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Necessary and sufficient conditions for non-singular invariant probability measures for Feller Markov chains☆

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Abstract

In this paper, we present necessary and sufficient conditions for the existence of a non-singular invariant probability measure for a Feller Markov chain taking values on a locally compact separable metric space. The necessary and sufficient condition is written in terms of the Foster's criterion with an extra requirement. Furthermore, we extend an assumption recently presented by the authors Costa and Dufour, Statist. Probab. Lett. 50 (3) (2000) 13–21, named T2 condition, which generalizes T-chain and irreducibility assumptions for Feller Markov chains on a locally compact separable metric space, and show that under this assumption the extra requirement on the Foster's criterion can be eliminated. © 2001 Elsevier Science B.V. All rights reserved

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1. Introduction

We consider Markov chains defined on a locally separable metric space satisfying the Feller property. This work is concerned with the existence of an invariant probability measure (i.p.m. for short) for those chains. This problem of great importance corresponds to a form of stochastic stability and it is the first step towards an ergodic analysis of the process. In general it is not a simple matter to determine whether a given process in a general state space has an invariant probability measure. For a small sample of the huge theory nowadays

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available on this subject, see Hernandez-Lerma and Lasserre (1998a), Lasserre (1997), Meyn and Tweedie (1992, 1993a, 1993b).

In Costa and Dufour (2000), it was shown that the Foster–Lyapunov criteria is equivalent to the existence of a non-singular i.p.m. under a new assumption which generalizes the concept of T-chain and irreducibility. The current paper can be considered as a continuation of Costa and Dufour (2000), in the sense that we present here a necessary and sufficient condition to ensure the existence of a non-singular i.p.m. without any additional assumptions. This condition is given in terms of the Foster–Lyapunov criteria and an extra technical condition.

More specifically, let $(X,\mathscr{B}(X))$ be a locally compact metric space equipped with its Borel σ-field and $\mathbb{N}_* \doteq \mathbb{N}-\{0\}$. For any $A\in\mathscr{B}(X)$, $A^{\rm c}\doteq X-A$ and $I_A(x)$ is the indicator function associated to A. Let P be a stochastic kernel defined on $(X,\mathscr{B}(X))$ satisfying the Feller hypothesis (i.e. P maps the space of continuous bounded functions into itself). We shall denote by $\{Y_n\}_{n\in\mathbb{N}}$ the Markov chain generated by the stochastic kernel P (i.e. $P(x,A)=P_x(Y_1\in A)$ for all $x\in X$ and $A\in\mathscr{B}(X)$). We shall write $P^n(x,A)=P_x(Y_n\in A)$ for all $n\geqslant 1$. Let us consider the following definitions.

Definition 1.1. A probability measure μ is said to be non-singular with respect to P if there exists an integer k such that the Radon–Nikodym derivative of $P^k(x,.)$ with respect to μ, that is, $\mathrm{d}P^k(x,.)/\mathrm{d}\mu\doteq p^k_\mu(x,.)$ (note that $p^k_\mu(.,.)$ can be chosen to be measurable on $X\times X$) satisfies $\int_X\int_X p^k_\mu(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)>0$.

Definition 1.2. The stochastic kernel $Q_n(.,.)$ is defined on $X\times\mathscr{B}(X)$ by

$$(\forall x\in X),\quad (\forall A\in\mathscr{B}(X)),\quad Q_n(x,A)\doteq\frac{1}{n}\sum_{k=0}^{n-1}P^k(x,A). \tag{1}$$

Definition 1.3. For any set $B\in\mathscr{B}(X)$, we define

$$\mathscr{V}(B)\doteq\{y\in X:\liminf_{n\to\infty}Q_n(y,B)=0\}.$$

Definition 1.4. For any set $A\in\mathscr{B}(X)$, we define

$$A^\circ\doteq\{y\in X\colon P^k(y,A)=0 \text{ for all } k\in\mathbb{N}_*\}. \tag{2}$$

$$\bar{A}\doteq[A^\circ]^{\rm c}, \tag{3}$$

Remark 1.5. If $A\subset B$ then $B^\circ\subset A^\circ$, (see Eq. (2)).

Our necessary and sufficient condition for the existence of a non-singular invariant probability measure for P reads as follows.

Theorem 1.6. *The following assertions are equivalent for a Feller Markov chain with stochastic kernel P*:

(i) *There exists a non-singular invariant probability measure for P.*

(ii) *There exist a real $\alpha>0$, a compact set C and a function $V:X\to[0,\infty]$ which is finite at least at one $x_0\in X$, such that*

$$PV(x)\leqslant V(x)-\alpha+I_C(x), \tag{4}$$

for all x in X, and for every closed subset with empty interior $B\subset C$ such that $\mathscr{V}(B)^{\rm c}\cap B\neq\emptyset$ we have $B^\circ\cap C=\emptyset$.

Since the latter condition may be difficult to check in practice, we introduce a new assumption under which it is shown that the Foster–Lyapunov criteria is equivalent to the existence of an i.p.m.. It must be pointed out that this new assumption includes the one introduced in Costa and Dufour (2000).

Our results are related to the problem studied by Hernandez-Lerma and Lasserre (2000). Indeed, a consequence of Theorem 3.3 presented in Hernandez-Lerma and Lasserre (2000) shows that if a Feller Markov chain has a unique i.p.m. then two cases may occur: the chain is positive recurrent or not. In the last case, the invariant probability measure is necessarily singular. One may regard our work as a special study of the first case. More specifically, the following Corollary holds from Theorem 3.3 (a) of Hernandez-Lerma and Lasserre (2000), and Theorem 1.6 (ii).

Corollary 1.7. *If condition* (ii) *of Theorem* 1.6 *holds and the invariant probability measure is unique then the Feller Markov chain* $\{Y_n\}_{n\in\mathbb{N}}$ *is positive Harris recurrent.*

The paper is organized as follow. In Section 2, some classical definitions related to Markov chains are recalled and some preliminaries are established. Our notation follows the same as the one in the book of Meyn and Tweedie (1992). The proof of Theorem 1.6 is given in Section 3 as well as a simpler sufficient condition.

2. Definitions and preliminaries results

We recall now some classical definitions related to Markov chains. For a complete exposition on the subject the reader is referred to the book of Meyn and Tweedie (1993a).

Definition 2.1. The first hitting time of the set $A\in\mathscr{B}(X)$ not including time zero, denoted by τ_A, is defined as

$$\tau_A \doteq \inf\{n \geqslant 1\colon\ Y_n \in A\}.$$

Definition 2.2. A set $A\in\mathscr{B}(X)$ is called absorbing if $P(x,A)=1$ for all $x\in A$.

Definition 2.3. If $a=\{a_k\}_{k=0}^{\infty}$ is a sampling distribution, then the stochastic kernel K_a is defined on $X\times\mathscr{B}(X)$ by

$$(\forall x\in X),\quad (\forall A\in\mathscr{B}(X)),\quad K_a(x,A)\doteq\sum_{k=0}^{\infty}a_kP^k(x,A). \tag{5}$$

Definition 2.4. For B a fixed set in $\mathscr{B}(X)$, the kernel $U_B(.,.)$ is defined on $X\times\mathscr{B}(X)$ by

$$U_B(x,A)\doteq\sum_{k=1}^{\infty}[(PI_{B^c})^{k-1}P](x,A). \tag{6}$$

Definition 2.5. A set $E\in\mathscr{B}(X)$ is said to be of maximal probability if for any invariant probability measure π for P, $\pi(X-E)=0$.

The ergodic decomposition of the state space X was presented in Yosida (1980, pp. 393–397) under the assumption that the chain is Feller and that P maps $C_c(X)$ into $C_c(X)$ (where $C_c(X)$ is the space of real-valued bounded continuous functions on X with compact support) Yosida (1980, p. 393) but, as pointed out in Hernandez-Lerma and Lasserre (1998b), this assumption is not really required. The purpose of the next Theorem is to introduce this ergodic decomposition, which will be required in the remaining of our paper. The proof of this Theorem can be found in Yosida (1980, pp. 393–397) and Hernandez-Lerma and Lasserre (1998b).

Theorem 2.6. *Suppose that π is an invariant probability measure for P. Then there exists a set $S \in \mathscr{B}(X)$ of maximal probability such that for each $x \in S, Q_n(x,.)$ converges weakly to an invariant probability measure $\lambda_x(.)$ of P as n converges to infinity. Moreover, for any $A \in \mathscr{B}(X)$,*

$$\pi(A) = \int_S \lambda_x(A)\pi(\mathrm{d}x).$$

Furthermore for each $x \in S$, there exist sets $T_x \in \mathscr{B}(X)$ and $\hat{T}_x \in \mathscr{B}(X)$ with $\hat{T}_x \subset T_x \subset S$ such that
(i) *for all $x \in S$ and $z \in S - T_x$, $T_z \cap T_x = \emptyset$,*
(ii) *for all $y \in T_x$*

$$\lambda_y(.) = \lambda_x(.) \tag{7}$$

and for $y \in S - T_x$, $\lambda_y(.) \neq \lambda_x(.)$,
(iii) *$\hat{T}_x$ is absorbing,*
(iv) *λ_x is the unique invariant probability measure for P satisfying $\lambda_x(\hat{T}_x) = 1$.*

According to the Lebesgue decomposition (Meyn and Tweedie, 1993a, p. 107), the transition probability kernel P^n admits a decomposition into its absolutely continuous and singular parts with respect to π. Therefore, for a fixed $y \in X$, we have for all $B \in \mathscr{B}(X)$

$$P^n(y,B) = P^n_\pi(y,B) + P^n_{\pi^\perp}(y,B)$$

with

$$P^n_\pi(y,B) = \int_B p^n_\pi(x,y)\pi(\mathrm{d}y)$$

where the density p^n_π is a measurable function on $X \times X$ for each n and $P^n_{\pi^\perp}(y,.)$ is orthogonal to π. Consequently, for each $y \in X$ there exists a set $L_{\pi,y}$ such that $\pi(L_{\pi,y}) = 1$ and $P^n_{\pi^\perp}(y,B) = P^n_{\pi^\perp}(y, B \cap (L_{\pi,y})^{\mathrm{c}})$ for all $B \in \mathscr{B}(X)$. We also define the set

$$I_\pi(X) \doteq \{y \in X\colon P^n_\pi(y,X) = 0 \text{ for all } n \in \mathbb{N}_*\}.$$

It is easy to check that $I_\pi(X) \in \mathscr{B}(X)$. It is clear by Definition 1.1 that π is non-singular if for some $n \in \mathbb{N}_*$,

$$\int_X P^n_\pi(y,X)\pi(\mathrm{d}y) > 0 \tag{8}$$

and therefore from (8), π is non-singular if and only if $\pi(I_\pi(X)^{\mathrm{c}}) > 0$. In the next results, we use the same notation and definitions as in Theorem 2.6.

Proposition 2.7. *For $x \in S$, consider $A \in \mathscr{B}(X)$ such that $A \subset \hat{T}_x$. Then*

$$\pi(A) = \lambda_x(A)\pi(\hat{T}_x).$$

Proof. Consider first $B \in \mathscr{B}(X)$ such that $B \subset T_x$. Then from Theorem 2.6,

$$\pi(B) = \int_S \lambda_y(B)\pi(\mathrm{d}y) = \int_{T_x} \lambda_y(B)\pi(\mathrm{d}y) + \int_{S-T_x} \lambda_y(B)\pi(\mathrm{d}y).$$

Suppose that $\lambda_y(B) > 0$ for some $y \in S - T_x$. Since $\lambda_y(T_y) = 1$, we should have $\lambda_y(B) = \lambda_y(B \cap T_y) > 0$. But $B \cap T_y = \emptyset$, which leads to a contradiction. Thus $\lambda_y(B) = 0$ for $y \in S - T_x$ and from (7),

$$\pi(B) = \lambda_x(B)\pi(T_x).$$

Since $\lambda_x(T_x - \hat{T}_x) = 0$, we have from the above equation with $B = T_x - \hat{T}_x$ that

$$\pi(T_x - \hat{T}_x) = \lambda_x(T_x - \hat{T}_x)\pi(T_x) = 0.$$

and thus $\pi(T_x) = \pi(\hat{T}_x)$. Therefore for $A \subset \hat{T}_x$,

$$\pi(A) = \lambda_x(A)\pi(T_x) = \lambda_x(A)\pi(\hat{T}_x)$$

and the result follows. □

The next result was presented in Proposition 3.1 of Hernandez-Lerma and Lasserre (2000).

Proposition 2.8. *For each $x \in S$ we have that either $\lambda_x(I_{\lambda_x}(X)) = 0$ or $\lambda_x(I_{\lambda_x}(X)) = 1$.*

The following corollary is immediate.

Corollary 2.9. *For each $x \in S$ we have that λ_x is non-singular if and only if $\lambda_x(I_{\lambda_x}(X)) = 0$.*

Proof. Recall that λ_x is non-singular if and only if $\lambda_x(I_{\lambda_x}(X)^{\mathrm{c}}) > 0$. From Proposition 2.8, λ_x is non-singular if and only if $\lambda_x(I_{\lambda_x}(X)^{\mathrm{c}}) = 1$. □

We can now establish the main result of this section.

Theorem 2.10. *If there exists a non-singular invariant probability measure π for P then there exists $x \in S$ such that λ_x is a non-singular invariant probability measure for P.*

Proof. Suppose π is a non-singular invariant probability measure for P and for every $x \in S$, λ_x is a singular invariant probability measure for P. From Corollary 2.9, $\lambda_x(I_{\lambda_x}(X)) = 1$ for every $x \in S$. Since $\pi(I_\pi(X)^{\mathrm{c}}) > 0$, we have from Theorem 2.6 that,

$$\pi(I_\pi(X)^{\mathrm{c}}) = \int_S \lambda_x(I_\pi(X)^{\mathrm{c}})\pi(\mathrm{d}x) > 0.$$

Select $x \in S$ such that $\lambda_x(I_\pi(X)^{\mathrm{c}}) > 0$. Therefore, since $\lambda_x(I_{\lambda_x}(X)) = 1$,

$$\lambda_x(I_\pi(X)^{\mathrm{c}}) = \lambda_x(I_\pi(X)^{\mathrm{c}} \cap I_{\lambda_x}(X) \cap \hat{T}_x) > 0$$

and we can find $y \in I_\pi(X)^{\mathrm{c}} \cap I_{\lambda_x}(X) \cap \hat{T}_x$. By the singularity of λ_x and since $y \in I_{\lambda_x}(X)$, there exists a set $L_{\lambda_x,y}$ such that $\lambda_x(L_{\lambda_x,y}) = 1$ and $P^n(y, L^{\mathrm{c}}_{\lambda_x,y}) = 1$ for all $n \in \mathbb{N}_*$. Since $y \in \hat{T}_x$, we have $P^n(y, L^{\mathrm{c}}_{\lambda_x,y} \cap \hat{T}_x) = 1$ for all $n \in \mathbb{N}_*$ (see statement (iii) in Theorem 2.6). Notice that, since $L^{\mathrm{c}}_{\lambda_x,y} \cap \hat{T}_x \subset \hat{T}_x$, we have from Proposition 2.7 that

$$\pi(L^{\mathrm{c}}_{\lambda_x,y} \cap \hat{T}_x) = \lambda_x(L^{\mathrm{c}}_{\lambda_x,y} \cap \hat{T}_x)\pi(\hat{T}_x) = 0.$$

But this implies that $y \in I_\pi(X)$, which is a contradiction since $y \in I_\pi(X)^{\mathrm{c}}$, showing the desired result. □

We conclude this section with two auxiliary results which will be required for the main results.

Proposition 2.11. *$\mathscr{V}(B)$ is either empty or an absorbing set.*

Proof. For $y \in \mathscr{V}(B)$, we have from Fatou's Lemma that $0 = \liminf_{n\to\infty} E_y(Q_n(Y_1, B)) \geqslant E_y(\liminf_{n\to\infty} Q_n(Y_1, B)) \geqslant 0$, showing that $E_y(\liminf_{n\to\infty} Q_n(Y_1, B)) = 0$, and thus $P(y, \mathscr{V}(B)) = 1$. □

Proposition 2.12. *For each $x \in S$ we have that if $\lambda_x(B) > 0$ then $\lambda_x(\mathscr{V}(B)) = 0$.*

Proof. According to the individual Ergodic theorem,

$$\lambda_x(B) = \int_S 1_B(y)\lambda_x(\mathrm{d}y) = \int_S \liminf_{n\to\infty} Q_n(y, B)\lambda_x(\mathrm{d}y) > 0.$$

and therefore $\lambda_x(\mathscr{V}(B)^{\mathrm{c}}) > 0$ (and thus $\lambda_x(\mathscr{V}(B)) < 1$). Suppose $\lambda_x(\mathscr{V}(B)) > 0$. From Proposition 10.4.6 in Meyn and Tweedie (1993a) and using (iv) of Theorem 2.6, we have that for all C in $\mathscr{B}(X)$

$$\lambda_x(C) = \frac{1}{c}\int_{\mathscr{V}(B)} U_{\mathscr{V}(B)}(y, C)\lambda_x(\mathrm{d}y),$$

where c is a normalizing positive constant. Then,

$$\lambda_x(\mathscr{V}(B)^{\mathrm{c}}) = \frac{1}{c}\int_{\mathscr{V}(B)} U_{\mathscr{V}(B)}(y, \mathscr{V}(B)^{\mathrm{c}})\lambda_x(\mathrm{d}y).$$

However, for $y \in \mathscr{V}(B)$, we have from (6) and Proposition 2.11 that

$$U_{\mathscr{V}(B)}(y, \mathscr{V}(B)^{\mathrm{c}}) \leqslant \sum_{k=1}^{\infty} P^k(y, \mathscr{V}(B)^{\mathrm{c}}) = 0.$$

and therefore $\lambda_x(\mathscr{V}(B)^{\mathrm{c}}) = 0$, in contradiction with the fact that $\lambda_x(\mathscr{V}(B)) < 1$, showing the desired result. □

3. Necessary and sufficient conditions

We begin by proving Theorem 1.6.

Proof of Theorem 1.6. Suppose first that (ii) holds. From (4) and Theorem 12.3.4 in Meyn and Tweedie (1993b) we know that there exists an invariant probability measure π for P such that $\pi(C) \geqslant \alpha$. Then from Theorem 2.6,

$$\pi(C) = \int_S \lambda_x(C)\pi(\mathrm{d}x) \geqslant \alpha,$$

and we can find $x \in S$ such that $\lambda_x(C) > 0$. For simplicity we shall suppress the subscript x and λ_x. As seen in Proposition 2.8, $\lambda(I(X)) = 0$ or $\lambda(I(X)) = 1$. Suppose that $\lambda(I(X)) = 1$. Then

$$\lambda(C) = \lambda(C \cap I(X) \cap \hat{T}) > 0,$$

and therefore we can find $z \in C \cap I(X) \cap \hat{T}$. Moreover, there exists a set $L_z \in \mathscr{B}(X)$ such that

$$(\forall n \in \mathbb{N}_*), \quad P^n(z, L_z) = 0 \quad \text{and} \quad \lambda(L_z) = 1. \tag{9}$$

Since $P^n(z, \text{int}\{L_z\}) = 0$ for all $n \in \mathbb{N}_*$ and $\text{int}\{L_z\}$ is open, it follows that

$$0 = \liminf_{n\to\infty} Q_n(z, \text{int}\{L_z\}) \geqslant \lambda_z(\text{int}\{L_z\}) = \lambda(\text{int}\{L_z\}), \tag{10}$$

implying that $\lambda(\text{int}\{L_z\}) = 0$. Therefore, $\lambda(L_z) = \lambda(L_z - \text{int}\{L_z\}) = 1$ and $\lambda(C) = \lambda(C \cap (L_z - \text{int}\{L_z\})) > 0$ so that we can find, by regularity of the measure λ, a closed set $B \subset C \cap (L_z - \text{int}\{L_z\})$ such that $\lambda(B) > 0$. Clearly $\text{int}\{B\} = \emptyset$. Since $B \subset L_z$, it follows from Remark 1.5 and Eq. (9) that $z \in L_z^\circ \subset B^\circ$, so that $z \in B^\circ \cap C \neq \emptyset$.

On the other hand, since from Proposition 2.12, $\lambda(\mathscr{V}(B)^c) = 1$, we get that $\lambda(B) = \lambda(B \cap \mathscr{V}(B)^c) > 0$, so that $B \cap \mathscr{V}(B)^c \neq \emptyset$. Therefore, we found a closed subset B of C with empty interior such that $B \cap \mathscr{V}(B)^c \neq \emptyset$ but $B^\circ \cap C \neq \emptyset$, which is a contradiction with the hypothesis. This shows that $\lambda(I(X)) = 0$ and from Corollary 2.9 we have that λ is a non-singular invariant probability measure for P.

Suppose now that (i) holds. From Theorem 2.10, there exists $x \in S$ such that λ_x is a non-singular invariant probability measure for P. Again let us suppress the subscript x and λ_x for notational simplicity. As shown in Theorems 2.6 and 3.9 of Costa and Dufour (2000), we can find a real $\alpha > 0$, a compact set $C \subset \hat{T}$ and a function $V: X \to [0, \infty]$ which is finite at least at one $x_0 \in X$, such that (4) is satisfied, and for all $D \in \mathscr{B}(X)$ with $\lambda(D) > 0$,

$$\sup_{z\in C} E_z[\tau_D] \leqslant M, \tag{11}$$

for some positive constant M. Consider B a closed subset of C with empty interior and $y \in B$ such that $\liminf_{n\to\infty} Q_n(y, B) > 0$. Since $Q_n(y, .)$ converges weakly to $\lambda(.)$ and B is closed, we have that

$$\lambda(B) \geqslant \limsup_{n\to\infty} Q_n(y, B) \geqslant \liminf_{n\to\infty}, Q_n(y, B) > 0,$$

and from (11), $B^\circ \cap C = \emptyset$. $\square$

Now we present a sufficient condition for the existence of a non-singular invariant probability measure for P which avoids the extra assumption in the Foster's criterion in Theorem 1.6. This hypothesis is related to the stochastic kernel P, and generalizes the condition T′ presented in Costa and Dufour (2000).

Condition T2. There exists a substochastic transition kernel T defined on $X \times \mathscr{B}(X)$ such that

(i) For all $x \in X$, $T(x, .) \ll K_a(x, .)$.

(ii) For all $x \in X$, $0 < T(x, X) \leqslant 1$.

(iii) There exists a countable sequence of sets $\{B_n\}_{n=1}^{\infty}$ in $\mathscr{B}(X)$ such that one of the conditions (a) or (b) below holds:

(a) for each compact set with empty interior C of X, there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that either

$$\{z \in X: T(z, \bar{C}) > 0\} = \bigcup_{i=1}^{\infty} B_{n_i} \tag{12}$$

or

$$\{z \in X: T(z, \bar{C}) = 0\} = \bigcup_{i=1}^{\infty} B_{n_i} \tag{13}$$

(b) the same as (a) replacing "compact set C of X" by "absorbing set E of P", and "C" by "E" in (12) and (13).

Since $0 < T(x,X) \leqslant 1$ there is no loss of generality in assuming that $T(x,X)=1$. Notice also that T-chains satisfy condition T2 since, as shown in Costa and Dufour (2000), they satisfy condition T′.

We shall prove the following Theorem.

Theorem 3.1. *Suppose that condition* T2 *is satisfied. Then there exists a non-singular invariant probability measure for P if and only if exist a real $\alpha > 0$, a compact set C and a function $V: X \to [0,\infty]$ which is finite at least at one $x_0 \in X$, such that Eq.* (4) *is satisfied.*

For $A \in \mathscr{B}(X)$, define

$$A^* \doteq \{y \in X\colon T(y,A) = 0\}, \tag{14}$$

$$\tilde{A} \doteq [A^*]^{\mathrm{c}}. \tag{15}$$

and we have the following remark:

Remark 3.2. For any $A \in \mathscr{B}(X)$, $A^{\circ} \subset A^*$, since from T2(i), $K_a(x,.) \gg T(x,.)$.

In the next results we consider λ_x for some $x \in S$. For notational simplicity, we suppress the subscript x and λ_x. The following result was proved in Proposition 3.4 of Costa and Dufour (2000):

Proposition 3.3. *For $A \in \mathscr{B}(X)$, $A^{\circ} \subset [\bar{A}]^{\circ}$ and if $\lambda(A) > 0$ then $\lambda(\bar{A}) = 1$.*

We have the following result.

Lemma 3.4. *Suppose that the set $A \in \mathscr{B}(X)$ is such that $\lambda(A) > 0$. Then $\lambda(\tilde{\bar{A}}) = 1$.*

Proof. From the previous lemma, we obtain that $\lambda(\bar{A}) = 1$. Using the fact that λ is an invariant measure, we have

$$\int_X K_a(y,A^{\circ})\lambda(\mathrm{d}y) = 0,$$

which gives $\lambda([A^{\circ}]^{\circ}) = 1$. From Remark 3.2, $[A^{\circ}]^{\circ} \subset [A^{\circ}]^*$, and we must have $\lambda([A^{\circ}]^*) = 1$, that is,

$$\int_X T(y,A^{\circ})\lambda(\mathrm{d}y) = 0. \tag{16}$$

However, by hypothesis, $T(y,X) = 1$ for all $y \in X$, and thus,

$$\int_X T(y,X)\lambda(\mathrm{d}y) = 1. \tag{17}$$

Combining Eqs. (16) and (17), we obtain that

$$\int_X T(y,\bar{A})\lambda(\mathrm{d}y) = 1,$$

so that

$$\lambda(\{y \in X\colon T(y,\bar{A}) = 1\}) = 1,$$

and since

$$\{y \in X \colon T(y,\bar{A}) = 1\} \subset \tilde{\bar{A}}$$

the result follows. □

Let us introduce the set

$$\Gamma \doteq \{n \in \mathbb{N} \colon \lambda(B_n) > 0\}.$$

Lemma 3.5. $I(X) \cap \hat{T} \subset \bigcup_{n\in\Gamma} [B_n]^\circ \cup [\bigcap_{n\in\Gamma^c} B_n^c]^\circ$.

Proof. We shall assume that (a) holds for condition T2(iii). The proof for case (b) is similar. If $y \in I(X)\cap\hat{T}$ then for $k \in \mathbb{N}_*$, $1 = P^k(y,X) = P^k_\perp(y,L^c_y)$ and so $P^k(y,L^c_y) = 1$, where $\lambda(L_y) = 1$. As in the proof of Theorem 1.6, there is no loss of generality in assuming that L_y is absorbing with empty interior (indeed, if this is not the case we have, as in (10), that $\lambda(\text{int}\{L_y\}) = 0$ and thus $\lambda(L_y - \text{int}\{L_y\}) = 1$ and by repeating the same arguments as in Yosida (1980, p. 396–397) we can find $\hat{L}_y \subset L_y - \text{int}\{L_y\}$ such that $\hat{L}_y$ is absorbing and $\lambda(\hat{L}_y) = 1$; then we could just re-define L_y as $\hat{L}_y$). Since λ is a regular measure (Theorem D.3.2 in Meyn and Tweedie (1993a)) we can find a compact set $C_y \subset L_y$ (clearly C has empty interior) such that $\lambda(C_y) > 0$. Consequently, from Remark 1.5,

$$y \in I(X) \cap \hat{T} \quad \Rightarrow \quad y \in [L_y]^\circ \subset [C_y]^\circ. \tag{18}$$

Now, from Proposition 3.3 and since $\lambda(C_y) > 0$, we obtain that

$$\lambda(\overline{C_y}) = 1, \tag{19}$$

$$[C_y]^\circ \subset [\overline{C_y}]^\circ \subset [\overline{\overline{C_y}}]^\circ. \tag{20}$$

Therefore, from Remarks 1.5 and 3.2

$$[C_y]^\circ \subset [\widetilde{\overline{C_y}}]^\circ. \tag{21}$$

Recalling that C_y is a compact set we have from condition T2(iii)(a) that either (12) or (13) hold for some subsequence $\{n_i(y)\}_{i=1}^\infty$. If (12) holds, then

$$\begin{aligned}\widetilde{\overline{C_y}} &= \{z \in X \colon T(z,\overline{C_y}) > 0\} \\ &= \bigcup_{i=1}^{\infty} B_{n_i(y)}.\end{aligned} \tag{22}$$

From Lemma 3.4 and using the fact that $\lambda(C_y) > 0$, it follows that $\lambda(\widetilde{\overline{C_y}}) = 1$. This means that there exists at least one integer (depending on y), denoted by $k(y)$, such that $\lambda(B_{k(y)}) > 0$. Consequently, using Eq. (22) and Remark 1.5, we have

$$[\widetilde{\overline{C_y}}]^\circ \subset [B_{k(y)}]^\circ \subset \bigcup_{n\in\Gamma} [B_n]^\circ. \tag{23}$$

Now, combining Eqs. (18), (21) and (23), we obtain that

$$y \in \bigcup_{n\in\Gamma} [B_n]^\circ. \tag{24}$$

Suppose now that (13) holds, that is,

$$[\overline{C_y}]^* = \{z \in X\colon T(z, \overline{C_y}) = 0\} = \bigcup_{i=1}^{\infty} B_{n_i(y)}. \tag{25}$$

From Lemma 3.4 and using the fact that $\lambda(C_y) > 0$, it follows that $\lambda(\widetilde{\overline{C_y}}) = 1$ and thus $\lambda([\overline{C_y}]^*) = 0$. This means that $\lambda(B_{n_i(y)}) = 0$ for all i and from (25)

$$\bigcap_{n\in\Gamma^c} B_n^c \subset \bigcap_{i=1}^{\infty} B_{n_i(y)}^c = \widetilde{\overline{C_y}}. \tag{26}$$

Note that $\lambda(\bigcap_{n\in\Gamma^c} B_n^c) = 1$ since $\lambda(B_n^c) = 1$ for each $n \in \Gamma^c$. From Remark 1.5 and (26) we obtain that

$$[\widetilde{\overline{C_y}}]^\circ \subset \left[\bigcap_{n\in\Gamma^c} B_n^c\right]^\circ, \tag{27}$$

and combining (18), (21) and (27), we obtain that

$$y \in \left[\bigcap_{n\in\Gamma^c} B_n^c\right]^\circ. \tag{28}$$

Since either (24) or (28) holds, we get the desired result. □

Proposition 3.6. $\lambda(I(X)) = 0$.

Proof. Since $\lambda(B_n) > 0$ for $n \in \Gamma$, we have from Proposition 3.3 that $\lambda([B_n]^\circ)=0$. Similarly, since $\lambda(\bigcap_{n\in\Gamma^c} B_n^c) = 1$, we have from Proposition 3.3 that $\lambda([\bigcap_{n\in\Gamma^c} B_n^c]^\circ) = 0$, and the result follows from Lemma 3.5 and (iv) of Theorem 2.6. □

Therefore, Theorem 3.1 follows from Theorem 2.10, Corollary 2.9, and Proposition 3.6.

It can be shown that the identity kernel does not satisfy the condition T′ presented in Costa and Dufour (2000). However, we can show (see the following example) that it satisfies the new condition T2.

Example. Let P be the identity kernel, and take $T = P$. Let us choose for $\{B_n\}_{n=1}^{\infty}$ in $\mathscr{B}(X)$ a countable basis for the open sets. Then for any compact set C, $\bar{C} = C$ and $[\bar{C}]^* = C^c = \bigcup_{i=1}^{\infty} B_{n_i}$ since C^c is an open set. Therefore, the identity kernel satisfies the condition T2.

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