Robust H_2-Control for Discrete-time Markovian Jump Linear Systems

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Abstract

This paper deals with the robust H_2-control of discrete-time Markovian jump linear systems. It is assumed that both the state and jump variables are available to the controller. Uncertainties satisfying some norm bounded conditions are considered on the parameters of the system. An upperbound for the H_2-control problem is derived in terms of an LMI optimization problem. For the case in which there are no uncertainties, we show that the convex formulation is equivalent to the existence of the mean square stabilizing solution for the set of coupled algebraic Riccati equations arising on the quadratic optimal control problem of discrete-time Markovian jump linear systems. Therefore, for the case with no uncertainties, the convex formulation considered in this paper imposes no extra conditions than those in the usual dynamic programming approach. Finally some numerical examples are presented to illustrate the technique.

Submitted in October 1997; Revised in January 1999

This work was supported in part by FAPESP (Research Council of the State of São Paulo) and CNPq (Brazilian National Research Council) under the grants No 97/4668-1 and 305173/88-0 respectively

1. Introduction

Several examples can be found in the literature nowadays showing the importance of the class of linear systems subject to abrupt changes in their structures. This is the case, for instance, of systems subject to random failures, repairs or sudden environmental disturbances, abrupt variation of the operating point on a nonlinear plant, etc. Markovian jump linear systems (MJLS), which comprise an important family of models subject to abrupt variations, are very often used to model the above class of systems. In this case the changes in the structure of the system are assumed to be modeled by a Markov chain, which takes the system into distinct linear forms in a state variable representation. Practical motivations as well as some theoretical results for MJLS can be found, for instance, in [1],[3]-[14],[16]-[26],[28],[29].

The quadratic optimal control problem for discrete-time MJLS has been analyzed in several papers using different approaches of analysis. In [1],[3],[5]-[7],[10]-[12],[18]-[22],[25] a dynamic programming approach has been adopted, leading to a set of coupled algebraic Riccati equations (CARE). These equations generalize the usual discrete-time algebraic Riccati equations (see [2],[15]), and results for the existence of maximal and mean square stabilizing solutions for CARE have been derived in, for instance, [6],[7],[10],[12]. Another approach that has been used is to apply convex programming. In [26] the authors obtain the maximal solution of the CARE via the solution of a LMI optimization problem. In [9] the authors use a convex approach to solve the H_2-control of a MJLS.

One question that may be asked is that if the convex approach in [9] is stronger or weaker than the dynamic programming formulation, in the sense that the existence of the solution to one problem implies or is implied by the existence of the solution to the other. As far as the authors are aware of, there are no theoretical results connecting the following problems: i) existence of the mean square stabilizing solution for the discrete-time CARE, and ii) the existence of the solution for the convex programming approach presented in [9]. Notice that the results in [26] are related to the maximal solution of the CARE, and not to the mean square stabilizing solution of the CARE. As noticed in [26], the maximal solution is not necessarily a mean square stabilizing solution, so that an extra condition would have to be imposed to guarantee that the maximal solution is also mean square stabilizing in general (the case with no jumps trivially shows this). For the continous-time case the equivalence between the mean square stabilizing solution of the CARE and the convex programming approach was considered in [13]. In this paper we start by addressing this point and show that these two problems, i) and ii) posed above, are in fact equivalent. Therefore, for the case with no uncertainties, the convex formulation imposes no extra conditions than those considered in the usual dynamic programming approach.

The advantages of the convex approach in analyzing H_2-control problems of MJLS over the dynamic programming approach had already been stressed in [9]. In that paper it was considered the H_2-control of MJLS with uncertainties on the transition probability distribution of the Markovian chain and the case in which the state of the Markov chain and/or the state of the system are not directly accessible to the controller. Convex programming problems were derived to handle these situations. In that paper the parameters of the matrices of the system were considered to be exactly known.

In this paper we consider the situation in which the controller has access to the state of the Markov chain and the state of the system, but there are uncertainties on the transition probability of the Markov chain as well as on the matrices of the system. The uncertainties on the matrices are considered to be of the norm bounded form. We shall call this problem with uncertainties as robust H_2-control of MJLS. The LMI optimization problem presented for the case with no uncertainties will be reformulated in order to provide a controller that stabilizes the closed loop MJLS in the mean square sense, and gives an upper bound for the robust H_2-control problem.

The paper is organized in the following way. Section 2 presents the notation that will be used throughout the work. Section 3 deals with previous results derived for mean square stability, CARE and H_2-control of MJLS, as well as some other auxiliary results. Section 4 presents the main theorem related to the equivalence between the dynamic programming and convex programming approaches. It shows that the existence of the mean square stabilizing solution for the CARE is equivalent to the existence of the solution of the H_2-control of MJLS, which in turns is also equivalent to the existence of the solution of a convex programming problem. Section 5 considers the robust H_2-control of MJLS, so that there are norm bound uncertainties on the parameters of the MJLS, as well as uncertainties on the transition probability matrix of the associated Markov chain. The LMI optimization problem of section 4 is reformulated so that its solution provides an upper bound mean square stabilizing solution for the robust H_2-control of the MJLS problem. Section 6 presents a numerical example and section 7 concludes the paper with some final remarks. Some tests for mean square stability and detectability are presented in the appendix.

2. Notation and Preliminary Results

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces we set $\mathbb{B}(\mathbb{X},\mathbb{Y})$ the Banach space of all bounded linear operators of $\mathbb{X}$ into $\mathbb{Y}$, with the uniform induced norm represented by $\|.\|$. For simplicity we shall set $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X},\mathbb{X})$. The spectral radius of an operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ will be denoted by $r_\sigma(\mathcal{T})$. If $\mathbb{X}$ is a Hilbert space then the inner product will be denoted by $\langle .;. \rangle$, and for $\mathcal{T} \in \mathbb{B}(\mathbb{X})$, $\mathcal{T}^*$ will denote the adjoint operator of $\mathcal{T}$. As usual, $\mathcal{T} \geq 0$ ($\mathcal{T} > 0$ respectively) will denote that the operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ will be positive semi-definite (positive definite). In particular we shall denote by $\mathbb{C}^n$ the n dimensional complex Euclidean spaces and by $\mathbb{B}(\mathbb{C}^n, \mathbb{C}^m)$ the normed bounded linear space of all $m \times n$ complex matrices, with $\mathbb{B}(\mathbb{C}^n) := \mathbb{B}(\mathbb{C}^n, \mathbb{C}^n)$.

Set $\mathbb{H}^{n,m}$ the linear space made up of all N-sequences of complex matrices $V = (V_1, \ldots, V_N)$ with $V_i \in \mathbb{B}(\mathbb{C}^n, \mathbb{C}^m)$, $i = 1, \ldots, N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$. For $V = (V_1, \ldots, V_N) \in \mathbb{H}^{n,m}$, we consider the following norm in $\mathbb{H}^{n,m}$; $\|V\|_2 := \left(\sum_{i=1}^{N} tr\left(V_i^* V_i\right)\right)^{\frac{1}{2}}$. It is easy to verify that $\left(\|.\|_2, \mathbb{H}^{n,m}\right)$ is a Hilbert space with the inner product given, for $V = (V_1, \ldots, V_N)$ and $S = (S_1, \ldots, S_N)$ in $\mathbb{H}^{n,m}$, by: $\langle V; S \rangle := \sum_{i=1}^{N} tr\left(V_i^* S_i\right)$.

We shall say that $V = (V_1, \ldots, V_N) \in \mathbb{H}^n$ is hermitian if $V_i = V_i^*$ for $i = 1, \ldots, N$, and shall write $\mathbb{H}^{n*} := \{ V = (V_1, \ldots, V_N) \in \mathbb{H}^n; V_i = V_i^*, \ i = 1, \ldots, N \}$. We set $\mathbb{H}^{n+} := \{ V = (V_1, \ldots, V_N) \in \mathbb{H}^n; V_i \geq 0, i = 1, \ldots, N \}$ and shall write, for $V = (V_1, \ldots, V_N) \in \mathbb{H}^n$ and $S = (S_1, \ldots, S_N) \in \mathbb{H}^n$, that $V \geq S$ if $V - S = (V_1 - S_1, \ldots, V_N - S_N) \in \mathbb{H}^{n+}$, and that $V > S$ if $V_i - S_i > 0$ for $i = 1, \ldots, N$.

For $\widetilde{A} = (\widetilde{A}_1, \ldots, \widetilde{A}_N) \in \mathbb{H}^n$ and matrix $\mathbb{P} = (p_{ij})$, $i, j = 1, \ldots, N$, with $p_{ij} \geq 0$ for all $i, j = 1, \ldots, N$, we define the following operators $\mathcal{E}(.) = (\mathcal{E}_1(.), \ldots, \mathcal{E}_N(.)) \in \mathbb{B}(\mathbb{H}^n)$, $\mathcal{L}(.) = (\mathcal{L}_1(.), \ldots, \mathcal{L}_N(.)) \in \mathbb{B}(\mathbb{H}^n)$ and $\mathcal{T}(.) = (\mathcal{T}_1(.), \ldots, \mathcal{T}_N(.)) \in \mathbb{B}(\mathbb{H}^n)$; for $V = (V_1, \ldots, V_N) \in \mathbb{H}^n$ and $i, j = 1, \ldots, N$,

$$\mathcal{E}_i(V) := \sum_{j=1}^{N} p_{ij} V_j \tag{1}$$

$$\mathcal{L}_i(V) := \widetilde{A}_i^* \mathcal{E}_i(V) \widetilde{A}_i \tag{2}$$

$$\mathcal{T}_j(V) := \sum_{i=1}^{N} p_{ij} \widetilde{A}_i V_i \widetilde{A}_i^* \tag{3}$$

and it is easy to verify that with the inner product given above we have from (1), (2) and (3) that $\mathcal{T} = \mathcal{L}^*$ (in particular, $r_\sigma(\mathcal{T}) = r_\sigma(\mathcal{L})$). It is also easy to check that the operators $\mathcal{E}, \mathcal{L}$, and $\mathcal{T}$ map $\mathbb{H}^{n*}$ into $\mathbb{H}^{n*}$ and $\mathbb{H}^{n+}$ into $\mathbb{H}^{n+}$.

For a matrix $Q \in \mathbb{B}(\mathbb{C}^n, \mathbb{C}^m)$ the generalized inverse of Q (or Moore-Penrose inverse of Q) is defined as the unique matrix $Q^\dagger \in \mathbb{B}(\mathbb{C}^m, \mathbb{C}^n)$ such that (see [27], page 13)

a) $QQ^\dagger Q = Q$,

b) $Q^\dagger Q Q^\dagger = Q^\dagger$

c) $(QQ^\dagger)^* = QQ^\dagger$ and

d) $(Q^\dagger Q)^* = Q^\dagger Q$.

We conclude this section with the following well known result used in LMI's, which will be useful in the sequel (see [27], page 13).

Remark 1: $W = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0$ if and only if either $Q \geq 0, S = QQ^\dagger S$ and $R - S^* Q^\dagger S \geq 0$ or $R \geq 0, S = SRR^\dagger$ and $Q - S R^\dagger S^* \geq 0$. Particularly, if $R > 0$, then $W \geq 0$ if and only if $Q \geq S R^{-1} S^*$.

3. Auxiliary Results

The goal of this section is to present some preliminary results regarding the mean square stability of MJLS, the CARE, and the H_2-control problem. This is achieved through subsections 3.1, 3.2 and 3.3 respectively.

3.1. Preliminaries

Consider the following Markovian jump linear system

$$\begin{cases} x(k+1) = \tilde{A}_{\theta(k)} x(k) & (4.a) \\ x(0) = x_0, \theta(0) = \theta_0 & (4.b) \end{cases}$$

where $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_N) \in \mathbb{H}^n$ and $\{\theta(k); k = 0, 1, \ldots\}$ is a discrete-time Markov chain with finite state space $\{1, \ldots, N\}$ and with transition probability matrix $\mathbb{P} = (p_{ij})$. We set $Q(k) = (Q_1(k), \ldots, Q_N(k)) \in \mathbb{H}^{n*}$, as

$$Q_j(k) := \mathrm{E}(x(k)x(k)^* 1_{\{\theta(k)=j\}}) \qquad (5)$$

where $1_{\{.\}}$ stands for the Dirac measure.

The following result, shown in [8], provides a connection between (3) and (5):

Proposition 1 *: For every* $k = 0, 1, \ldots,$ $Q(k+1) = \mathcal{T}(Q(k))$.

We make the following definition:

Definition 1 : Model (4) is mean square stable (MSS) if $E(\| x(k) \|^2) \to 0$ as $k \to \infty$ for any initial condition x_0 and initial distribution for θ_0.

The next result has been proved in [8]:

Proposition 2 *: The following assertions are equivalent:*

a) Model (4) is MSS

b) $r_\sigma(\mathcal{T}) < 1$.

c) $r_\sigma(\mathcal{L}) < 1$.

d) There exists $\alpha \in (0,1)$ and $a \in \mathbb{R}$, $a > 0$, such that for each $k = 0, 1 \ldots,$

$$E(\| x(k) \|^2) \le a\, \alpha^k.$$

e) (coupled Lyapunov equations) given any $S = (S_1, \ldots, S_N) > 0$ in $\mathbb{H}^{n+}$ there exists $P = (P_1, \ldots, P_N) > 0$ in $\mathbb{H}^{n+}$ satisfying $P - \mathcal{T}(P) = S$ with $P = \sum_{k=0}^{\infty} \mathcal{T}^k(S)$.

f) (adjoint coupled Lyapunov equations) given any $S = (S_1, \ldots, S_N) > 0$ in $\mathbb{H}^{n+}$ there exists $P = (P_1, \ldots, P_N) > 0$ in $\mathbb{H}^{n+}$ satisfying $P - \mathcal{L}(P) = S$ with $P = \sum_{k=0}^{\infty} \mathcal{L}^k(S)$.

Moreover if $r_\sigma(\mathcal{T}) < 1$ then for any $S \in \mathbb{H}^n$ there exists a unique $P \in \mathbb{H}^n$ such that $P - \mathcal{T}(P) = S$. If $S \ge T \ge 0$ (> 0 respectively) and $L - \mathcal{T}(L) = T$, then $P \ge L \ge 0$ (> 0). These results also hold replacing $\mathcal{T}$ by $\mathcal{L}$.

3.2. Coupled Algebraic Riccati Equations

Let us consider now the following controlled discrete-time Markovian jump linear system,

$$\begin{cases} x(k+1) = A_{\theta(k)} x(k) + B_{\theta(k)} u(k) \\ x(0) = x_0, \theta(0) = \theta_0 \end{cases}$$

where $A = (A_1, \ldots, A_N) \in \mathbb{H}^n$ and $B = (B_1, \ldots, B_N) \in \mathbb{H}^{m,n}$. It is desired to find $\{u(k); k \ge 0\}$ which minimizes the following functional

$$J(x(0), \theta(0), u) = \frac{1}{2} \sum_{k=0}^{\infty} \mathrm{E}\left\{ \begin{pmatrix} x(k)^* & u(k)^* \end{pmatrix} \begin{pmatrix} Q_{\theta(k)} & L_{\theta(k)} \\ L^*_{\theta(k)} & R_{\theta(k)} \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \right\}.$$

We assume in this paper that $\begin{pmatrix} Q_i & L_i \\ L_i^* & R_i \end{pmatrix} \ge 0$, and $R_i > 0$, for $i = 1, \ldots, N$.

For $X = (X_1,\dots,X_N) \in \mathbb{H}^n$ such that $B_i^* \mathcal{E}_i(X)B_i + R_i > 0$, $i = 1,\dots,N$, set $\mathcal{R}(X) = (\mathcal{R}_1(X),\dots,\mathcal{R}_N(X))$ as

$$\mathcal{R}_i(X) := -X_i + A_i^* \mathcal{E}_i(X)A_i + Q_i - (A_i^* \mathcal{E}_i(X)B_i + L_i)(B_i^* \mathcal{E}_i(X)B_i + R_i)^{-1}(B_i^* \mathcal{E}_i(X)A_i + L_i^*).$$

and introduce the following notation

$$\mathbb{M} := \{ X = (X_1,\dots,X_N) \in \mathbb{H}^{n*}; \; B_i^* \mathcal{E}_i(X)B_i + R_i > 0, i = 1,\dots,N, \text{ and } \mathcal{R}(X) \geq 0 \}.$$

Associated to the above optimization problem we have the following discrete-time coupled algebraic Riccati equations (CARE) (see [10]),

$$\mathcal{R}_i(X) = 0, i = 1,\dots,N \tag{6}$$

where $X = (X_1,\dots,X_N) \in \mathbb{H}^{n*}$ is said to be an hermitian solution for the CARE if $B_i^* \mathcal{E}_i(X)B_i + R_i$ is invertible and X satisfies the equation above. Before we present the main result for the CARE, we make the following definitions.

Definition 2 : We say that (A,B) is mean square stabilizable if there exists $F = (F_1,\dots,F_N) \in \mathbb{H}^{n,m}$ such that model (4) is MSS with $\tilde{A}_i = A_i + B_i F_i$. In this case we say that F stabilizes (A,B) in the mean square sense and set

$$\mathbb{K} := \{F \in \mathcal{H}^{n,m}; F \text{ stabilizes } (A,B) \text{ in the mean square sense}\}.$$

We also say that (C,A) is mean square detectable if there exists $H = (H_1,\dots,H_N) \in \mathbb{H}^{s,n}$ such that model (4) is MSS with $\tilde{A}_i = A_i + H_i C_i$.

Tests for mean square stabilizability and detectability are presented in the appendix.

Definition 3 : For $F = (F_1,\dots,F_N) \in \mathbb{H}^{m,n}$ and $X = (X_1,\dots,X_N) \in \mathbb{H}^{n*}$ such that $B_i^* \mathcal{E}_i(X)B_i + R_i > 0, i = 1,\dots,N$, $\mathcal{S}(F) = (\mathcal{S}_1(F),\dots,\mathcal{S}_N(F))$, $\mathcal{D}(X) = (\mathcal{D}_1(X),\dots,\mathcal{D}_N(X))$ and $\mathcal{F}(X) = (\mathcal{F}_1(X),\dots,\mathcal{F}_N(X))$ are defined, for $i = 1,\dots,N$, as

$$\mathcal{S}_i(F) := \begin{pmatrix} I & F_i^* \end{pmatrix} \begin{pmatrix} Q_i & L_i \\ L_i^* & R_i \end{pmatrix} \begin{pmatrix} I \\ F_i \end{pmatrix}$$
$$\mathcal{D}_i(X) := B_i^* \mathcal{E}_i(X)B_i + R_i$$
$$\mathcal{F}_i(X) := -(B_i^* \mathcal{E}_i(X)B_i + R_i)^{-1}(B_i^* \mathcal{E}_i(X)A_i + L_i^*).$$

Definition 4 : We say that $X=(X_1,\dots,X_N)\in\mathbb{H}^n$ is a mean square stabilizing solution for the CARE if $X\geq 0$, $\mathcal{R}(X)=0$ and $\mathcal{F}(X)\in\mathbb{K}$.

The proof of the following proposition is straightforward but otherwise long, and therefore will be omitted.

Proposition 3 : *Suppose that* $X\in\mathbb{M}$ *and for some* $\widehat{F}=(\widehat{F}_1,\dots,\widehat{F}_N)\in\mathbb{H}^{n,m}$, $\widehat{X}=(\widehat{X}_1,\dots,\widehat{X}_N)\in\mathbb{H}^{n+}$ *satisfies for* $i=1,\dots,N$

$$\widehat{X}_i-(A_i+B_i\widehat{F}_i)^*\mathcal{E}_i(\widehat{X})(A_i+B_i\widehat{F}_i)=\mathcal{S}_i(\widehat{F}).$$

Then, for $i=1,\dots,N$,

$$(\widehat{X}_i-X_i)-(A_i+B_i\widehat{F}_i)^*\mathcal{E}_i(\widehat{X}-X)(A_i+B_i\widehat{F}_i)=\mathcal{R}_i(X)+ (\widehat{F}_i-\mathcal{F}_i(X))^*D_i(X)(\widehat{F}_i-\mathcal{F}_i(X)). \quad (7)$$

We can now state the following Theorem, regarding the existence of a maximal solution of (6) in $\mathbb{M}$, proved in [12].

Theorem 1: *Suppose that* (A,B) *is mean square stabilizable. Then for* $l=0,1,2,\dots,$ *there exists* $X^l=(X^l_1,\dots,X^l_N)$ *which satisfies the following properties:*

a) $X^0\geq X^1\geq\cdots\geq X^l\geq 0$ *(8.a)*
b) $r_\sigma(\mathcal{L}^l)<1$, *where* $\mathcal{L}^l(.)=(\mathcal{L}^l_1(.),\dots,\mathcal{L}^l_N(.))$ *and for* $i=1,\dots,N$,

$$\begin{aligned}\mathcal{L}^l_i(.)&:=A^{l*}_i\mathcal{E}_i(.)A^l_i,\\ A^l_i&:=A_i+B_iF^l_i,\\ F^l_i&:=\mathcal{F}_i(X^{l-1})\ \text{for}\ l=1,2,\dots.\end{aligned} \quad (8.b)$$

c) $X^l_i-A^{l*}_i\mathcal{E}_i(X^l)A^l_i=\mathcal{S}_i(F^l),\ i=1,\dots,N.$ *(8.c)*

Moreover there exists $X^+=(X^+_1,\dots,X^+_N)\in\mathbb{M}$ *such that* $\mathcal{R}(X^+)=0$, $X^+\geq X$ *for any* $X\in\mathbb{M}$ *and* $X^l\to X^+$ *as* $l\to\infty$. *Furthermore* $r_\sigma(\mathcal{L}^+)\leq 1$, *where* $\mathcal{L}^+(.)=(\mathcal{L}^+_1(.),\dots,\mathcal{L}^+_N(.))$ *is defined as* $\mathcal{L}^+_i(.)=A^{+*}_i\mathcal{E}_i(.)A^+_i$, *for* $i=1,\dots,N$, *and* $A^+_i=A_i+B_i\mathcal{F}_i(X^+)$.

Remark 2 : It has been shown in [12] that there exists at most one mean square stabilizing solution for the CARE (6), which will coincide with the maximal solution X^+ of Theorem 1.

3.3. The H_2-Norm

Consider now the controlled discrete-time Markovian jump linear system $\mathcal{G}$ with $x_0 = 0$, input $w(k)$, and output $z(k)$, given by

$$\mathcal{G} = \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + J_{\theta(k)}w(k) & \text{(9.a)} \\ x(0) = 0, \theta(0) = \theta_0 & \text{(9.b)} \\ z(k) = \mathcal{C}_{\theta(k)}x(k) + D_{\theta(k)}u(k) & \text{(9.c)} \end{cases}$$

where $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_N) \in \mathbb{H}^{n,p}$, $D = (D_1, \ldots, D_N) \in \mathbb{H}^{m,p}$, and $J = (J_1, \ldots, J_N) \in \mathbb{H}^{r,n}$ with $D_i^* D_i > 0$and $J_i J_i^* > 0$ for $i = 1, \ldots, N$. For $F = (F_1, \ldots, F_N) \in \mathbb{K}$, we define $\mathcal{G}_F$ as system (9) above with feedback control law $u(k) = F_{\theta(k)}x(k)$. The definition of the H_2-norm of system $\mathcal{G}_F$, presented in [9], is given by

$$\| \mathcal{G}_F \|_2^2 := \sum_{s=1}^{r} \sum_{j=1}^{N} \| z_{s,j} \|_2^2$$

where $z_{s,j}$ represents the output sequence $(z(0), z(1), \ldots)$ given by (9.c) when: a) the input sequence is given by $w = \{w(0), w(1), \ldots\}$, $w(0) = e_s$, $w(k) = 0, k > 0$, $\{e_1, \ldots, e_r\}$ forms a basis for $\mathbb{C}^r$ and, b) $\theta(0) = \theta(1) = j$.

For the deterministic case ($N = 1$ and $p_{11} = 1$) the above definition reduces to the usual H_2-norm. For $F \in \mathbb{K}$, let $\mathcal{P}(F) = (\mathcal{P}_1(F), \ldots, \mathcal{P}_N(F)) \in \mathbb{H}^{n+}$ and $U(F) = (U_1(F), \ldots, U_N(F)) \in \mathbb{H}^{n+}$ be the unique solution of the discrete-time coupled gramian of observability and controllability respectively (recall Proposition 2 for existence and uniqueness)

$$\mathcal{P}_i(F) = (A_i + B_i F_i)^* \mathcal{E}_i(\mathcal{P}(F))(A_i + B_i F_i) + \mathcal{S}_i(F) \tag{10}$$

$$U_i(F) = \sum_{i=1}^{N} p_{ij}(A_i + B_i F_i) U_i(F)(A_i + B_i F_i)^* + J_j J_j^*. \tag{11}$$

Since $\mathcal{S}_i(F) \geq 0$ and $J_j J_j^* > 0$, we get that $\mathcal{P}_i(F) \geq 0$ and $U(F) > 0$ (see Proposition 2). Define $Q_i = \mathcal{C}_i^* \mathcal{C}_i$, $L_i = \mathcal{C}_i^* D_i$, $R_i = D_i^* D_i$. The next result is an adaptation of the result proved in [9], and represents a characterization of the H_2-norm, for $F \in \mathbb{K}$, in terms of the solution of the observability and controllability gramians.

Proposition 4 :

$$\| \mathcal{G}_F \|_2^2 = \sum_{j=1}^{N} tr(J_j^* \mathcal{P}_j(F) J_j)$$

$$= \sum_{i=1}^{N} tr\left(\begin{pmatrix} Q_i & L_i \\ L_i^* & R_i \end{pmatrix} \begin{pmatrix} U_i(F) & U_i(F)F_i^* \\ F_iU_i(F) & F_iU_i(F)F_i^* \end{pmatrix} \right).$$

4. Equivalence Results

The goal of this section is to present the equivalence among the mean square stabilizing solution of the CARE, the H_2-control problem, and an LMI optimization problem. From now on we shall consider all matrices real. Define the convex set

$$\Psi = \{ W = (W_1, \ldots, W_N);\ \text{for } j = 1, \ldots, N, W_j = \begin{pmatrix} W_{j1} & W_{j2} \\ W_{j2}^* & W_{j3} \end{pmatrix} \geq 0, W_{j1} > 0,$$

$$\sum_{i=1}^{N} p_{ij}(A_iW_{i1}A_i^* + B_iW_{i2}^*A_i^* + A_iW_{i2}B_i^* + B_iW_{i3}B_i^*) - W_{j1} + J_jJ_j^* \leq 0 \}$$

and the set $\hat{\Psi} \subseteq \Psi$

$$\hat{\Psi} = \{ W = (W_1, \ldots, W_N) \subseteq \Psi;\ \text{for } j = 1, \ldots, N, W_{j3} = W_{j2}^* W_{j1}^{-1} W_{j2} \text{ and}$$

$$\sum_{i=1}^{N} p_{ij}(A_iW_{i1}A_i^* + B_iW_{i2}^*A_i^* + A_iW_{i2}B_i^* + B_iW_{i3}B_i^*) - W_{j1} + J_jJ_j^* = 0 \}.$$

Define also the cost function

$$\mu(W) = \sum_{j=1}^{N} tr\left(\begin{pmatrix} Q_i & L_i \\ L_i^* & R_i \end{pmatrix} \begin{pmatrix} W_{i1} & W_{i2} \\ W_{i2}^* & W_{i3} \end{pmatrix} \right).$$

For any $F = (F_1, \ldots, F_N) \in \mathbb{K}$, let $U(F) = (U_1(F), \ldots, U_N(F)) > 0$ be as in (11). Define

$$\mathcal{U}(F) = \left(\begin{pmatrix} U_1(F) & U_1(F)F_1^* \\ F_1U_1(F) & F_1U_1(F)F_1^* \end{pmatrix}, \ldots, \begin{pmatrix} U_N(F) & U_N(F)F_N^* \\ F_NU_N(F) & F_NU_N(F)F_N^* \end{pmatrix} \right).$$

From Remark 1 and equation (11) it is immediate that $\mathcal{U}(F) \in \hat{\Psi}$. For any $W = (W_1, \ldots, W_N) \subseteq \Psi$, let $\mathcal{V}(W) = (W_{12}^* W_{11}^{-1}, \ldots, W_{N2}^* W_{N1}^{-1})$. Since $W_{i3} \geq W_{i2}^* W_{i1}^{-1} W_{i2}$ (see Remark 1), we get that

$$\sum_{i=1}^{N} p_{ij}(A_i + B_i W_{i2}^* W_{i1}^{-1}) W_{i1} (A_i + B_i W_{i2}^* W_{i1}^{-1})^* - W_{j1} + J_j J_j^* \leq$$
$$\sum_{i=1}^{N} p_{ij}(A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* + B_i W_{i3} B_i^*) - W_{j1} + J_j J_j^* \leq 0$$

and thus from Proposition 2, $\mathcal{V}(W) \in \mathbb{K}$. This defines mappings $\mathcal{U} : \mathbb{K} \to \hat{\Psi}$ and $\mathcal{V} : \Psi \to \mathbb{K}$. We have the following proposition.

Proposition 5 : *The following assertions hold:*

a) $\mathcal{VU} = \mathcal{I}$, *b)* $\mathcal{UV} = \mathcal{I}$ *on* $\hat{\Psi}$ *and c)* $\mathcal{UV}(W) \leq W$ *for any* $W \in \Psi$.

Proof: It is immediate to check that $\mathcal{V}(\mathcal{U}(F)) = F$, showing a). From the uniqueness of the solution $U(F)$ of (11), stated in Proposition 2, we get that for any $W \in \hat{\Psi}$, $U(\mathcal{V}(W)) = (W_{11}, \dots, W_{N1})$, so that $\mathcal{U}(\mathcal{V}(W)) = W$, showing b). Let us now prove c). We have that

$$\sum_{i=1}^{N} p_{ij}(A_i + B_i W_{i2}^* W_{i1}^{-1}) U_i(\mathcal{V}(W)) (A_i + B_i W_{i2}^* W_{i1}^{-1})^* - U_j(\mathcal{V}(W)) + J_j J_j^* = 0$$

and

$$\sum_{i=1}^{N} p_{ij}(A_i + B_i W_{i2}^* W_{i1}^{-1}) W_{i1} (A_i + B_i W_{i2}^* W_{i1}^{-1})^* - W_{j1} + J_j J_j^* \leq$$
$$\sum_{i=1}^{N} p_{ij}(A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* + B_i W_{i3} B_i^*) - W_{j1} + J_j J_j^* \leq 0$$

and thus, from Proposition 2, $U_j(\mathcal{V}(W)) \leq W_{j1}$, $j = 1, \dots, N$. Let us show now that

$$W_i - (\mathcal{UV})_i(W) = \begin{pmatrix} W_{i1} - U_i(\mathcal{V}(W)) & W_{i2} - U_i(\mathcal{V}(W)) W_{i1}^{-1} W_{i2} \\ W_{i2}^* - W_{i2}^* W_{i1}^{-1} U_i(\mathcal{V}(W)) & W_{i3} - W_{i2}^* W_{i1}^{-1} U_i(\mathcal{V}(W)) W_{i1}^{-1} W_{i2} \end{pmatrix} \geq 0. \quad (12)$$

Indeed, $U_i(\mathcal{V}(W)) \leq W_{i1}$,

$$(W_{i1} - U_i(\mathcal{V}(W)))(W_{i1} - U_i(\mathcal{V}(W)))^\dagger (W_{i1} - U_i(\mathcal{V}(W))) W_{i1}^{-1} W_{i2} = W_{i2} - U_i(\mathcal{V}(W)) W_{i1}^{-1} W_{i2},$$

and

$$W_{i3} - W_{i2}^* W_{i1}^{-1} U_i(\mathcal{V}(W)) W_{i1}^{-1} W_{i2} - W_{i2}^* W_{i1}^{-1} (W_{i1} - U_i(\mathcal{V}(W)))$$
$$(W_{i1} - U_i(\mathcal{V}(W)))^\dagger (W_{i1} - U_i(\mathcal{V}(W))) W_{i1}^{-1} W_{i2} =$$
$$W_{i3} - W_{i2}^* W_{i1}^{-1} W_{i2} \geq 0$$

and from Remark 1 we obtain that (12) holds. □

We can establish now the main result of this section, showing the equivalence between the existence of the mean square stabilizing solution for the CARE, the H_2-control problem, and the convex problem.

Theorem 2 : *The following assertions are equivalent:*

a) There exists the mean square stabilizing solution $P = (P_1, \ldots, P_N) \geq 0$ *for the CARE given by (6).*

b) There exists $F = (F_1, \ldots, F_N) \in \mathbb{K}$ *such that*

$$\parallel \mathcal{G}_F \parallel_2^2 = min\{ \parallel \mathcal{G}_K \parallel_2; K \in \mathbb{K}\}.$$

c) There exists $W = (W_1, \ldots, W_N) \in \Psi$ *such that*

$$\mu(W) = min\{\mu(V); V \in \Psi\}.$$

Moreover,

1) if P *satisfies a) then* $\mathcal{F}(P)$ *satisfies b) and* $\mathcal{U}(\mathcal{F}(P))$ *satisfies c).*

2) if F *satisfies b) then* $\mathcal{P}(F)$ *satisfies a) and* $\mathcal{U}(F)$ *satisfies c).*

3) if W *satisfies c) then* $\mathcal{P}(\mathcal{V}(W))$ *satisfies a) and* $\mathcal{V}(W)$ *satisfies b).*

Proof : The second part of the proof will follow immediately from the first one. Let us show first that b) is equivalent to c). Indeed, from Propositions 4 and 5 it is immediate that

$$min\{ \parallel \mathcal{G}_K \parallel_2; K \in \mathbb{K}\} = min\{\mu(V); V \in \hat{\Psi}\},$$

and since $\hat{\Psi} \subseteq \Psi$, it is clear that $min\{\mu(V); V \in \hat{\Psi}\} \geq min\{\mu(V); V \in \Psi\}$. On the other hand, for any $V \in \Psi$, we have from Proposition 5 that $\mathcal{U}\mathcal{V}(V) \leq V$, with $\mathcal{U}\mathcal{V}(V) \in \hat{\Psi}$. Therefore, since $\mu(V) \geq \mu(\mathcal{U}\mathcal{V}(V))$, we get that $min\{\mu(V); V \in \hat{\Psi}\} \leq min\{\mu(V); V \in \Psi\}$. This shows that

$$min\{ \parallel \mathcal{G}_K \parallel_2; K \in \mathbb{K}\} = min\{\mu(V); V \in \Psi\},$$

completing the proof of the equivalence between b) and c).

Let us now prove the equivalence between a) and b). Suppose that $P \geq 0$ is the mean square stabilizing solution of the CARE. From (6) it is easy to show that

P satisfies

$$P_i - (A_i + B_i\mathcal{F}_i(P))^* \mathcal{E}_i(P)(A_i + B_i\mathcal{F}_i(P)) = \mathcal{S}_i(\mathcal{F}(P))$$

and therefore, from uniqueness (see Proposition 2), $\mathcal{P}(\mathcal{F}(P)) = P$. For any $K \in \mathbb{K}$ and $\mathcal{P}(K)$ as in (10) we have, according to Proposition 3 (see (7)) that,

$$\begin{aligned}&(\mathcal{P}_i(K) - P_i) - (A_i + B_iK_i)^* \mathcal{E}_i(\mathcal{P}(K) - P)(A_i + B_iK_i) = \\ &(K_i - \mathcal{F}_i(P))^* \mathcal{D}_i(P)(K_i - \mathcal{F}_i(P)),\end{aligned} \tag{13}$$

and from Proposition 2, $\mathcal{P}(K) \geq P$. From Proposition 4,

$$\| \mathcal{G}_K \|_2^2 = \sum_{j=1}^{N} tr(J_j^* \mathcal{P}_j(K) J_j) \geq \sum_{j=1}^{N} tr(J_j^* P_j J_j) = \| \mathcal{G}_{\mathcal{F}(P)} \|_2^2,$$

showing that $\mathcal{F}(P)$ satisfies b). On the other hand, suppose that there exists $F = (F_1, \ldots, F_N) \in \mathbb{K}$ satisfying b). Then clearly (A,B) is mean square stabilizable and there exists the maximal solution $P \geq 0$ to the CARE (6). Moreover, according to Theorem 1, we can find a sequence $P^l \in \mathbb{H}^{n+}$ such that $F^l = \mathcal{F}(P^{l-1}) \in \mathbb{K}$, $l = 1, 2, \ldots$, equations (8) are satisfied, and P^l converges to P. Start with $F^0 = F$, so that $P^0 = \mathcal{P}(F) \geq P^l \geq P$. From optimality of F, Proposition 4, and equations (8) we get that

$$\| \mathcal{G}_{F^l} \|_2^2 = \sum_{j=1}^{N} tr(J_j^* P_j^l J_j) \geq \sum_{j=1}^{N} tr(J_j^* \mathcal{P}_j(F) J_j) = \| \mathcal{G}_F \|_2^2$$

that is,

$$\sum_{j=1}^{N} tr(J_j^* (\mathcal{P}_j(F) - P_j^l) J_j) = 0$$

and since $J_j J_j^* > 0$, we can conclude that $\mathcal{P}(F) = P^l = P$. From equation (13) and recalling that $\mathcal{D}_i(P) > 0$, we get that $F = \mathcal{F}(P) \in \mathbb{K}$, proving that $P \geq 0$ is indeed the mean square stabilizing solution for the CARE (6). $\square$

5. Robust H_2-Control

In this section we consider the following MJLS with uncertainties,

$$\mathcal{G} = \begin{cases} x(k+1) = (A_{\theta(k)} + \Delta A_{\theta(k)})x(k) \\ \qquad\qquad + (B_{\theta(k)} + \Delta B_{\theta(k)})u(k) + J_{\theta(k)}w(k) \\ x(0) = 0, \theta(0) = \theta_0 \\ z(k) = \mathcal{C}_{\theta(k)}x(k) + D_{\theta(k)}u(k) \end{cases}$$

where ΔA_i, ΔB_i, $i = 1, \ldots, N$ are the uncertainties satisfying the following norm bounded condition,

$$\Delta A_i = E_i \Delta_i M_i, \quad \Delta B_i = L_i \Delta_i N_i, \quad \Delta_i \Delta_i^* \leq I$$

for $i = 1, \ldots, N$. The transition probability of the Markov chain $\mathbb{P}$ is assumed to belong to a convex set $\mathbb{D} = \{\mathbb{P}; \mathbb{P} = \sum_{t=1}^{r} \alpha^t \mathbb{P}^t, \ \ \alpha^t \geq 0, \sum_{t=1}^{r} \alpha^t = 1\}$, where $\mathbb{P}^t = (p_{ij}^t)$ are known transition probability matrices. We shall redefine the convex set Ψ in the following way:

$$\Psi = \bigcap_{t=1}^{r} \Psi^t$$

where Ψ^t, $t = 1, \ldots, r$, are defined as

$$\Psi^t = \{ W = (W_1, \ldots, W_N); \text{ for } i = 1, \ldots, N, W_i = \begin{pmatrix} W_{i1} & W_{i2} \\ W_{i2}^* & W_{i3} \end{pmatrix} \geq 0, W_{i1} > 0, \\ \mathcal{H}_i^t(W) \geq 0\}$$

with, for $i, j = 1, \ldots, N$

$$\mathcal{D}_i(W) = \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W_{i3} N_i^* \end{pmatrix}$$

$$\mathcal{X}_{ij}^t(W) = \begin{pmatrix} \sqrt{p_{ij}^t}(A_i W_{i1} + B_i W_{i2}^*) M_i^* & \sqrt{p_{ij}^t}(A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix}$$

$$\mathcal{Z}_j^t(W) = \sum_{i=1}^{N} p_{ij}^t \Big(A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* + B_i W_{i3} B_i^* + E_i E_i^* + L_i L_i^* \Big) + J_j J_j^*$$

$$\mathcal{H}_j^t(W) = \begin{pmatrix} W_{j1} - \mathcal{Z}_j^t(W) & \mathcal{X}_{1j}^t(W) & \ldots & \mathcal{X}_{Nj}^t(W) \\ \mathcal{X}_{1j}^t(W)^* & I - \mathcal{D}_1(W) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_{Nj}^t(W)^* & 0 & \ldots & I - \mathcal{D}_N(W) \end{pmatrix}.$$

Proposition 6 : *Suppose that there exists $W \in \Psi$. Then*

$$W_{j1} \geq \sum_{i=1}^{N} p_{ij}(A_i + B_i F_i + \Delta A_i + \Delta B_i F_i) W_{i1} (A_i + B_i F_i + \Delta A_i + \Delta B_i F_i)^* + J_j J_j^*, \; j = 1, \ldots, N$$

where for $i = 1, \ldots, N$, $\Delta A_i = E_i \Delta_i M_i$, $\Delta B_i = L_i \Delta_i N_i, \Delta_i$ satisfies $\Delta_i \Delta_i^* \leq I$, $\mathbb{P} \in \mathbb{D}$, and,

$$F_i = W_{i2}^* W_{i1}^{-1}.$$

Proof: We have that if $W \in \Psi$ then $W \in \Psi^t$ for each $t = 1, \ldots, r$, and from Remark 1, $\mathcal{H}_j^t(W) \geq 0$ if and only if

$$\begin{pmatrix} I - \mathcal{D}_1(W) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & I - \mathcal{D}_N(W) \end{pmatrix} \geq 0,$$

$$W_{j1} - \mathcal{Z}_j^t(W) - \begin{pmatrix} \mathcal{X}_{1j}^t(W) & \ldots & \mathcal{X}_{Nj}^t(W) \end{pmatrix} \begin{pmatrix} I - \mathcal{D}_1(W) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & I - \mathcal{D}_N(W) \end{pmatrix}^{\dagger} \begin{pmatrix} \mathcal{X}_{1j}^t(W)^* \\ \vdots \\ \mathcal{X}_{Nj}^t(W)^* \end{pmatrix} \geq 0 \quad (14)$$

and

$$\begin{pmatrix} \mathcal{X}^t_{1j}(W) & \dots & \mathcal{X}^t_{Nj}(W) \end{pmatrix} = \begin{pmatrix} \mathcal{X}^t_{1j}(W) & \dots & \mathcal{X}^t_{Nj}(W) \end{pmatrix}$$
$$\begin{pmatrix} I-\mathcal{D}_1(W) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I-\mathcal{D}_N(W) \end{pmatrix}^{\dagger} \begin{pmatrix} I-\mathcal{D}_1(W) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I-\mathcal{D}_N(W) \end{pmatrix}. \quad (15)$$

Equations (14) and (15) canbe rewritten as

$$0 \le W_{j1} - J_j J_j^* - \sum_{i=1}^{N} p^t_{ij}\Big(A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* + B_i W_{i3} B_i^*$$
$$+ E_i E_i^* + L_i L_i^* + \mathcal{Y}_i(W)\Big),$$
$$\mathcal{Y}_i(W) = \begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix}$$
$$\left(I - \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W'_{i3} N_i^* \end{pmatrix} \right)^{\dagger} \begin{pmatrix} M_i (A_i W_{i1} + B_i W_{i2}^*)^* \\ N_i (A_i W_{i2} + B_i W_{i3})^* \end{pmatrix}. \quad (16)$$

and

$$\begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix} =$$
$$\begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix}$$
$$\left(I - \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W'_{i3} N_i^* \end{pmatrix} \right)^{\dagger} \left(I - \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W'_{i3} N_i^* \end{pmatrix} \right). \quad (17)$$

Write now

$$T_i(W) = \begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix}$$

$$\left(\left(I - \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W'_{i3} N_i^* \end{pmatrix} \right)^{\dagger} \right)^{\frac{1}{2}}$$

$$- \begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \left(I - \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W'_{i3} N_i^* \end{pmatrix} \right)^{\frac{1}{2}}.$$

Therefore, from equation (17) and the properties of the generalized inverse seen in section 2,

$$0 \le T_i(W)T_i(W)^* = \mathcal{Y}_i(W) +$$

$$\begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \left(I - \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W_{i3}' N_i^* \end{pmatrix} \right) \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} -$$

$$\begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix} \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} -$$

$$\begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \begin{pmatrix} M_i (A_i W_{i1} + B_i W_{i2}^*)^* \\ N_i (A_i W_{i2} + B_i W_{i3})^* \end{pmatrix} \tag{18}$$

and from $\Delta_i \Delta_i^* \le I$, we get that

$$\begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} \le E_i E_i^* + L_i L_i^* \tag{19}$$

so that, from (18) and (19),

$$\mathcal{Y}_i(W) + E_i E_i^* + L_i L_i^* \ge \begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W_{i3}' N_i^* \end{pmatrix} \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} +$$

$$\begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix} \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} +$$

$$\begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \begin{pmatrix} M_i (A_i W_{i1} + B_i W_{i2}^*)^* \\ N_i (A_i W_{i2} + B_i W_{i3})^* \end{pmatrix}. \tag{20}$$

From (20) and (16), and recalling that $W_{i3} \geq W_2^* W_{i1}^* W_{i2}$ we get,

$$
\begin{aligned}
W_{j1} \geq\ & J_j J_j^* + \sum_{i=1}^{N} p_{ij}^t \Big(A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* \\
& + B_i W_{i3} B_i^* + E_i E_i^* + L_i L_i^* + \mathcal{Y}_i(W) \Big) \\
& \geq J_j J_j^* + \sum_{i=1}^{N} p_{ij}^t \Big(A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* + B_i W_{i3} B_i^* \\
& + \begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \begin{pmatrix} M_i W_{i1} M_i^* & M_i W_{i2} N_i^* \\ N_i W_{i2}^* M_i^* & N_i W_{i3}' N_i^* \end{pmatrix} \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} \\
& + \begin{pmatrix} (A_i W_{i1} + B_i W_{i2}^*) M_i^* & (A_i W_{i2} + B_i W_{i3}) N_i^* \end{pmatrix} \Delta_i^* \begin{pmatrix} E_i^* \\ L_i^* \end{pmatrix} \\
& + \begin{pmatrix} E_i & L_i \end{pmatrix} \Delta_i \begin{pmatrix} M_i (A_i W_{i1} + B_i W_{i2}^*)^* \\ N_i (A_i W_{i2} + B_i W_{i3})^* \end{pmatrix} \Big) \\
& = J_j J_j^* + \sum_{i=1}^{N} p_{ij}^t \Big((A_i + \Delta A_i) W_{i1} (A_i + \Delta A_i)^* \\
& + (B_i + \Delta B_i) W_{i3} (B_i + \Delta B_i)^* \\
& + (A_i + \Delta A_i) W_{i2} (B_i + \Delta B_i)^* + (B_i + \Delta B_i) W_{i2}^* (A_i + \Delta A_i)^* \Big) \\
& \geq J_j J_j^* + \sum_{i=1}^{N} p_{ij}^t (A_i + B_i F_i + \Delta A_i \\
& + \Delta B_i F_i) W_{i1} (A_i + B_i F_i + \Delta A_i + \Delta B_i F_i)^*. \qquad (21)
\end{aligned}
$$

Since $p_{ij} = \sum_{t=1}^{r} \alpha^t p_{ij}^t$, for some $\alpha^t \geq 0, \sum_{t=1}^{r} \alpha^t = 1$, and (21) is satisfied for every $t = 1, \ldots, r$, (since $W \in \bigcap_{t=1}^{r} \Psi^t$) we have, after multiplying by α^t and taking the sum over $t = 1, \ldots, r$, the desired result. □

Remark 3: Notice that for the case in which there are no uncertainties on the matrices A_i, B_i, and on the transition probability $\mathbb{P}$ (that is, $E_i = 0, L_i = 0$, $M_i = 0$, $N_i = 0$, $\mathbb{D} = \{\mathbb{P}\}$), the restriction $\mathcal{H}_j^1(W) \geq 0$ reduces to

$$
\begin{aligned}
& 0 \leq W_{j1} - \mathcal{Z}_j^1(W) = \\
& W_{j1} - \sum_{i=1}^{N} p_{ij} (A_i W_{i1} A_i^* + B_i W_{i2}^* A_i^* + A_i W_{i2} B_i^* + B_i W_{i3} B_i^*) - J_j J_j^*
\end{aligned}
$$

and thus the set Ψ coincides with the one in section 4.

Theorem 3 : *Suppose that there exists* $W = (W_1, \dots, W_N) \in \Psi$ *such that*

$$\mu(W) = min\{\mu(V); V \in \Psi\}.$$

Then for $F = (F_1, \dots, F_N)$ *defined as* $F_i = W_{i2}^* W_{i1}^{-1}$, $i = 1, \dots, N$, *we have that system* $\mathcal{G}_F$ *is MSS and*

$$\| \mathcal{G}_F \|_2^2 \leq \mu(W).$$

Proof: For any Δ_i satisfying $\Delta_i \Delta_i^* \leq I$, let us denote $\Delta = (\Delta_1, \dots, \Delta_N)$ and for $V = (V_1, \dots, V_N)$, let $\mathcal{T}_\Delta(V) = (\mathcal{T}_{\Delta 1}(V), \dots, \mathcal{T}_{\Delta N}(V))$, where,

$$\mathcal{T}_{\Delta_j}(V) := \sum_{i=1}^{N} p_{ij}(A_i + B_i F_i + \Delta A_i + \Delta B_i F_i) V_i (A_i + B_i F_i + \Delta A_i + \Delta B_i F_i)^*.$$

Let us write $P = (P_1, \dots, P_N)$, $P_i = W_{i1}$, and $JJ^* = (J_1 J_1^*, \dots, J_N J_N^*)$ $i = 1, \dots, N$. From Proposition 6 we have that whatever Δ_i satisfying $\Delta_i \Delta_i^* \leq I$, and $\mathbb{P} \in \mathbb{D}$,

$$P \geq \mathcal{T}_\Delta(P) + JJ^*$$

and from Proposition 2 we get that $\mathcal{G}_F$ is MSS. From Proposition 5,

$$\mathcal{U}\mathcal{V}(W) \leq W$$

and from Proposition 4,

$$\| \mathcal{G}_F \|_2^2 = \mu(\mathcal{U}\mathcal{V}(W)) \leq \mu(W)$$

completing the proof of the Theorem. □

6. Numerical Example

Consider the following example, adapted from [9]. The MJLS has three operating modes, described by:

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2.2308+\delta & 2.5462+\delta \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1.5049 & -1.0709 \\ -1.0709 & 1.6160 \\ 0 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 1 \\ -38.9103+\delta & 2.5462+\delta \end{pmatrix}, \quad C_2 = \begin{pmatrix} 10.2036 & -10.3952 \\ -10.3952 & 11.2819 \\ 0 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 1 \\ 4.6384+\delta & -4.7455+\delta \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1.7355 & -1.2255 \\ -1.2255 & 1.6639 \\ 0 & 0 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0 \\ 0 \\ 1.6125 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 \\ 0 \\ 1.0794 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 \\ 0 \\ 1.0540 \end{pmatrix},$$

$$B_1 = B_2 = B_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad J_1 = J_2 = J_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both state and jump variables are assumed available. Two cases are considered:

i) $\delta = 0$ and $\mathbb{P}$ is exactly known, given by

$$\mathbb{P} = \begin{pmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{pmatrix}.$$

For this case, the optimal solution is given by $\mu_1 = 4124$and controllers

$$F_1 = \begin{pmatrix} 2.2153 & -1.5909 \end{pmatrix} \quad F_2 = \begin{pmatrix} 38.8637 & -38.8864 \end{pmatrix} \quad F_3 = \begin{pmatrix} -4.6176 & 5.6267 \end{pmatrix}.$$

Note that this result is equivalent to obtaining the maximal solution of the associated CARE (see [12],[26] or [1]) or using the convex approach of [9].

ii) $\delta \in [-0.1, 0.1]$ and $\mathbb{P} \in \mathbb{D}$, where $\mathbb{D}$ is the polytope defined by the transition probability matrices

$$\mathbb{P}^1 = \begin{pmatrix} 0.51 & 0.25 & 0.24 \\ 0.14 & 0.55 & 0.31 \\ 0.10 & 0.18 & 0.72 \end{pmatrix}, \quad \mathbb{P}^2 = \begin{pmatrix} 0.83 & 0.09 & 0.08 \\ 0.46 & 0.39 & 0.15 \\ 0.42 & 0.02 & 0.56 \end{pmatrix},$$
$$\mathbb{P}^3 = \begin{pmatrix} 0.50 & 0.25 & 0.25 \\ 0.20 & 0.50 & 0.30 \\ 0.30 & 0.30 & 0.40 \end{pmatrix}, \quad \mathbb{P}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the uncertainties defined by δ, we have that

$$E_1 = E_2 = E_3 = \begin{pmatrix} 0 \\ 0.3162 \end{pmatrix}, \quad M_1 = M_2 = M_3 = \begin{pmatrix} 0.3162 & 0.3162 \end{pmatrix},$$

$$L_1 = L_2 = L_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad N_1 = N_2 = N_3 = 0.$$

The optimal solution in this case is μ_2=6840 and $F_1 = \begin{pmatrix} 2.2281 & -2.4440 \end{pmatrix}$, $F_2 = \begin{pmatrix} 38.8998 & -39.7265 \end{pmatrix}$, $F_3 = \begin{pmatrix} -4.6360 & 4.8930 \end{pmatrix}$. As expected, due to the uncertainties involved in the design, $\mu_2 > \mu_1$.

7. Final Remarks

In this paper we have considered the robust H_2-control problem of Markovian jump linear systems (MJLS). It is assumed that both the state and jump variables are available to the controller. Robustness here is in the sense that the system is considered to have uncertainties on the transition probability of the Markov chain as well as on the matrices of the system. The uncertainties on the matrices are of the norm bounded form. An LMI optimization problem was proposed which provides a mean square stabilizing controller for the closed loop MJLS, as well as an upper bound for the H_2-norm of the system. For the case with no uncertainties it was shown that the existence of a solution for the resulting LMI optimization problem is equivalent to the existence of the mean square stabilizing solution for the discrete-time coupled algebraic Riccati equations (CARE) associated to the quadratic cost control problem for MJLS. Therefore, for the case with no uncertainties, the convex approach imposes no extra conditions than those usually required for the dynamic programming approach, which is associated to the CARE. This result differs from the one recently published in the literature ([26]), which connects the maximal solution for the CARE and a convex problem. The convex problem with no uncertainties presented here (see section 4) is related to the mean square stabilizing solution of the CARE and thus is different from the one in [26].

Appendix
Stabilizability and Detectability Tests

Proposition 5 and Theorem 2 suggest the following mean square stabilizability and detectability tests for a MJLS, based on LMI´s.

Proposition A-1: Stabilizability test. The pair (A,B) is mean square stabilizable if and only if the convex set Ψ defined in section 4 is not empty.

Proof: If $F \in \mathbb{K} \neq \emptyset$ then as seen in section 4, $\mathcal{U}(F) \in \hat{\Psi} \subset \Psi \neq \emptyset$. On the other hand, if $W \in \Psi \neq \emptyset$ then again from section 4, $\mathcal{V}(W) \in \mathbb{K} \neq \emptyset$. □

Proposition A-2: Detectability test. The pair (C, A) is mean square detectable if and only if there are $R = (R_1, \ldots, R_N) \in \mathbb{H}^{n+}$, $Z = (Z_1, \ldots, Z_N) \in \mathbb{H}^{n+}$, $S = (S_1, \ldots, S_N) \in \mathbb{H}^{s,n}$, and $V = (V_1, \ldots, V_N) \in \mathbb{H}^{s+}$ such that for $i = 1, \ldots, N$,

$$A_i^* Z_i A_i + C_i^* S_i^* A_i + A_i^* S_i C_i + C_i^* V_i C_i - R_i < 0$$

$$\begin{pmatrix} Z_i & S_i \\ S_i^* & V_i \end{pmatrix} \geq 0$$

$$Z_i \geq \mathcal{E}_i(R)$$

$$R_i > 0$$

$$Z_i > 0 .$$

Proof: Analogous to the proof of Proposition A-1.

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