x) - \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) \right.$$
$$\left. + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds \right\} e^{\alpha t + \Lambda^v(t,x)}$$

we conclude from Proposition 4.5 that $L(h_1,h_2)(\phi(t,x))$ is absolutely continuous. From the above equation,

$$\int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds$$
$$+ e^{-\alpha t - \Lambda^v(t,x)} L(h_1,h_2)(\phi(t,x)) = M^v(h_1,h_2)(t,x). \tag{10}$$

With $v(0) = a \in U(x)$, we get from (10) that

$$f(x,a) + \lambda(x,a)Q(h_2)(x,a) - (\alpha + \lambda(x,a))L(h_1,h_2)(x) + \mathcal{X}L(h_1,h_2)(x) = \mathcal{X}M^v(h_1,h_2)(0,x) \geq 0$$

since $M^v(h_1,h_2)(t,x)$ is non-decreasing in t. This shows that b) of the Theorem holds, since that a was arbitrary in $U(x)$, and the minimum comes from the Remark 2.1. Clearly we have

$$L(h_1,h_2)(x) \leq h_1(x)$$

showing a) of the Theorem. Let us suppose now that

$$L(h_1,h_2)(x) < h_1(x).$$

Then we must have that for all ν sufficiently small, $0 < \nu < t^*(x)$,

$$L(h_1,h_2)(x) = \inf_{v \in V(x)} M^v(h_1,h_2)(\nu,x),$$

and from equation (10)

$$\inf_{v \in V(x)} \left(\int_0^\nu e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds \right.$$
$$\left. + e^{-\alpha\nu - \Lambda^v(\nu,x)} L(h_1,h_2)(\phi(\nu,x)) \right) = L(h_1,h_2)(x).$$

Since $L(h_1,h_2)(x)$ is absolutely continuous,

$$L(h_1,h_2)(\phi(\nu,x)) = L(h_1,h_2)(x) + \int_0^\nu \mathcal{X}L(h_1,h_2)(\phi(s,x))ds$$

and thus,

$$0 = \inf_{v \in V(x)} \left(\frac{1}{\nu} \int_0^\nu e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds - (\alpha + \right.$$
$$\left. \frac{1}{\nu}\int_0^\nu \lambda^v(\phi(s,x))ds) L(h_1,h_2)(\phi(\nu,x)) + \frac{1}{\nu}\int_0^\nu \mathcal{X}L(h_1,h_2)(\phi(s,x))ds + o(\nu) \right)$$

where $o(\nu)$ are terms going to zero as ν goes to zero uniformly in $v \in V(x)$. From standards arguments in dynamic programming (see, for instance, Theorem 6.1$'$ in [9], page 19), we get, by taking the limit as ν goes to zero, and using Remark 2.1, that

$$0 = min_{a \in U(x)}\{f(x,a) - \lambda(x,a)\int_E (L(h_1,h_2)(x) - h_2(y))Q(dy;x,a)\} \\ - \alpha L(h_1,h_2)(x) + \mathcal{X} L(h_1,h_2)(x).$$

Finally as in the proof of Proposition 4.4, we get that

$$lim_{t \uparrow t^*(x)} L(h_1,h_2)(\phi(t,x)) = h_1(z) \wedge min_{a \in U(z)}\{r(z,a) + Qh_2(z,a)\}$$

where $z = \phi(t^*(x),x)$.

($\Leftarrow$) Consider $h \in \mathbb{B}^{ac}(E)$ satisfying a), b), c) and d). From Remark 2.1, there exists a measurable selector $u^* \in \mathcal{U}$ such that for each $x \in E$, $z \in \partial^* E$

$$\begin{aligned} f^{u^*}(x) - \lambda^{u^*}(x)\int_E (h(x) - h_2(y))Q^{u^*}(dy;x) \\ = min_{a \in U(x)}\{f(x,a) - \lambda(x,a)\int_E (h(x) - h_2(y))Q(dy;x,a)\} \\ r^{u^*}(z) + Q^{u^*}h_2(z) = min_{a \in U(z)}\{r(z,a) + Qh_2(z,a)\}. \end{aligned} \tag{11}$$

From the fact that $h(y) \le h_1(y)$ for all $y \in E$, from continuity along trajectories of $h(\phi(.,x))$, $h_1(\phi(.,x))$, and the existence of the limit as $t \uparrow t^*(x)$ (whenever $t^*(x) < \infty$), we have that one of the three possibilities below holds:
i) $h(\phi(t,x)) < h_1(\phi(t,x))$ for all $t \in \mathcal{I}_0(x)$
ii) $h(x) = h_1(x)$
iii) for some $\nu \in \mathcal{I}_0(x)$, we have for all $t \in [0,\nu)$ that

$$h(\phi(t,x)) < h_1(\phi(t,x)), \quad h(\phi(\nu,x)) = h_1(\phi(\nu,x)). \tag{12}$$

Suppose that i) holds. For the case in which $t^*(x) < \infty$, we have from d) that

$$h(\phi(t^*(x),x)) = r^{u^*}(\phi(t^*(x),x)) + Q^{u^*}h_2(\phi(t^*(x),x)). \tag{13}$$

From c) of the Theorem, for all $s \in [0, t^*(x))$,

$$\mathcal{X} h(\phi(s,x),x) - (\alpha + \lambda^{u^*}(\phi(s,x)))h(\phi(s,x)) + f^{u^*}(\phi(s,x)) + \lambda^{u^*}(\phi(s,x))Q^{u^*}h_2(\phi(s,x)) = 0$$

that is,

$$\begin{aligned} \mathcal{X}(e^{-\alpha s - \Lambda^{u^*}(s,\phi(s,x))}h(\phi(s,x))) + e^{-\alpha s - \Lambda^{u^*}(s,\phi(s,x))}(f^{u^*}(\phi(s,x)) + \\ \lambda^{u^*}(\phi(s,x))Q^{u^*}h_2(\phi(s,x))) = 0. \end{aligned} \tag{14}$$

Taking the integral in (14) over s, from 0 to $t^*(x)$, and using (13) if $t^*(x) < \infty$, we get that

$$h(x) = K^{v(u^*)}(h_2)(x).$$

On the other hand, for any $v \in V(x)$ and $s \in [0, t^*(x))$, we have from b) that,

$$\begin{aligned} \mathcal{X}(e^{-\alpha s - \Lambda^{v}(s,\phi(s,x))}h(\phi(s,x),x)) + e^{-\alpha s - \Lambda^{v}(s,\phi(s,x))}(f^{v}(\phi(s,x)) + \\ \lambda^{v}(\phi(s,x))Q^{v}h_2(\phi(s,x))) \ge 0. \end{aligned} \tag{15}$$

Taking the integral in (15) over s, from 0 to $t < t^*(x)$, and from the fact that $h(\phi(t,x)) < h_1(\phi(t,x))$, $h(\phi(t^*(x),x)) \leq (r^v(\phi(t^*(x),x)) + Q^v h_2(\phi(t^*(x),x))$ (from d) of the Theorem) if $t^*(x) < \infty$, we obtain that

$$h(x) < J^v(h_1,h_2)(t,x),\ h(x) \leq K^v(h_2)(x)$$

and thus

$$h(x) = L(h_1,h_2)(x) = K^{v(u^*)}(h_2)(x).$$

Suppose now that iii) holds. Then from c) again and for all $s \in [0,\nu)$, we have that (14) holds. Taking the integral over s from 0 to ν, and recalling that $h(\phi(\nu,x)) = h_1(\phi(\nu,x))$, we obtain that

$$h(x) = J^{v(u^*)}(h_1,h_2)(\nu,x).$$

On the other hand, for any $v \in V(x)$ and $s \in [0,t^*(x))$, we have from b) that (15) holds. Recalling from a) that $h(\phi(t,x)) \leq h_1(\phi(t,x))$ for all $t \in [0,t^*(x))$, and from d) for the case $t^*(x) < \infty$, that $h(\phi(t^*(x),x)) \leq (r^v(\phi(t^*(x),x)) + Q^v h_2(\phi(t^*(x),x))$, we obtain that

$$h(x) \leq J^v(h_1,h_2)(t,x), h(x) \leq K^v(h_2)(x),$$

and thus

$$h(x) = L(h_1,h_2)(x) = J^{v(u^*)}(h_1,h_2)(\nu,x). \tag{16}$$

Finally notice that ii) is a particular case of the situation considered above, when $\nu = 0$. □

An immediate consequence of the above Theorem is the following Corollary.

Corollary 4.1. *Under the conditions of Theorem 4.1 and $u^* \in \mathcal{U}$ as defined in (11), one of the three possibilities below hold:*

i) $h(\phi(t,x)) < h_1(\phi(t,x))$ *for all* $t \in \mathcal{I}_0(x)$;
in this case $h(x) = K^{v(u^*)}(h_2)(x) = \underset{v\in V(x)}{inf}\ K^v(h_2)(x) \leq \underset{v\in V(x)}{inf}\ J_0^v(h_1,h_2)(x)$.

ii) $h(x) = h_1(x)$

iii) for some $\nu \in \mathcal{I}_0(x)$, *we have*
$h(\phi(t,x)) < h_1(\phi(t,x))$ *for all* $t \in [0,\nu)$, *and* $h(\phi(\nu,x)) = h_1(\phi(\nu,x))$; *in this case*
$h(x) = J^{v(u^*)}(h_1,h_2)(\nu,x) = \underset{v\in V(x)}{inf}\ J_0^v(h_1,h_2)(x) \leq \underset{v\in V(x)}{inf}\ K^v(h_2)(x)$.

Proof. Immediate from equations (11)-(16). □

5 Main Results

In this section we shall present the main result of the paper, showing that the value function $\rho(x)$ of the optimal stopping with continuous control problem posed in (3) can be uniquely characterized as the fixed point of the fist jump operator $L(\varphi,.)$ in $\mathbb{B}^{ac}(E)$ or, equivalently, as the unique solution in $\mathbb{B}^{ac}(E)$ of a set of variational inequalities. As a by-product, we also derive an ϵ-optimal solution for the problem.
We start by considering the truncated problem (4). Theorem 5.1 shows that the truncated value

function $\rho_n(x)$ can be obtained from iterations of the first jump operator $L(\varphi,.)$. Theorem 5.2 then follows by taking the appropriate limit as $n \to \infty$.
The results of this section follow the ideas presented in [11]. For $n = 1, 2, \ldots$, define the functions

$$\vartheta_{n+1} = L(\varphi, \vartheta_n),\ \vartheta_0 = \varphi.$$

From Theorem 4.1, we know that for each n, $\vartheta_n \in \mathbb{B}^{ac}(E)$ (recall from Section 2, condition f) that $\varphi \in \mathbb{B}^{ac}(E)$). Moreover as shown in the next Proposition, $\{\vartheta_n\}_n$ is a sequence of monotone decreasing functions in $\mathbb{B}^{ac}(E)$.

Proposition 5.1. *For each $n = 1, 2, \ldots$ and $x \in E$, $\vartheta_n(x) \le \vartheta_{n-1}(x)$.*

Proof. For $n = 1$,

$$\vartheta_1 = L(\varphi, \varphi) \le \varphi.$$

Suppose now that $\vartheta_n \le \vartheta_{n-1}$. Then

$$\vartheta_{n+1} = L(\varphi, \vartheta_n) \le L(\varphi, \vartheta_{n-1}) = \vartheta_n$$

□

For each $n = 1, 2, \ldots$, define $\xi_n \in \mathcal{U}$ such that for each $x \in E$, $z \in \partial^* E$

$$\begin{aligned} f^{\xi_n}(x) - \lambda^{\xi_n}(x) \int_E (\vartheta_n(x) - \vartheta_{n-1}(y)) Q^{\xi_n}(dy;x) \\ = min_{a \in U(x)} \{ f(x,a) - \lambda(x,a) \int_E (\vartheta_n(x) - \vartheta_{n-1}(y)) Q(dy;x,a) \} \\ r^{\xi_n}(z) + Q^{\xi_n} \vartheta_{n-1}(z) = min_{a \in U(z)} \{ r(z,a) + Q\vartheta_{n-1}(z,a) \}. \end{aligned} \tag{17}$$

As seen in Remark 2.1, such a choice is always possible. Set $\psi_n = (\xi_n, \xi_{n-1}, \ldots, \xi_1, u, u, \ldots) \in \Psi$, where we pick up an arbitrary $u \in \mathcal{U}$. For arbitrary $\epsilon > 0$, set for $n = 1, 2, \ldots$,

$$s_{n-1,\epsilon}(x) = \begin{cases} t^*(x) & \text{if } K^{v(\xi_n)}(\vartheta_{n-1})(x) \le J_0^{v(\xi_n)}(\varphi, \vartheta_{n-1})(x) \\ inf\Big\{ 0 \le s < t^*(x); J^{v(\xi_n)}(\varphi, \vartheta_{n-1})(s,x) & \\ \le J_0^{v(\xi_n)}(\varphi, \vartheta_{n-1})(x) + \epsilon \Big\} & \text{if } J_0^{v(\xi_n)}(\varphi, \vartheta_{n-1})(x) < K^{v(\xi_n)}(\vartheta_{n-1})(x) \end{cases} \tag{18}$$

and

$$\begin{aligned} S_{1,\epsilon} &= s_{o,\epsilon}(X_0) \wedge T_1^{\psi_1} \\ S_{n+1,2\epsilon} &= \begin{cases} s_{n,\epsilon}(X_0) \wedge T_1^{\psi_{n+1}} & \text{if } T_1^{\psi_{n+1}} > s_{n,\epsilon}(X_0) \\ T_1^{\psi_{n+1}} + \theta(T_1^{\psi_{n+1}}) S_{n,\epsilon} & \text{if } T_1^{\psi_{n+1}} \le s_{n,\epsilon}(X_0) \end{cases} \end{aligned} \tag{19}$$

where $\theta(T_1^{\psi_{n+1}}) S_{n,\epsilon}$ indicates that we are considering the stopping time $S_{n,\epsilon}$ for the Markov process $\{X_t^{\psi_n}\}$ starting at the point $Z_1^{\psi_{n+1}}$.

Theorem 5.1. *For any $x \in E$, $\epsilon > 0$, $n = 1, 2, \ldots$, we have that*

$$\begin{aligned} &a)\ \vartheta_n(x) \le \rho_n(x) \\ &b)\ \mathcal{J}_n^{\psi_n, S_{n,\epsilon}}(x) \le \vartheta_n(x) + \epsilon \end{aligned} \tag{20}$$

Proof. First of all notice from the definitions (17)-(19) that, according to the notation and results of sub-section 3.3, $\psi'_{n+1} = \psi_n$, $S'_{n+1,2\epsilon} = S_{n,\epsilon}$, and since there is no dependence of ψ'_{n+1}, $S'_{n+1,2\epsilon}$ on X_0, $T_1^{\psi_{n+1}}$, we have that,

$$\mathcal{J}_n^{\psi_n,S_{n,\epsilon}}\left(Z_1^{\psi_{n+1}}\middle|x, T_1^{\psi_{n+1}}\right) = \mathcal{J}_n^{\psi_n,S_{n,\epsilon}}\left(Z_1^{\psi_{n+1}}\right).$$

From (7), and (18), (19), we have for $n = 0, 1, \ldots$

$$\mathcal{J}_{n+1}^{\psi_{n+1},S_{n+1,2\epsilon}}(x) = \begin{cases} J^{v(\xi_{n+1})}(\varphi, \mathcal{J}_n^{\psi_n,S_{n,\epsilon}})(s_{n,\epsilon}(x), x) & \text{if } s_{n,\epsilon}(x) < t^*(x) \\ K^{v(\xi_{n+1})}(\mathcal{J}_n^{\psi_n,S_{n,\epsilon}})(x) & \text{if } s_{n,\epsilon}(x) = t^*(x) \end{cases} \tag{21}$$

where $\mathcal{J}_0^{\psi,\tau} = \varphi$. Let us show that a) and b) hold by using induction in n. For $n = 1$,it is easy to see from (7) and $\mathcal{J}_0^{\psi,\tau} = \varphi$ that

$$\mathcal{J}_1^{\psi,\tau}(x) \geq L(\varphi, \varphi)(x) = \vartheta_1(x),$$

for arbitrary $\psi \in \Psi$, $\tau \in \mathcal{M}^\psi$ showing a) for $n = 1$. For b), let us suppose first that condition i) in Corollary 4.1 holds with $h_1 = h_2 = \varphi$, and thus

$$\vartheta_1(x) = K^{v(\xi_1)}(\varphi)(x) = \mathop{inf}_{v\in V(x)} K^v(\varphi)(x) \leq \mathop{inf}_{v\in V(x)} J_0^v(\varphi, \varphi)(x) \leq J_0^{v(\xi_1)}(\varphi, \varphi)(x),$$

and from the definition (18), $s_{0,\epsilon}(x) = t^*(x)$. From (21) for $n = 0$, we obtain that

$$\mathcal{J}_1^{\psi_1,S_{1,\epsilon}}(x) = K^{v(\xi_1)}(\varphi)(x) = \vartheta_1(x), \tag{22}$$

and therefore b) holds. If conditions ii) or iii) of Corollary 4.1 holds then for some $\nu \in \mathcal{I}_0(x)$, we have

$$\begin{aligned}\vartheta_1(x) = J^{v(\xi_1)}(\varphi, \varphi)(\nu, x) &= \mathop{inf}_{v\in V(x)} J_0^v(\varphi, \varphi)(x) \\ &= J_0^{v(\xi_1)}(\varphi, \varphi)(x) \leq \mathop{inf}_{v\in V(x)} K^v(h_2)(x) \leq K^{v(\xi_1)}(\varphi)(x).\end{aligned}$$

If we have $J^{v(\xi_1)}(\varphi, \varphi)(\nu, x) = K^{v(\xi_1)}(\varphi)(x)$, then (22) holds and therefore we have b). If $J^{v(\xi_1)}(\varphi, \varphi)(\nu, x) < K^{v(\xi_1)}(\varphi)(x)$ then since $J^{v(\xi_1)}(\varphi, \varphi)(\nu, x) < J_0^{v(\xi_1)}(\varphi, \varphi)(x) + \epsilon$, it is clear from the definition of $s_{0,\epsilon}(x)$ that $0 \leq s_{0,\epsilon}(x) < \nu$ if $\nu > 0$, $s_{0,\epsilon}(x) = 0$ if $\nu = 0$. In both cases, $0 \leq s_{0,\epsilon}(x) < t^*(x)$, and from equation (21) and the definition of $s_{0,\epsilon}(x)$,

$$\mathcal{J}_1^{\psi_1,S_{1,\epsilon}}(x) = J^{v(\xi_1)}(\varphi, \varphi)(s_{0,\epsilon}(x), x) \leq J_0^{v(\xi_1)}(\varphi, \varphi)(x) + \epsilon = \vartheta_1(x) + \epsilon$$

completing the induction argument for $n = 1$.
Suppose now that a) and b) holds for n. For arbitrary $\psi \in \Psi$, $\tau \in \mathcal{M}^\psi$, we have from a), the induction hypothesis for n (so that $\mathcal{J}_n^{\psi',\tau'}(Z_1^{\psi_{n+1}}\Big|x, T_1^{\psi_{n+1}}) \geq \vartheta_n(Z_1^{\psi_{n+1}})$), and (7), that

$$\mathcal{J}_{n+1}^{\psi,\tau}(x) \geq L(\varphi, \vartheta_n)(x) = \vartheta_{n+1}(x)$$

showing a) for $n + 1$. Suppose that b) holds for n, and consider first that condition i) in Corollary 4.1 is true with $h_1 = \varphi$, $h_2 = \vartheta_n$, so that

$$\vartheta_{n+1}(x) = K^{v(\xi_{n+1})}(\vartheta_n)(x) = \mathop{inf}_{v\in V(x)} K^v(\vartheta_n)(x) \leq \mathop{inf}_{v\in V(x)} J_0^v(\varphi, \vartheta_n)(x) \leq J_0^{v(\xi_{n+1})}(\varphi, \vartheta_n)(x).$$

From the definition (18), $s_{n,\epsilon}(x) = t^*(x)$. From (21) for $n+1$, using the induction hypothesis $\mathcal{J}_n^{\psi_n, S_{n,\epsilon}}(y) \leq \vartheta_n(y) + \epsilon$ for all $y \in E$, and noticing that $K^{v(\xi_{n+1})}(\vartheta_n + \epsilon)(x) \leq K^{v(\xi_{n+1})}(\vartheta_n)(x) + \epsilon$, we obtain that

$$\begin{aligned} \mathcal{J}_{n+1}^{\psi_{n+1}, S_{n+1,2\epsilon}}(x) &= K^{v(\xi_{n+1})}(\mathcal{J}_n^{\psi_n, S_{n,\epsilon}})(x) \\ &\leq K^{v(\xi_{n+1})}(\vartheta_n + \epsilon)(x) \leq K^{v(\xi_{n+1})}(\vartheta_n)(x) + \epsilon = \vartheta_{n+1}(x) + \epsilon \end{aligned} \tag{23}$$

and therefore b) holds. If conditions ii) or iii) of Corollary 4.1 holds then for some $\nu \in \mathcal{I}_0(x)$, we have that

$$\begin{aligned} \vartheta_{n+1}(x) = J^{v(\xi_{n+1})}(\varphi, \vartheta_n)(\nu, x) &= \inf_{v \in V(x)} J_0^v(\varphi, \vartheta_n)(x) \\ &= J_0^{v(\xi_{n+1})}(\varphi, \vartheta_n)(x) \leq \inf_{v \in V(x)} K^v(\vartheta_n)(x) \leq K^{v(\xi_1)}(\vartheta_n)(x). \end{aligned}$$

If we have $J^{v(\xi_{n+1})}(\varphi, \vartheta_n)(\nu, x) = K^{v(\xi_{n+1})}(\vartheta_n)(x)$, then (23) holds and therefore we have b) . If $J^{v(\xi_{n+1})}(\varphi, \vartheta_n)(\nu, x) < K^{v(\xi_{n+1})}(\vartheta_n)(x)$ then since $J^{v(\xi_{n+1})}(\varphi, \vartheta_n)(\nu, x) < J_0^{v(\xi_{n+1})}(\varphi, \vartheta_n)(x) + \epsilon$, it is clear from the definition of $s_{n,\epsilon}(x)$ that $0 \leq s_{n,\epsilon}(x) < \nu$ if $\nu > 0$, $s_{n,\epsilon}(x) = 0$ if $\nu = 0$. In both cases, $0 \leq s_{n,\epsilon}(x) < t^*(x)$, and from equation (21), the definition of $s_{n,\epsilon}(x)$, and the fact that $J^{v(\xi_{n+1})}(\varphi, \vartheta_n + \epsilon) \leq J^{v(\xi_{n+1})}(\varphi, \vartheta_n) + \epsilon$,

$$\begin{aligned} \mathcal{J}_{n+1}^{\psi_{n+1}, S_{n+1,2\epsilon}}(x) &= J^{v(\xi_{n+1})}(\varphi, \mathcal{J}_n^{\psi_n, S_{n,\epsilon}})(s_{n,\epsilon}(x), x) \\ &\leq J^{v(\xi_{n+1})}(\varphi, \vartheta_n + \epsilon)(s_{n,\epsilon}(x), x) \\ &\leq J^{v(\xi_{n+1})}(\varphi, \vartheta_n)(s_{n,\epsilon}(x), x) + \epsilon \\ &\leq J_0^{v(\xi_{n+1})}(\varphi, \vartheta_n)(x) + 2\epsilon = \vartheta_{n+1}(x) + 2\epsilon \end{aligned}$$

completing the induction argument for $n+1$. □

From a) and b) of Theorem 5.1, it is immediate that for $n = 0, 1, \ldots$

$$\vartheta_n(x) = \rho_n(x),$$

and thus (see subsection 3.1 for definition of d_1)

$$\|\vartheta_n\| = \|\rho_n\| \leq \frac{\|f\|}{\alpha} + \|\varphi\| + d_1.$$

Therefore the monotone decreasing sequence of functions $\vartheta_n(.)$ converges to a function $\vartheta \in \mathbb{B}(E)$. We have the following result.

Proposition 5.2. *$\vartheta \in \mathbb{B}^{ac}(E)$ and satisfies*

$$\vartheta(x) = L(\varphi, \vartheta)(x)$$

for all $x \in E$.

Proof. From the monotone convergence Theorem,

$$\begin{aligned} \vartheta(x) = \inf_n \vartheta(x) &= \inf_{v \in V(x)} \Big(\inf_{0 \leq t < t^*(x)} \inf_n J^v(\varphi, \vartheta_n)(t, x) \Big) \wedge \inf_{v \in V(x)} \Big(\inf_n K^v(\vartheta_n)(x) \Big) \\ &= \inf_{v \in V(x)} \Big(\inf_{0 \leq t < t^*(x)} J^v(\varphi, \vartheta)(t, x) \Big) \wedge \inf_{v \in V(x)} \Big(K^v(\vartheta)(x) \Big) = L(\varphi, \vartheta)(x). \end{aligned}$$

From Proposition 4.4, we have that $\vartheta \in \mathbb{B}^{ac}(E)$. □

Consider $u^* \in \mathcal{U}$ such that for each $x \in E$

$$\begin{aligned} f^{u^*}(x) - \lambda^{u^*}(x)\int_E (\vartheta(x) - \vartheta(y))Q^{u^*}(dy;x) &= min_{a\in U(x)}\{f(x,a) \\ &\quad - \lambda(x,a)\int_E (\vartheta(x)-\vartheta(y))Q(dy;x,a)\} \\ r^{u^*}(z) + Q^{u^*}\vartheta(z) &= min_{a\in U(z)}\{r(z,a) + Q\vartheta(z,a)\} \end{aligned} \tag{24}$$

As seen in Remark 2.1, such a choice is always possible to be made. In terms of the infinitesimal operator defined in sub-section 3.2, we have

$$\begin{aligned} &\mathcal{X}\vartheta(x) - \alpha\vartheta(x) + min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E(\vartheta(x)-\vartheta(y))Q(dy;x,a)\} = \\ &\mathcal{A}^{u^*}\vartheta(x) - \alpha\vartheta(x) + f^{u^*}(x). \end{aligned} \tag{25}$$

For arbitrary $\epsilon > 0$, set

$$\tau_\epsilon = inf\{t \geq 0; e^{-\alpha t}\varphi(X_t^{u^*}) \leq e^{-\alpha t}\vartheta(X_t^{u^*}) + \epsilon\} \in \mathcal{M}^{u^*}. \tag{26}$$

From boundedness of φ, ϑ, it follows that $\tau_\epsilon < \infty$ w.p.1.

Proposition 5.3. *For all $x \in E$,*

$$\vartheta(x) = E_x^{u^*}\Big(\int_0^{\tau_\epsilon} e^{-\alpha s} f^{u^*}(X_s^{u^*})ds + \int_0^{\tau_\epsilon} e^{-\alpha s} r^{u^*}(X_{s-}^{u^*})dp^{u^**}(s) + e^{-\alpha\tau_\epsilon}\vartheta(X_{\tau_\epsilon}^{u^*})\Big). \tag{27}$$

Proof. Let us write

$$\begin{aligned} \mathcal{W}_n(x) = E_x^{u^*}\Big(&\int_0^{\tau_\epsilon\wedge T_n^{u^*}} e^{-\alpha s} f^{u^*}(X_s^{u^*})ds + \int_0^{\tau_\epsilon\wedge T_n^{u^*}} e^{-\alpha s} r^{u^*}(X_{s-}^{u^*})dp^{u^**}(s) \\ &+ e^{-\alpha\tau_\epsilon\wedge T_n^{u^*}}\vartheta(X_{\tau_\epsilon\wedge T_n^{u^*}}^{u^*})\Big). \end{aligned} \tag{28}$$

We want to show first that for each $n = 0, 1, \ldots$

$$\vartheta(x) = \mathcal{W}_n(x). \tag{29}$$

Using the same arguments as those in equation (6) of Proposition 3.2, we have that

$$\begin{aligned} \mathcal{W}_n(x) = E_x^{u^*}\Bigg(&\Big(\int_0^{s_\epsilon(x)} e^{-\alpha s} f^{u^*}(\phi(s,x))ds + e^{-\alpha s_\epsilon(x)}\varphi(\phi(s_\epsilon(x),x))\Big)1_{\{T_1^{u^*} > s_\epsilon(x)\}} + \\ &\Big(\int_0^{T_1^{u^*}} e^{-\alpha s} f^{\psi}(\phi(s,x))ds + \int_0^{T_1^{u^*}} e^{-\alpha s} r^{\psi}(\phi(s^-,x))dp^{u^**}(s) + \\ &e^{-\alpha T_1^{u^*}}\mathcal{W}_{n-1}(Z_1^{u^*})\Big)1_{\{T_1^{u^*}\leq s_\epsilon(x)\}}\Bigg) \end{aligned} \tag{30}$$

where, from the representation theorem for $\mathcal{F}_t^{u^*}$-stopping times we have that,

$$\tau_\epsilon \wedge T_1^{u^*} = s_\epsilon(X_0) \wedge T_1^{u^*}$$

and, from (26), the measurable function $s_\epsilon(.) : E \to [0,\infty]$ is defined as

$$s_\epsilon(x) = inf\{0 \le t \le t^*(x); e^{-\alpha t}\varphi(\phi(t,x)) \le e^{-\alpha t}\vartheta(\phi(t,x)) + \epsilon\}. \tag{31}$$

From (31) it is immediate to see that, if $s_\epsilon(x) = t^*(x)$, then for $0 \le t < t^*(x)$,

$$e^{-\alpha t}\varphi(\phi(t,x)) > e^{-\alpha t}\vartheta(\phi(t,x)) + \epsilon > e^{-\alpha t}\vartheta(\phi(t,x)), \tag{32}$$

that is, for all $t \in \mathcal{I}_0(x)$,

$$\varphi(\phi(t,x)) > \vartheta(\phi(t,x)) \tag{33}$$

and in this case, from the same arguments as in equation (14), we get that

$$\vartheta(x) = K^{v(u^*)}(\vartheta)(x). \tag{34}$$

Similarly, if $0 < s_\epsilon(x) < t^*(x)$, then for all $0 \le t < s_\epsilon(x)$, (32) and (33) are satisfied, and from the same arguments as in equations (14) to (16),

$$\vartheta(x) = L^{v(u^*)}(\varphi,\vartheta)(s_\epsilon(x), x). \tag{35}$$

If $s_\epsilon(x) = 0$ it is obvious that (35) also holds. By direct calculation of (30) it follows that

$$\mathcal{W}_n(x) = \begin{cases} J^{v(u^*)}(\varphi, W_{n-1})(s_\epsilon(x), x) & \text{if } s_\epsilon(x) < t^*(x) \\ K^{v(u^*)}(W_{n-1})(x) & \text{if } s_\epsilon(x) = t^*(x) \end{cases} \tag{36}$$

where $W_0 = \vartheta$. Let us show by induction in n that (29) holds. For $n = 1$, it is immediate from (34), (35) and (36) that (29) holds. Similar suppose that (29) holds for n. Then from (36),

$$\mathcal{W}_{n+1}(x) = \begin{cases} J^{v(u^*)}(\varphi, \vartheta)(s_\epsilon(x), x) & \text{if } s_\epsilon(x) < t^*(x) \\ K^{v(u^*)}(\vartheta)(x) & \text{if } s_\epsilon(x) = t^*(x) \end{cases} \tag{37}$$

and again from (34), (35) and (37) we have that (29) holds. Taking the limit as $n \to \infty$ in (28), and using the bounded convergence theorem, we deduce from (29) that (27) holds, (since by the assumption in sub-section 3.2, $T_n^{u^*} \to \infty$, w.p.1), completing the proof of the Proposition. □

We can now prove the main result of the paper, which characterizes the value function $\rho(x)$ as the unique fixed point of $L(\varphi, .)$ in $\mathbb{B}^{ac}(E)$ or, equivalently, as the unique solution in $\mathbb{B}^{ac}(E)$ of a set of variational inequalities. Moreover, we also derive an ϵ-optimal solution for the problem.

Theorem 5.2. *The following assertions are equivalent:*
1) There exists a unique solution ϑ *in the class* $\mathbb{B}^{ac}(E)$ *to the system*

$$w(x) = L(\varphi, w)(x)$$

for all $x \in E$.
2) There exists a unique solution ϱ *in the class* $\mathbb{B}^{ac}(E)$ *to the system,*

2.a) $w(x) \le \varphi(x)$
2.b) $\mathcal{X}w(x) - \alpha w(x)$
$+ min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (w(x) - w(y))Q(dy;x,a)\} \ge 0$
2.c) $\Big(\varphi(x) - w(x)\Big)\Big(\mathcal{X}w(x) - \alpha w(x) +$
$min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (w(x) - w(y))Q(dy;x,a)\}\Big) = 0$

for all $x \in E$, and

$$2.d)\ w(z) = \varphi(z) \wedge min_{a\in U(z)}\{r(z,a) + Qw(z,a)\}$$

for $z \in \partial^* E$.
Moreover, $\rho(x) = \varrho(x) = \vartheta(x)$ for all $x \in E$, and given any $\epsilon > 0$, we have that

$$\mathcal{J}^{u^*,\tau_\epsilon}(x) \leq \rho(x) + \epsilon \tag{38}$$

where $u^* \in \mathcal{U}$ is as defined in (24) and $\tau_\epsilon \in \mathcal{M}^{u^*}$ as in (26).

Proof. As seen in Proposition 5.2, there exists $\vartheta \in \mathbb{B}^{ac}(E)$ which satisfies $\vartheta(x) = L(\varphi, \vartheta)(x)$, and from Theorem 4.1, ϑ satisfies conditions 2.a), 2.b), 2.c) and 2.d). It is enough to show now that $\vartheta(x) = \rho(x)$ and that (38) is satisfied. From Theorem 5.1, for arbitrary $\psi \in \Psi$, $\tau \in \mathcal{M}^\psi$

$$\mathcal{J}_n^{\psi,\tau}(x) \geq \rho_n(x) = \vartheta_n(x) \geq \vartheta(x).$$

Taking the limit as $n \to \infty$,we have from the bounded convergence theorem, and the fact that by hypothesis $T_n^\psi \to \infty$ as $n \to \infty$, that

$$\lim_{n\to\infty} \mathcal{J}_n^{\psi,\tau}(x) = \mathcal{J}^{\psi,\tau}(x) \geq \vartheta(x) \tag{39}$$

and from (39) is clear that $\rho(x) \geq \vartheta(x)$. On the other hand, from (27) in Proposition 5.3,

$$\begin{aligned}
\vartheta(x) &= E_x^{u^*}\Big(\int_0^{\tau_\epsilon} e^{-\alpha s} f^{u^*}(X_s^{u^*})ds + \int_0^{\tau_\epsilon} e^{-\alpha s} r^{u^*}(X_{s-}^{u^*})dp^{u^**}(s) + e^{-\alpha\tau_\epsilon}\vartheta(X_{\tau_\epsilon}^{u^*})\Big)\\
&\geq E_x^{u^*}\Big(\int_0^{\tau_\epsilon} e^{-\alpha s} f^{u^*}(X_s^{u^*})ds + \int_0^{\tau_\epsilon} e^{-\alpha s} r^{u^*}(X_{s-}^{u^*})dp^{u^**}(s) + e^{-\alpha\tau_\epsilon}\varphi(X_{\tau_\epsilon}^{u^*})\Big) - \epsilon\\
&= \mathcal{J}^{u^*,\tau_\epsilon}(x) - \epsilon \geq \rho(x) - \epsilon
\end{aligned} \tag{40}$$

where the last inequality follows from the definition (26) of τ_ϵ, right-continuity of $\{X_t^{u^*}\}$, and continuity along trajectories of the functions $\varrho(.)$ and $\varphi(.)$. Since ϵ is arbitrary, we get from (40) that $\vartheta(x) \geq \rho(x)$ and, together with (39) we obtain that $\vartheta(x) = \rho(x)$. From (40) we get (38), completing the proof of the Theorem. $\square$

Appendix

Proof of Proposition 4.1: From the semigroup property of the flow $\phi(t,x)$ we have that

$$\phi(s, \phi(t,x)) = \phi(s+t, x),\ t^*(\phi(t,x)) = t^*(x) - t. \tag{41}$$

For an arbitrary $w' \in V(\phi(t,x))$, define $w \in V_t(x)$ as

$$w(s) = w'(s-t), \text{for } t \leq s,\ s \in \mathcal{I}_0(x). \tag{42}$$

Then for any $0 \leq s < t^*(\phi(t,x))$ and $v \in V(x)$ we have from (41) and (42) that

$$\begin{aligned}
\Lambda^{w'}(\ell, \phi(t,x)) &= \int_0^\ell \lambda(\phi(\nu, \phi(t,x)), w'(\nu))d\nu = \int_0^\ell \lambda(\phi(\nu+t, x), w'(\nu))d\nu\\
&= \int_t^{\ell+t} \lambda(\phi(\nu,x), w(\nu))d\nu\\
&= \Lambda^{q_t(v,w)}(\ell+t, x) - \Lambda^v(t,x),
\end{aligned}$$

and

$$
\begin{aligned}
J^{w'}(h_1,h_2)(s,\phi(t,x)) &= e^{\alpha t+\Lambda^v(t,x)}\bigg(\int_t^{t+s} e^{-\alpha\ell-\Lambda^{q_t(v,w)}(\ell,x)}(f^{q_t(v,w)}(\phi(\ell,x)) \\
&\quad + \lambda^{q_t(v,w)}(\phi(\ell,x))Q^{q_t(v,w)}h_2(\phi(\ell,x)))d\ell \\
&\quad + e^{-\alpha(s+t)-\Lambda^{q_t(v,w)}(s+t,x)}h_1(\phi(s+t,x))\bigg) \\
&= e^{\alpha t+\Lambda^v(t,x)}\bigg(J^{q_t(v,w)}(h_1,h_2)(s+t,x) - \\
&\quad \int_0^t e^{-\alpha\ell-\Lambda^v(\ell,x)}(f^v(\phi(\ell,x))+\lambda^v(\phi(\ell,x))Q^v h_2(\phi(\ell,x)))d\ell\bigg)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
K^{w'}(h_2)(\phi(t,x)) =& e^{\alpha t+\Lambda^v(t,x)}\bigg(K^{q_t(v,w)}(h_2)(x) - \\
& \int_0^t e^{-\alpha\ell-\Lambda^v(\ell,x)}(f^v(\phi(\ell,x))+\lambda^v(\phi(\ell,x))Q^v h_2(\phi(\ell,x)))d\ell\bigg).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
L(h_1,h_2)(\phi(t,x)) &= \inf_{w'\in V(\phi(t,x))}\bigg(\Big\{\inf_{0\le s<t^*(x)-t} J^{w'}(h_1,h_2)(s,\phi(t,x))\Big\}\wedge K^{w'}(h_2)(\phi(t,x))\bigg) \\
&= \bigg\{\inf_{w\in V_t(x)}\bigg(\Big(\inf_{t\le s<t^*(x)} J^{q_t(v,w)}(h_1,h_2)(s,x)\Big)\wedge K^{q_t(v,w)}(h_2)(x)\bigg) - \\
&\quad \int_0^t e^{-\alpha\ell-\Lambda^v(\ell,x)}(f^v(\phi(\ell,x))+\lambda^v(\phi(\ell,x))Q^v h_2(\phi(\ell,x)))d\ell\bigg\}e^{\alpha t+\Lambda^v(t,x)}
\end{aligned}
$$

proving (8). $\square$

Proof of Proposition 4.2: For $0\le t<s$, $s\in\mathcal{I}_0(x)$, let $v_t, v_s\in V(x)$ be such that $v_t(\ell)=v_s(\ell)$ for $\ell\in[0,t)\cup\mathcal{I}_s(x)$. Then for any $\nu\in[t,t^*(x))$,

$$
\begin{aligned}
\Lambda^{v_t}(\nu,x) &= \int_0^\nu \lambda^{v_t}(\phi(\ell,x))d\ell = \int_0^t \lambda^{v_t}(\phi(\ell,x))d\ell \\
&\quad + \int_t^{s\wedge\nu}\lambda^{v_t}(\phi(\ell,x))d\ell + \int_{s\wedge\nu}^{\nu}\lambda^{v_t}(\phi(\ell,x))d\ell \\
&\le \int_0^t \lambda^{v_s}(\phi(\ell,x))d\ell + \int_t^{s\wedge\nu}\lambda^{v_s}(\phi(\ell,x))d\ell + \int_{s\wedge\nu}^{\nu}\lambda^{v_s}(\phi(\ell,x))d\ell \\
&\quad + \int_t^{s\wedge\nu}\left|\lambda^{v_s}(\phi(\ell,x))-\lambda^{v_t}(\phi(\ell,x))\right|d\ell \le \Lambda^{v_s}(\nu,x)+2\left\|\lambda\right\|(s-t)
\end{aligned}
$$

and similarly $\Lambda^{v_s}(\nu,x)\le\Lambda^{v_t}(\nu,x)+2\left\|\lambda\right\|(s-t)$, which shows that

$$
\left|\Lambda^{v_t}(\nu,x)-\Lambda^{v_s}(\nu,x)\right|\le 2\left\|\lambda\right\|(s-t).
$$

From this, we conclude that

$$
\begin{aligned}
e^{-\Lambda^{v_t}(\nu,x)}-e^{-\Lambda^{v_s}(\nu,x)} &\le e^{-\Lambda^{v_t}(\nu,x)}-e^{-\Lambda^{v_t}(\nu,x)}e^{-2\|\lambda\|(s-t)} = e^{-\Lambda^{v_t}(\nu,x)}(1-e^{-2\|\lambda\|(s-t)}) \\
&\le (1-e^{-2\|\lambda\|(s-t)}) \le 2\left\|\lambda\right\|(s-t)
\end{aligned}
$$

and similarly, $e^{-\Lambda^{v_s}(\nu,x)} - e^{-\Lambda^{v_t}(\nu,x)} \leq 2\|\lambda\|(s-t)$, showing that

$$\left|e^{-\Lambda^{v_s}(\nu,x)} - e^{-\Lambda^{v_t}(\nu,x)}\right| \leq 2\|\lambda\|(s-t).$$

We have now that

$$\begin{aligned}
&J^{v_t}(h_1,h_2)(\nu,x) - J^{v_s}(h_1,h_2)(\nu,x) \\
&= \int_t^s e^{-(\alpha\ell+\Lambda^{v_t}(\ell,x))}\Bigg(\Big\{f^{v_t}(\phi(\ell,x)) + \lambda^{v_t}(\phi(\ell,x))Q^{v_t}h_2(\phi(\ell,x))\Big\} - \\
&\Big\{f^{v_s}(\phi(\ell,x)) + \lambda^{v_s}(\phi(\ell,x))Q^{v_s}h_2(\phi(\ell,x))\Big\}\Bigg)d\ell + \\
&\int_t^\nu \Big\{e^{-(\alpha\ell+\Lambda^{v_t}(\ell,x))} - e^{-(\alpha\ell+\Lambda^{v_s}(\ell,x))}\Big\}\Bigg(f^{v_s}(\phi(\ell,x)) + \lambda^{v_s}(\phi(\ell,x))Q^{v_s}h_2(\phi(\ell,x))\Bigg)d\ell + \\
&e^{-\alpha\nu}h_1(\phi(\nu,x))(e^{-\Lambda^{v_t}(\nu,x)} - e^{-\Lambda^{v_s}(\nu,x)}) \\
&\leq 2(\|f\| + \|\lambda\|\,\|h_2\|)(s-t) + (\|f\| + \|\lambda\|\,\|h_2\|)\int_t^\nu e^{-\alpha\ell}\left|e^{-\Lambda^{v_t}(\ell,x)} - e^{-\Lambda^{v_s}(\ell,x)}\right|d\ell + \\
&\|h_1\|\left|e^{-\Lambda^{v_t}(\nu,x)} - e^{-\Lambda^{v_s}(\nu,x)}\right| \\
&\leq \Big(\Big\{\|f\| + \|\lambda\|\,\|h_2\|\Big\}\Big\{2 + \frac{1}{\alpha}\|\lambda\|\Big\} + 2\|h_1\|\,\|\lambda\|\Big)(s-t) \leq c(s-t)
\end{aligned}$$

and similarly,

$$J^{v_s}(h_1,h_2)(\nu,x) - J^{v_t}(h_1,h_2)(\nu,x) \leq c(s-t)$$

showing that indeed,

$$\left|J^{v_t}(h_1,h_2)(\nu,x) - J^{v_s}(h_1,h_2)(\nu,x)\right| \leq c(s-t).$$

By repeating the arguments above we get that

$$\left|K^{v_t}(h_2)(x) - K^{v_s}(h_2)(x)\right| \leq c(s-t)$$

completing the proof of the Proposition. □

Proof of Proposition 4.3: We have two possibilities:

$$\begin{aligned}
&\text{A)}M^v(h_1,h_2)(t_i,x) < M^v(h_1,h_2)(s_i,x) \\
&\text{or} \\
&\text{B) } M^v(h_1,h_2)(t_i,x) = M^v(h_1,h_2)(s_i,x).
\end{aligned}$$

The result is clearly true if B) holds, by making $\bar{t}_i = s_i$. Let us suppose that A) holds, that is, $M^v(h_1,h_2)(t_i,x) < M^v(h_1,h_2)(s_i,x)$. In this case we also have two possibilities:

$$\text{A1) } M^v(h_1,h_2)(t_i,x) = \inf_{v_{t_i}\in V_{t_i}(x)}\Big(\inf_{t_i\leq s<t^*(x)} J^{q_{t_i}(v,v_{t_i})}(h_1,h_2)(s,x)\Big)$$

or

$$\text{A2) } M^v(h_1,h_2)(t_i,x) = \inf_{v_{t_i}\in V_{t_i}(x)} K^{q_{t_i}(v,v_{t_i})}(h_2)(x).$$

Consider first A1). Then we can find $v_{t_i} \in V_{t_i}(x)$ and $t_i \leq \bar{t}'_i < t^*(x)$ such that

$$J^{q_{t_i}(v,v_{t_i})}(h_1,h_2)(\bar{t}'_i,x) \leq M^v(h_1,h_2)(t_i,x)+\zeta.$$

Moreover, let $v_{s_i} \in V_{s_i}(x)$ be defined as

$$v_{s_i}(\ell) = v_{t_i}(\ell) \text{ for all } \ell \in \mathcal{I}_{s_i}(x)$$

so that

$$q_{s_i}(v,v_{s_i})(\ell) = q_{t_i}(v,v_{t_i})(\ell) \text{ for all } \ell \notin [t_i,s_i).$$

Indeed, $q_{s_i}(v,v_{s_i})(\ell) = v(\ell) = q_{t_i}(v,v_{t_i})(\ell)$ for $\ell \in [0,t_i)$, and $q_{s_i}(v,v_{s_i})(\ell) = v_{t_i}(\ell) = q_{t_i}(v,v_{t_i})(\ell)$ for $\ell \geq s_i$; for $\ell \in [t_i,s_i)$, $q_{t_i}(v,v_{t_i})(\ell) = v_{t_i}(\ell)$ and $q_{s_i}(v,v_{s_i})(\ell) = v(\ell)$. We get that

$$M^v(h_1,h_2)(s_i,x) \leq J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{s}_i,x)$$

where $\bar{s}_i = max\{\bar{t}'_i, s_i\}$. Note that

$$J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{s}_i,x) - J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{t}'_i,x) = J^v(h_1,h_2)(\bar{s}_i,x) - J^v(h_1,h_2)(\bar{t}'_i,x)$$

(indeed, if $\bar{t}'_i \geq s_i$ then $\bar{s}_i = \bar{t}'_i$ and the above expression is zero; otherwise, $\bar{s}_i = s_i$ and $q_{s_i}(v,v_{s_i})(\ell)$ $= v(\ell)$ for $\ell \in [0,s_i)$ and $\ell \in [0,\bar{t}'_i)$). Therefore, from Proposition 4.2,

$$\begin{aligned}
&\left|J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{s}_i,x) - J^{q_{t_i}(v,v_{t_i})}(h_1,h_2)(\bar{t}'_i,x)\right| \\
&\leq \left|J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{s}_i,x) - J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{t}'_i,x)\right| + \\
&\left|J^{q_{t_i}(v,v_{t_i})}(h_1,h_2)(\bar{t}'_i,x) - J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{t}'_i,x)\right| \\
&= \left|J^v(h_1,h_2)(\bar{s}_i,x) - J^v(h_1,h_2)(\bar{t}'_i,x)\right| + \\
&\left|J^{q_{t_i}(v,v_{t_i})}(h_1,h_2)(\bar{t}'_i,x) - J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{t}'_i,x)\right| \\
&\leq \left|J^v(h_1,h_2)(\bar{s}_i,x) - J^v(h_1,h_2)(\bar{t}'_i,x)\right| + c(s_i-t_i).
\end{aligned}$$

Defining $\bar{t}_i = min\{\bar{t}'_i, s_i\}$, it follows that

$$\begin{aligned}
0 &\leq M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x) \\
&\leq J^{q_{s_i}(v,v_{s_i})}(h_1,h_2)(\bar{s}_i,x) - J^{q_{t_i}(v,v_{t_i})}(h_1,h_2)(\bar{t}'_i,x) + \zeta \\
&\leq \left|J^v(h_1,h_2)(\bar{s}_i,x) - J^v(h_1,h_2)(\bar{t}'_i,x)\right| + c(s_i-t_i) + \zeta \\
&\leq \left|J^v(h_1,h_2)(s_i,x) - J^v(h_1,h_2)(\bar{t}_i,x)\right| + c(s_i-t_i) + \zeta.
\end{aligned}$$

Similarly, if A2) holds, then by the same arguments as before, we can find $v_{t_i} \in V_{t_i}(x)$ such that

$$K^{q_{t_i}(v,v_{t_i})}(h_2)(x) \leq M^v(h_1,h_2)(t_i,x)+\zeta$$

and defining $v_{s_i} \in V_{s_i}(x)$ as before, we get from Proposition 4.2 that

$$K^{q_{t_i}(v,v_{t_i})}(h_2)(x) - K^{q_{s_i}(v,v_{s_i})}(h_2)(x) \leq c(s_i-t_i).$$

Since

$$M^v(h_1,h_2)(s_i,x) \le K^{q_{s_i}(v,v_{s_i})}(h_2)(x)$$

we get that

$$\begin{aligned}
0 &\le M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x) \\
&\le K^{q_{s_i}(v,v_{s_i})}(h_2)(x) - K^{q_{t_i}(v,v_{t_i})}(h_2)(x) + \zeta \\
&\le c(s_i - t_i) + \zeta.
\end{aligned}$$

and the result holds by making $\bar{t}_i = s_i$. This completes the proof of the Proposition. $\square$

Proof of Proposition 4.4: Consider $v \in V(x)$ and $t \in [0, t^*(x))$. Given $\epsilon > 0$, take $0 < \delta < \frac{\epsilon}{3c}$, such that

$$\Big| J^v(h_1,h_2)(t,x) - J^v(h_1,h_2)(s,x) \Big| < \frac{\epsilon}{6} \quad \text{whenever } \Big| t-s \Big| < \delta \text{ and } s \in [0, t^*(x)).$$

Note that such δ exists from the fact that $J^v(.,x)$ is continuous in $[0, t^*(x))$. We want to show first that

$$0 \le M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x)) = \Big| M^v(h_1,h_2)(t,x) - M^v(h_1,h_2)(s,x) \Big| < \epsilon,$$

where $s_i = max\{t,s\}$ and $t_i = min\{t,s\}$. Choose in Proposition 4.3, $\zeta = \frac{\epsilon}{3}$. Then for some $\bar{t}_i$ such that $0 \le t_i \le \bar{t}_i \le s_i < t^*(x)$,

$$\begin{aligned}
0 &\le M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x) \\
&\le \Big| J^v(h_1,h_2)(s_i,x) - J^v(h_1,h_2)(\bar{t}_i,x) \Big| + c(s_i - t_i) + \frac{\epsilon}{3} \\
&\le \Big| J^v(h_1,h_2)(s_i,x) - J^v(h_1,h_2)(t,x) \Big| + \Big| J^v(h_1,h_2)(t,x) - J^v(h_1,h_2)(\bar{t}_i,x) \Big| + c(s_i - t_i) + \frac{\epsilon}{3} \\
&\le \frac{\epsilon}{3} + c\delta + \frac{\epsilon}{3} \le \epsilon.
\end{aligned}$$

where the second inequality follows from the fact that $0 \le s_i - t = \Big| t - s_i \Big| \le \Big| t - s \Big| \le \delta$ and that $\Big| \bar{t}_i - t \Big| \le \Big| t - s \Big| \le \delta$. This shows continuity of $M^v(h_1,h_2)(.,x)$ in $[0, t^*(x))$. From (8) in Proposition 4.1 we have continuity of $L(h_1,h_2)(\phi(.,x))$ in $[0,t^*(x))$.From Proposition 4.2,

$$\begin{aligned}
&\left| \inf_{w \in V_t(x)} \left\{ \inf_{t \le s < t^*(x)} J^{q_t(v,w)}(h_1,h_2)(s,x) \right\} - J^v(h_1,h_2)(t^*(x),x) \right| \\
&\le \inf_{w \in V_t(x)} \left\{ \inf_{t \le s < t^*(x)} \Big| J^{q_t(v,w)}(h_1,h_2)(s,x) - J^v(h_1,h_2)(t^*(x),x) \Big| \right\} \\
&\le \inf_{w \in V_t(x)} \left\{ \inf_{t \le s < t^*(x)} \Big| J^{q_t(v,w)}(h_1,h_2)(s,x) - J^v(h_1,h_2)(s,x) \Big| \right\} + \\
&\inf_{t \le s < t^*(x)} \Big| J^v(h_1,h_2)(s,x) - J^v(h_1,h_2)(t^*(x),x) \Big| \\
&\le c(t^*(x) - t) + \inf_{t \le s < t^*(x)} \Big| J^v(h_1,h_2)(s,x) - J^v(h_1,h_2)(t^*(x),x) \Big|
\end{aligned}$$

and since $J^v(h_1,h_2)(t,x) \to J^v(h_1,h_2)(t^*(x),x)$ as $t \uparrow t^*(x)$, we conclude from the above expression that

$$\inf_{w\in V_t(x)} \left\{ \inf_{t\le s<t^*(x)} J^{q_t(v,w)}(h_1,h_2)(s,x) \right\} \to J^v(h_1,h_2)(t^*(x),x) \text{ as } t \uparrow t^*(x).$$

Similarly, we can show that

$$\inf_{w\in V_t(x)} \left\{ K^{q_t(v,w)}(h_2)(x) \right\} \to K^v(h_2)(x) \text{ as } t \uparrow t^*(x)$$

where $v \in V(x)$ is such that, for $z = \phi(t^*(x),x)$,

$$r(z,v(z)) + Qh_2(z,v(z)) = min_{a\in U(z)}\{r(z,a) + Qh_2(z,a)\}.$$

By taking the limit as $t \to t^*(x)$ in (8), we finally get from the above equations that

$$lim_{t\uparrow t^*(x)} L(h_1,h_2)(\phi(t,x)) = h_1(z) \wedge min_{a\in U(z)}\{r(z,a) + Qh_2(z,a)\}$$

completing the proof of the Proposition. □

Proof of Proposition 4.5: Consider $v \in V(x)$ arbitrary. We want to show that $M^v(h_1,h_2)(.,x)$ is absolutely continuous in $[0,t^*(x))$, that is (see [14]), given any $\epsilon > 0$, we can find $\delta > 0$ such that for arbitrary $\{(t_i,s_i)\}_{i=1}^n$ satisfying $\sum_{i=1}^n (s_i - t_i) < \delta$, (t_i,s_i) disjoint intervals of $[a,b] \subset [0,t^*(x))$, we have that

$$\sum_{i=1}^{n} (M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x)) = \sum_{i=1}^{n} \Big| M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x) \Big| < \epsilon.$$

It is easy to check that $J^v(s,x)$ is absolutely continuous in $s \in [0,t^*(x))$ (it is the sum of 2 absolutely continuous). Therefore we can find $\delta > 0$, $\delta < \frac{\epsilon}{3c}$, such that

$$\sum_{i=1}^{n'} \Big| J^v(h_1,h_2)(t'_i,x) - J^v(h_1,h_2)(s'_i,x) \Big| < \frac{\epsilon}{3} \text{ whenever } \sum_{i=1}^{n'} (s'_i - t'_i) < \delta \tag{43}$$

for any disjoint intervals $\{(t'_i,s'_i)\}_{i=1}^{n'}$ of $[a,b] \subset [0,t^*(x))$. Consider an arbitrary $\{(t_i,s_i)\}_{i=1}^n$ satisfying $\sum_{i=1}^n (s_i - t_i) < \delta$, (t_i,s_i) disjoint intervals of $[a,b] \subset [0,t^*(x))$. Setting $\zeta = \frac{\epsilon}{3n}$ in (11) of Proposition 4.3, we have that

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} (M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x)) \\ &\le \sum_{i=1}^{n} \Big| J^v(h_1,h_2)(s_i,x) - J^v(h_1,h_2)(\bar{t}_i,x) \Big| + c\sum_{i=1}^{n} (s_i - t_i) + \frac{\epsilon}{3}. \end{aligned}$$

But since $\sum_{i=1}^n (s_i - \bar{t}_i) \le \sum_{i=1}^n (s_i - t_i) < \delta$, we can conclude from (43) that

$$\sum_{i=1}^{n} \Big| J^v(h_1,h_2)(s_i,x) - J^v(h_1,h_2)(\bar{t}_i,x) \Big| < \frac{\epsilon}{3}$$

and thus

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n}(M^v(h_1,h_2)(s_i,x) - M^v(h_1,h_2)(t_i,x)) \\
&\leq \frac{\epsilon}{3} + c\sum_{i=1}^{n}(s_i - t_i) + \frac{\epsilon}{3} \\
&\leq \frac{2\epsilon}{3} + c\delta \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

This shows that $M^v(h_1,h_2)(t,x)$ is absolutely continuous in $t \in [0, t^*(x))$. □

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