Chapter 6

A LINEAR MATRIX INEQUALITIES APPROACH TO ROBUST MEAN-SEMIVARIANCE PORTFOLIO OPTIMIZATION

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Abstract The main goal of this chapter is to formulate a robust mean-semivariance portfolio selection problem in terms of a linear matrix inequalities (LMI) optimization problem. We consider different forms of calculating the mean and semivariance of the tracking error. It is desired to minimize an objective function defined as a convex combination of the risk function minus the expected return of the tracking error.

Keywords: mean-semivariance, portfolio optimization, linear matrix inequalities, computational tool.

1. Introduction

Mean-Variance portfolio selection is a classical financial problem introduced by [Markowitz, 1959] in which it is desired to reduce risks by diversifying assets allocation. The main goal is to maximize the expected return for a given level of risk or minimize the expected risk for a given level of expected

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return. Optimal portfolio selection is the most used and well known tool for economic allocation of capital (see [Campbell et al., 1997], [Elton and Gruber, 1995], [Jorion, 1992]). More recently it has been extended to include tracking error optimization (see [Roll, 1992]) and semivariance models (see [Hanza and Janssen, 1998]). In this case the professional money manager is judged by total return performance relative to a pre specified benchmark portfolio, usually a broadly diversified index of assets. The allocation decision problem is based on the difference between the manager's return and the benchmark return, the so-called tracking error. In the semivariance model the risk function is taken as a convex combination of the semivariances (below and above the expected return) of the tracking error. As examples of previous works on tracking error we can mention [Roll, 1992] who considers the problem of minimizing the volatility of the tracking error, and [Rudolf et al., 1999], who used linear models for tracking error minimization. Other objective functions can also be considered ([Zenios, 1993]), and a unifying approach to these methodologies can be found in [Duarte Jr., 1999].

As pointed out in [Rustem et al., 1995], for the optimal mean-variance strategy to be useful the set of expected return of the component assets and the covariance matrix should be sufficiently precise. Small changes in expected returns can produce large changes in asset allocation decisions, as shown by [Black and Litterman, 1991]. This lack of robustness with respect to the inherent inaccuracy of the expected returns and covariance matrix estimates motivated the studies of robust financial optimization. In this case the optimization is performed not for a single value of covariance matrix and expected returns but instead a set of possible values for these parameters (called scenarios) are considered. Robust min-max portfolio strategies were considered in [Rustem et al., 1995], where it is desired to find the portfolio which minimizes the worst case (maximum over the scenarios) value of the objective function. Although a worst case solution among all scenarios is used, in practice this can be a good strategy, mainly when analyzing periods of great volatility and emerging markets. In [Costa and Paiva, 2002] it was proposed a linear matrix inequality approach to solve numerically a robust optimal portfolio selection problem. The expected returns of the risky and risky-free assets as well as the covariance matrix of the risky assets were assumed to belong to a convex polytope defined by some vertices provided by the asset manager. Two problems were considered: the first one was to find a portfolio of minimum worst case volatility of the tracking error with guaranteed fixed minimum target expected performance. The second one was to find a portfolio of maximum worst case target expected performance with guaranteed fixed maximum volatility of the tracking error. These two problems were shown to be equivalent to solving LMI (linear matrix inequalities) optimization problems. Other papers related to robust finance algorithms are [Howe and Rustem, 1997] and [Howe et al., 1996].

Due to the large number of fast and reliable computational techniques available for LMI optimization programming nowadays (see [Oliveira et al., 1994]), this approach has shown to be an important tool to derive numerical algorithms in robust control and related problems. In particular, algorithms using LMI optimization programming for obtaining control for uncertain systems have been recently presented in the literature ([Ghaoui and Niculescu, 2000], [Boyd et al., 1994], [Costa et al., 1997], [Geromel et al., 1991]). We refer the reader to these references for further details on the practical implementation of LMI algorithms.

The main goal of this chapter is to use LMI to solve a robust tracking error optimization problem under the mean-semivariance criterion and considering different possible values for the expected returns and semivariances. We assume that the expected returns of the risky and risky free assets belong to a convex polytope defined by some vertices, provided by the asset manager. For the calculation of the semivariances, we consider estimates based on historical data of the asset returns. Weights declining over time are applied to the set of data points. We assume that the matrix formed by the weights belong to a convex polytope defined by some vertices provided by the asset manager. The idea behind these hypothesis is that these parameters (expected returns and semivariances) are not exactly known or cannot be confidentially estimated. The objective value function which it is desired to minimize is a convex combination of the semivariance of the tracking error minus the expected return of the tracking error. The robust tracking error optimization problem in this case is to find the portfolio which minimizes the worst case (maximum over all possible convex combinations of the parameters) value of the objective function. As pointed out in remark 1 below, the optimization problem considered here is different from the ones studied in [Costa and Paiva, 2002]. We believe that the technique presented in this chapter represents a computational tool in the direction of overcoming the main limitations of standard mean-semivariance optimization mentioned above, with special focus on real problems.

The chapter is organized in the following way. The notation, basic results, and problem formulation that will be considered throughout the work are presented in Section 2. Section 3 presents the LMI formulation for the mean-semivariance optimization problem for the robust portfolio tracking error. Two numerical examples in the São Paulo stock exchange (BOVESPA) are presented in Section 4. The chapter is concluded in Section 5 with some final remarks. All the proofs are presented in the Appendix at the end of the chapter.

2. Preliminaries

2.1. LMI and Some Definitions

We will denote by $\mathbb{R}$ and $\mathbb{R}^n$ the space of real values and n-dimensional real vectors respectively. For a matrix Q, we set Q' as the transpose of Q and $Q \geq 0$ ($Q > 0$ respectively) means that the symmetric matrix Q must be positive semi-definite (positive definite). For $x \in \mathbb{R}^n$ and $\mathbf{0}$ the n-dimensional zero vector the inequality $x \geq \mathbf{0}$ means that each element of the vector x is greater than or equal to zero.

We recall now some basic facts on LMI. A (non strict) LMI has the following form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \geq 0$$

where $x = \begin{pmatrix} x_1^T & \dots & x_n^T \end{pmatrix}^T \in \mathbb{R}^n$ are the variables and the symmetric matrices $F_i, i = 0, \dots, n$ are given. A key result for converting nonlinear convex inequalities into LMI formulation is the Schur complement, which we present next (see [Saberi et al., 1995], page 13).

Proposition 1. *Suppose $R > 0$. Then*

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0 \tag{6.1}$$

if and only if

$$Q - SR^{-1}S' \geq 0. \tag{6.2}$$

Notice that the first inequality in Proposition 1 is in the form of an LMI, and the second inequality is in the form of a nonlinear convex inequality. An LMI optimization problem consists of finding a feasible x (that is, find x such that $F(x) \geq 0$) which minimizes (or maximizes) a linear function $c(x)$. As pointed out in [Boyd et al., 1994], [Ghaoui and Niculescu, 2000], there are nowadays very efficient numerical algorithms for solving LMI optimization problems (see for instance [Oliveira et al., 1994]), with the global optimum found in polynomial time ([Boyd et al., 1994], [Ghaoui and Niculescu, 2000]). These methods can be roughly divided in two classes; interior-point methods and non-differentiable optimization methods. Interior-point methods can be viewed as a reduction of the original problem to a sequence of differentiable unconstrained minimization sub-problems, with each of these sub-problems being efficiently solved, perhaps approximately. When the number of variables in the optimization problem is not too large (say, less than 1000) or some sparsity property of the problem can be exploited, primal-dual interior-points methods are highly efficient ([Ghaoui and Niculescu, 2000]). For a brief description on how primal-dual interior-points methods work, see [Ghaoui and Niculescu, 2000]. When the number of variables is sufficiently large and sparsity cannot be exploited, it may be advantageous to use non-differentiable optimization problems. Opposite to interior-point methods, which transform the original problem into a sequence of smooth ones, the non-smooth analysis directly handles the problem with its non-differentiable functions. Yet the problem is convex and all theoretical tools from convex analysis can be applied, so that first-order and second order generalized derivatives can be explicitly quantified and numerically implemented. For further properties of these methods, see [Ghaoui and Niculescu, 2000], chapter 3.

An immediate Corollary from Proposition 1 is the following:

Corollary 1. *Suppose $Q \geq 0$ is an n by n matrix, $x(i) \in \mathbb{R}^n$, and $\sum_{i=1}^{k} \alpha(i) = 1, \alpha(i) \geq 0, i = 1, \ldots, k$. Then*

$$\left\| Q^{\frac{1}{2}} \Big(\sum_{i=1}^{k} \alpha(i) x(i) \Big) \right\|^2 \leq \sum_{i=1}^{k} \alpha(i) \left\| Q^{\frac{1}{2}} x(i) \right\|^2.$$

In what follows it is convenient to define, for $\mathbb{X}$ a space of real vectors or matrices, and a collection of points $v(i) \in \mathbb{X}$, $i = 1, \ldots, \kappa$, the convex polytope $Con\{v(1), \ldots, v(\kappa)\}$ as

$$Con\{v(1), \ldots, v(\kappa)\} = \left\{ v \in \mathbb{X}; v = \sum_{i=1}^{\kappa} \alpha(i) v(i), \sum_{i=1}^{\kappa} \alpha(i) = 1, \alpha(i) \geq 0 \right\}. \tag{6.3}$$

The above formulation will be useful in characterizing the robustness in the financial model.

For a vector $x \in \mathbb{R}^n$, we define the vectors $[x]_+ \in \mathbb{R}^n$ and $[x]_- \in \mathbb{R}^n$ as follows

$$[x]_+ = \begin{pmatrix} max\{0, x_1\} \\ \vdots \\ max\{0, x_n\} \end{pmatrix},$$

and $[x]_- = x - [x]_+$, where x_i is the i^{th} component of the vector x. Let Λ be a diagonal n by n matrix with positive components. It is easy to see that

$$[x]'_+ \Lambda [x]_- = 0. \tag{6.4}$$

In what follows we define for $x \in \mathbb{R}^n, y \in \mathbb{R}^n, u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$, and Λ a diagonal n by n matrix with positive components,

$$\Upsilon(x, u) = \Lambda^{\frac{1}{2}} (u - x),$$

$$\Delta(y, v) = \Lambda^{\frac{1}{2}} (v + y),$$

$$g(x,y,u,v) = \theta\|\Upsilon(x,u)\|^2 + (1-\theta)\|\Delta(y,v)\|^2$$

where $\theta \in [0,1]$. Suppose it is desired to find $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$, $u \geq \mathbf{0}$ and $v \geq \mathbf{0}$ such that minimizes $g(x,y,u,v)$. Let

$$g(x,y) = \min_{u\geq \mathbf{0},\, v\geq \mathbf{0}} g(x,y,u,v). \tag{6.5}$$

Proposition 2. *For any $u \geq \mathbf{0}, v \geq \mathbf{0}$,*

$$g(x,y,u,v) \geq \theta\|\Lambda^{\frac{1}{2}}[x]_-\|^2 + (1-\theta)\|\Lambda^{\frac{1}{2}}[x]_+\|^2$$

and the minimum in (6.5) is achieved at $u = [x]_+$, $v = -[y]_-$ and $g(x,y) = \theta\|\Lambda^{\frac{1}{2}}[x]_-\|^2 + (1-\theta)\|\Lambda^{\frac{1}{2}}[y]_+\|^2$.

2.2. Financial Model

Let us consider a financial model in which there are N risky assets represented by the random return vector A (taking values in $\mathbb{R}^N$), and a risk-free asset with return $r \in \mathbb{R}$. Whenever convenient we shall write $A(t)$ to denote the return vector at a certain time t and by $A_i(t)$ the i^{th} component of this vector $A(t)$. The random return vector A has mean vector $\mu \in \mathbb{R}^N$, and $N \times N$ dimensional covariance matrix Ω. Therefore A can be written as

$$A = \mu + \varepsilon \tag{6.6}$$

where ε is a random vector with zero mean and covariance matrix Ω. It is convenient to define the vector $\varphi \in \mathbb{R}^{N+1}$ as follows:

$$\varphi = \begin{pmatrix} \mu \\ r \end{pmatrix}.$$

A portfolio ω will be a vector in $\mathbb{R}^N$ in which its components represent the weights on the risky assets A, that is, the i^{th} entry ω_i of ω is the portfolio's proportion invested in asset i. We assume that it belongs to a set Γ, defined in terms of the following LMI:

$$\Gamma = \left\{ \omega \in \mathbb{R}^N ; F_0 + \sum_{i=1}^{N} \omega_i F_i \geq 0 \right\}$$

where F_i, $i = 0, \ldots, N$ are given symmetric matrices. The set Γ is suitable for representing constraints like the sum of the portfolio components is equal or less than 1, and no short sales are permitted, that is, constraints of the form

$$\omega' \mathbf{1} \leq 1, 0 \leq \omega_i, \, i = 1, \ldots, N$$

where $\mathbf{1}$ represents the N vector formed by 1 in all positions. In this case, the matrices F_i would be $(N+1)\times(N+1)$ dimensional. Matrix F_0 would be formed by zeroes except for the element $(1,1)$, which would be 1. For $i=1,\dots,N$, the matrices F_i would be

$$F_1 = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, F_N = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

We consider that $(1-\omega'\mathbf{1})$ is invested in the a risk-free asset r. Therefore the return of the investor is

$$\omega' A + (1-\omega'\mathbf{1})r. \tag{6.7}$$

We will also consider a fixed portfolio ω_B provided by the manager, called benchmark portfolio. The return of the benchmark portfolio is as in (6.7), that is,

$$\omega'_B A + (1-\omega'_B\mathbf{1})r. \tag{6.8}$$

From (6.7) and (6.8) we have that the difference between the return of the investor's portfolio ω and the benchmark portfolio, defined as the tracking error $e(\omega)$, is

$$e(\omega) = (\omega-\omega_B)'A + (\omega_B-\omega)'\mathbf{1}r. \tag{6.9}$$

The expected value of the tracking error $e(\omega)$, denoted by $\rho(\omega)$, follows from (6.6) and (6.9), and is given by

$$\rho(\omega) = (\omega-\omega_B)'\mu + (\omega_B-\omega)'\mathbf{1}r = (\omega-\omega_B)'(\mu-\mathbf{1}r) \tag{6.10}$$

and the variance (volatility), denoted by $\sigma^2(\omega)$, by

$$\sigma^2(\omega) = (\omega-\omega_B)'\Omega(\omega-\omega_B). \tag{6.11}$$

As mentioned earlier, we shall be interested in this chapter in measuring risk through a convex combination of semivariances of the tracking error return below and above the average, which are defined as follows: the semivariance below average is defined by

$$\eta^-(\omega) = E\Big((min\{0, e(\omega)-\rho(\omega)\})^2\Big) = E\Big((min\{0, (\omega-\omega_B)'(A-\mu)\})^2\Big) \tag{6.12}$$

and the semivariance above average is defined by

$$\begin{aligned}\eta^+(\omega) &= E\Big((max\{0, e(\omega)-\rho(\omega)\})^2\Big)\\ &= E\Big((max\{0, (\omega-\omega_B)'(A-\mu)\})^2\Big).\end{aligned} \tag{6.13}$$

From (6.12) and (6.13), the risk function is defined, for $\theta \in [0,1]$, as

$$\theta\eta^-(\omega) + (1-\theta)\eta^+(\omega). \tag{6.14}$$

Clearly for $\theta = \frac{1}{2}$, we have that $\frac{1}{2}\Big(\eta^-(\omega)+\eta^+(\omega)\Big) = \frac{1}{2}\sigma^2(\omega)$. The parameter θ represents the degree of risk aversion of the investor. For more details and properties of this risk function we refer to [Hanza and Janssen, 1995], [Duarte Jr., 1999], [Markowitz, 1959].

Suppose that a set of past data $a_i(t)$, $t = 1,\ldots T$, is available, where $a_i(t)$ represents the observed value of the random return $A_i(t)$ at time t. Let us define the matrix $\mathcal{A}$ as follows:

$$\mathcal{A} = \begin{bmatrix} a_1(1) & \ldots & a_N(1) \\ \vdots & \ddots & \vdots \\ a_1(T) & \ldots & a_N(T) \end{bmatrix}. \tag{6.15}$$

Thus the t^{th} row of $\mathcal{A}$ represents the observed value of $A(t)$. Let

$$\Lambda = \begin{bmatrix} \lambda(1) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda(T) \end{bmatrix}$$

where $\lambda(t)$ represents the weight of the data observed at time t when estimating the parameters $\eta^-(\omega), \eta^+(\omega)$. We assume that $\sum_{t=1}^{T}\lambda(t) = 1$. From equations (6.12), (6.13), (6.15), an estimate for $\eta^-(\omega), \eta^+(\omega)$ using $\mathcal{A}$, the weights Λ, and the mean μ, denoted by $\eta^-_{\mu,\Lambda}(\omega), \eta^+_{\mu,\Lambda}(\omega)$ respectively, would be as follows:

$$\begin{aligned} \eta^-_{\mu,\Lambda}(\omega) &= \left\| \Lambda^{\frac{1}{2}}[(\mathcal{A} - \mathbf{1}\mu')(\omega - \omega_B)]_- \right\|^2 \\ &= \sum_{t=1}^{T}\lambda(t)\left(\left[\sum_{i=1}^{N}(a_i(t)-\mu_i)(\omega_i - \omega_{iB})\right]_-\right)^2, \end{aligned} \tag{6.16}$$

$$\eta^+_{\mu,\Lambda}(\omega) = \sum_{t=1}^{T}\lambda(t)\left(\left[\sum_{i=1}^{N}(a_i(t)-\mu_i)(\omega_i - \omega_{iB})\right]_+\right)^2. \tag{6.17}$$

2.3. Robust Mean-Semivariance Portfolio Selection

In this subsection we introduce the robust mean-semivariance portfolio selection problem we want to solve. If we had accurate estimates for the parameters μ, r and $\eta^-_{\mu,\Lambda}(\omega), \eta^+_{\mu,\Lambda}(\omega)$ then the mean-semivariance portfolio selection problem would be as follows. For $\delta \in [0,1], \theta \in [0,1]$:

$$\inf_{\omega\in\Gamma} f_{\varphi,\Lambda}(\omega) \tag{6.18}$$

where

$$f_{\varphi,\Lambda}(\omega) = -(1-\delta)(\omega-\omega_B)'(\mu-\mathbf{1}r)+\delta(\theta\eta^{-}_{\mu,\Lambda}(\omega)+(1-\theta)\eta^{+}_{\mu,\Lambda}(\omega)) \quad (6.19)$$

(recall that $\varphi = \binom{\mu}{r}$). Note that δ represents the investor's risk aversion. As δ increases from 0 to 1 so does the emphasis on caution. It is clear from equations (6.16) and (6.17) that the estimates for $\eta^{-}(\omega), \eta^{+}(\omega)$ would rely on Λ, φ. Different values of Λ, φ would lead to different values for the objective function (6.19). Let us consider the situation in which we want to take into account in the optimization problem (6.18) all possible values of the parameters Λ, φ inside the convex polytopes $\Xi = Con\{\varphi(1), \ldots, \varphi(\kappa)\}$, and $\Sigma = Con\{\Lambda(1), \ldots, \Lambda(\nu)\}$. We assume again that $\sum_{t=1}^{T} \lambda(k,t) = 1, k = 1, \ldots, \nu$, where $\lambda(k,t)$ are the elements in the diagonal of $\Lambda(k)$. Then it is easy to check that $\sum_{t=1}^{T} \lambda(t) = 1$, for any $\Lambda \in \Sigma$. It is important to stress that we assume here that the values $\varphi(1), \ldots, \varphi(\kappa)$ and $\Lambda(1), \ldots, \Lambda(\nu)$ are provided by the asset manager, and we shall not be concerned on how to obtain them. The parameter $\mu(i)$ could represent different estimations for the return of the risky assets, due to different forms of numerical calculations or due to different future scenarios expectations over the period in which the portfolio will be kept unchanged. Similarly the parameters $r(i)$ could represent the different values that the one-day interest rate could take over the period in which the portfolio will be kept unchanged. The parameters $\Lambda(k)$ denotes the different weights we could consider in the estimation of the risk function in equations (6.16), (6.17). The robust version of the portfolio optimization problem posed in (6.18), (6.19) is as follows:

$$\inf_{\omega\in\Gamma} \sup_{\varphi\in\Xi,\Lambda\in\Sigma} f_{\varphi,\Lambda}(\omega). \quad (6.20)$$

Therefore we are looking for a portfolio ω^* such that

$$\sup_{\varphi\in\Xi,\Lambda\in\Sigma} f_{\varphi,\Lambda}(\omega^*) \le \sup_{\varphi\in\Xi,\Lambda\in\Sigma} f_{\varphi,\Lambda}(\omega)$$

for each $\omega \in \Gamma$, that is, there exists $\varphi \in \Xi, \Lambda \in \Sigma$ such that

$$\sup_{\varphi\in\Xi,\Lambda\in\Sigma} f_{\varphi,\Lambda}(\omega^*) \le f_{\varphi,\Lambda}(\omega).$$

Notice that it would be possible to have for some $\bar{\varphi} \in \Xi, \bar{\Lambda} \in \Sigma$,

$$f_{\bar{\varphi},\bar{\Lambda}}(\omega) \le f_{\bar{\varphi},\bar{\Lambda}}(\omega^*) \le \sup_{\varphi\in\Xi,\Lambda\in\Sigma} f_{\varphi,\Lambda}(\omega^*). \quad (6.21)$$

The larger the sets Ξ, Σ are, the more likely to find $\bar{\varphi} \in \Xi, \bar{\Lambda} \in \Sigma$ satisfying (6.21) will be. Thus by specifying a large number of scenarios we will deteriorate the performance of the method since it will lead to a perhaps unnecessarily pessimistic solution ω^* which minimizes the maximum (worst case) value of the objective function over the sets Ξ, Σ. The success of the method will therefore rely on a judicious choice of the scenarios to be considered in the min-max problem.

Remark 1. *In [Costa and Paiva, 2002] the following two robust optimization problems were considered: The first problem was to find a portfolio ω of minimum (worst case) volatility of the tracking error with guaranteed fixed minimum target expected performance τ. By target expected performance we mean a real number τ provided by the investor which represents the minimum value that the tracking error expectation could achieve. The second problem was to find a portfolio ω of maximum (worst case) expected value of the tracking error conditional on a maximum value ϑ^2 for the volatility of the tracking error. The value ϑ^2, provided by the manager, represents the maximum value that the volatility of the tracking error could achieve. In problem (6.20) with $\theta = \frac{1}{2}$ it is desired to find a portfolio of minimum (worst case) convex combination of the volatility and minus the expected tracking error, with no performance indexes (τ and ϑ) requirements. In other words, it is desired to solve the following problem:*

$$\min_{\omega} \max_{\Omega,\varphi} [-(1-\delta)\rho(\omega) + \frac{\delta}{2}\sigma^2(\omega)] \textit{ subject to } \omega \in \Gamma.$$

Therefore even for the case in which $\theta = \frac{1}{2}$ and $\delta \in [0,1]$, problem (6.20), considered here and the ones considered in [Costa and Paiva, 2002] are different.

In what follows it will be useful to make the following definitions for $u \geq \mathbf{0}, v \geq \mathbf{0}, \omega \in \Gamma, \varphi \in \Xi, \Lambda \in \Sigma$:

$$\Upsilon_{\varphi,\Lambda}(\omega,u) = \left\{\Lambda^{\frac{1}{2}}(u - (\mathcal{A} - \mathbf{1}\mu')(\omega - \omega_B))\right\}$$

$$\Delta_{\varphi,\Lambda}(\omega,v) = \left\{\Lambda^{\frac{1}{2}}(v + (\mathcal{A} - \mathbf{1}\mu')(\omega - \omega_B))\right\}.$$

Define also

$$\begin{aligned} g_{\varphi,\Lambda}(\omega,u,v) \quad &= \quad -(1-\delta)(\omega - \omega_B)'(\mu - \mathbf{1}r) + \delta\theta\|\Upsilon_{\varphi,\Lambda}(\omega,u)\|^2 \\ &\quad\quad +\delta(1-\theta)\|\Delta_{\varphi,\Lambda}(\omega,v)\|^2. \end{aligned}$$

Proposition 3. *For each $\omega \in \Gamma$ fixed,*

$$\sup_{\varphi \in \Xi, \Lambda \in \Sigma} f_{\varphi,\Lambda}(\omega) = \min_{u \geq \mathbf{0}, v \geq \mathbf{0}} \max_{k=1,\ldots,\kappa, \ell=1,\ldots,\nu} g_{\varphi(k),\Lambda(\ell)}(\omega,u,v). \tag{6.22}$$

3. LMI Formulation

We will now propose an LMI formulation for solving the robust mean-semivariance portfolio optimization problem posed in (6.20). We shall show in

Theorem 1 that the LMI formulation for problem (6.20) is:

$$\min z \tag{6.23}$$

subject to:

$$\begin{bmatrix} z+(1-\delta)(\omega-\omega_B)'(\mu(i)-\mathbf{1}r(i)) & \mathcal{S}_1(i,k,\omega,u) & \mathcal{S}_2(i,k,\omega,v) \\ \mathcal{S}_1(i,k,\omega,u)' & I & 0 \\ \mathcal{S}_2(i,k,\omega,v)' & 0 & I \end{bmatrix} \geq 0$$

$$i=1,\ldots,\kappa, k=1,\ldots,\nu \tag{6.24}$$

$$u \geq \mathbf{0}, v \geq \mathbf{0}, \omega \in \Gamma. \tag{6.25}$$

where:

$$\begin{aligned} \mathcal{S}_1(i,k,\omega,u) &= (\delta\theta)^{\frac{1}{2}}\Upsilon_{\varphi(i),\Lambda(k)}(\omega,u)', \\ \mathcal{S}_2(i,k,\omega,v) &= (\delta(1-\theta))^{\frac{1}{2}}\Delta_{\varphi(i),\Lambda(k)}(\omega,v)' \end{aligned}$$

and the variables are : u, v, z, ω.

The next result shows that the LMI problem (6.23) to (6.25) solves the robust portfolio optimization problem (6.20).

Theorem 1. *$\overline{\omega} \in \Gamma$ is an optimal solution to the inf sup problem posed in (6.20) if and only if $\overline{u}, \overline{v}, \overline{z}, \overline{\omega}$ is an optimal solution to the LMI problem (6.23) to (6.25). Moreover*

$$\begin{aligned} \overline{z} &= \max_{i=1,\ldots,\kappa,k=1,\ldots,\nu} \Big\{ -(1-\delta)(\overline{\omega}-\omega_B)'(\mu(i)-\mathbf{1}r(i)) \\ &+ \delta\theta\|\Lambda(k)^{\frac{1}{2}}\left[(\mathcal{A}-\mathbf{1}\mu(i)')(\overline{\omega}-\omega_B)\right]_-\|^2 \\ &+ \delta(1-\theta)\|\Lambda(k)^{\frac{1}{2}}\left[(\mathcal{A}-\mathbf{1}\mu(i)')(\overline{\omega}-\omega_B)\right]_+\|^2 \Big\}, \end{aligned} \tag{6.26}$$

$$\overline{u} = \left[(\mathcal{A}-\mathbf{1}\mu(\iota)')(\overline{\omega}-\omega_B)\right]_+, \tag{6.27}$$

$$\overline{v} = -\left[(\mathcal{A}-\mathbf{1}\mu(\iota)')(\overline{\omega}-\omega_B)\right]_- \tag{6.28}$$

and $\iota \in \{1,\ldots,\kappa\}, \xi \in \{1,\ldots,\nu\}$ reaches the maximum in equation (6.26).

Remark 2. *The optimization problem posed in (6.23) to (6.25) is equivalent to solving the following non-linear optimization problem:*

$$\begin{aligned} &\min z \\ &\text{subject to:} \\ &z+(1-\delta)(\omega-\omega_B)'(\mu(i)-\mathbf{1}r(i))+ \\ &\delta\theta(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))'\Lambda(k)(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))+ \\ &\delta(1-\theta)(v+(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))'\Lambda(k)\times \\ &(v+(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B)) \geq 0 \\ &\omega \in \Gamma, u \geq \mathbf{0}, v \geq \mathbf{0},\ i=1,\ldots,\kappa,\ k=1,\ldots,\nu. \end{aligned}$$

Therefore other non-linear programming algorithms could be used to solve the problem. The advantage of formulating it as an LMI optimization is that these problems are tractable both from theoretical and numerical point of view and very efficient codes are now widely available (see, for instance, [Oliveira et al., 1994]).

For the special case in which $\theta = 1$, the LMI optimization problem (6.23) to (6.25) reduces to:

$$\begin{aligned}
&\min z \\
&\text{subject to:} \\
&\quad \begin{bmatrix} z + (1-\delta)(\omega - \omega_B)'(\mu(i) - \mathbf{1}r(i)) & \mathcal{S}_1(i,k,\omega,u) \\ \mathcal{S}_1(i,k,\omega,u)' & I \end{bmatrix} \geq 0 \\
&\quad i = 1,\ldots,\kappa, k = 1,\ldots,\nu \\
&\quad u \geq \mathbf{0}, \omega \in \Gamma,
\end{aligned}$$

and similarly for $\theta = 0$. For the case in which $\theta = \frac{1}{2}$ the LMI optimization problem (6.23) to (6.25) can be simplified to

$$\begin{aligned}
&\min z \\
&\text{subject to:} \\
&\quad \begin{bmatrix} z + (1-\delta)(\omega - \omega_B)'(\mu(i) - \mathbf{1}r(i)) & \delta^{\frac{1}{2}}(\omega - \omega_B)'\Omega(i,k) \\ \delta^{\frac{1}{2}}\Omega(i,k)(\omega - \omega_B) & \Omega(i,k) \end{bmatrix} \geq 0 \\
&\quad i = 1,\ldots,\kappa, k = 1,\ldots,\nu, \omega \in \Gamma
\end{aligned} \tag{6.29}$$

where $\Omega(i,k) = (\mathcal{A} - \mathbf{1}\mu(i)')'\Lambda(k)(\mathcal{A} - \mathbf{1}\mu(i)')$.

4. Numerical Examples

In this section we present 2 numerical examples based on the LMI formulation.

Example 1: Our first example compares the possible values of $f_{\varphi,\Lambda}(\omega)$ and $f_{\varphi,\Lambda}(\omega^*)$, where ω^* is the optimal solution of the minimization problem in (6.20) and ω another portfolio, when we vary φ and Λ. In particular we will see cases in which equation (6.21) holds. We consider 11 risky assets and one risk-free asset, and take the benchmark portfolio to be $\omega_B = (1/11,\ldots,1/11)'$. These assets were collected from the Bovespa (Brazilian Stock Exchange) Index. Our value function is to minimize $-(1-\delta)(\omega - \omega_B)'(\mu - \mathbf{1}r) + \delta(\omega - \omega_B)'\Omega(\omega - \omega_B)$. We impose short-selling constraints of the form $\omega \geq \mathbf{0}$, and $\omega'\mathbf{1} \leq 1$, and consider two portfolios. *Portfolio 1* is as in (6.29) with $\kappa = 1, \nu = 1$, and *Portfolio 2* (robust) is as in (6.29) with $\kappa = 2, \nu = 2$. For simplicity we took $\Omega(i,k) = \Omega(k)$. For *Portfolio 2* we calculated $\mu(1)$ as the mean of the

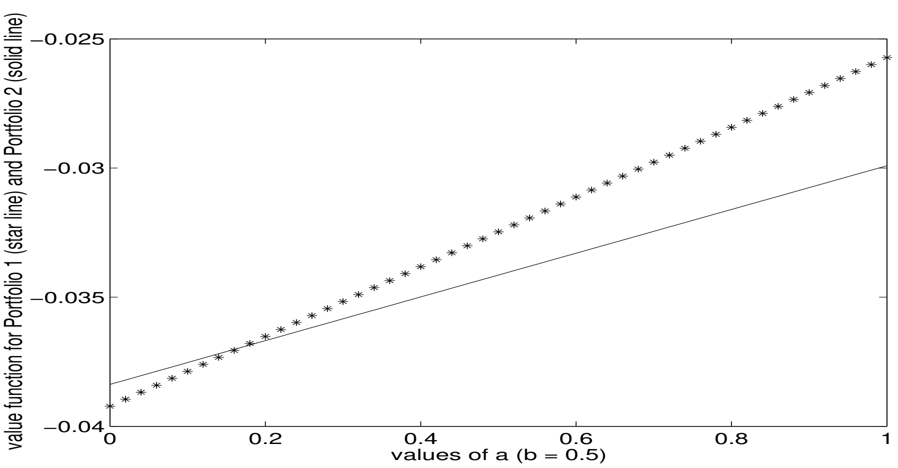

Figure 6.1. Expected Value Function for Portfolios 1 (star line) and 2 (solid line) when $b = 0.5$ and a varies from 0 to 1

previously 100 daily observed asset returns, and for $\mu(2)$ we considered the mean with a decaying factor of 0.95. For $\Omega(1)$ and $\Omega(2)$ we estimated the covariance matrices using the 100 daily past observed asset returns, the mean $\mu(1)$, and the matrices $\Lambda(1) = \frac{1}{100}I$ and $\Lambda(2)$ with $\lambda(2,t) = 0.9^{(T-t)}/c$ (where c is a normalizing factor). For *Portfolio 1* we considered the mean $\mu(1)$ and covariance matrix $\Omega(1)$. In the optimization problem we considered $\delta = 0.95$. In the figures 6.1 and 6.2 we compare the value functions for *Portfolios 1* and *2* when $\Omega = b\Omega(1) + (1-b)\Omega(2)$ and $\mu = a\mu(1) + (1-a)\mu(2)$, where $a, b \in [0,1]$. Figure 6.1 shows the results when $b = 0.5$ and a varies from 0 to 1, and figure 6.2 shows the results for the case in which $a = 0.5$ and b varies from 0 to 1. Notice that, as indicated in equation (6.21), it is possible to have some values of μ and Ω such that the value function of *Portfolio 1* will be better (lower) than the value function of *Portfolio 2*. We can see this in figure 6.1 for the values of a between 0 and around 0.15, and in figure 6.2 for the values of b between around 0.65 and 1. For the other values of a and b, the value function of *Portfolio 2* is smaller than the one of *Portfolio 1*.

Example 2: In the second example we use real data collected from the Bovespa to calculate a robust portfolio that tracks the Bovespa Index, based on the LMI formulation presented above. For this we selected the 47 most important stocks of the index during the period considered, which together represent about 95% of the Bovespa Index. We calculated the optimal portfolio once a week from January 2, 2000 to January 2, 2001, which represents 52 weeks, and used the last 30 observations to build the matrix $\mathcal{A}$ of returns. We generated 3 different sets of portfolios, all with $\delta = 0.9$ and $\theta = 1.0$ (semivariance below

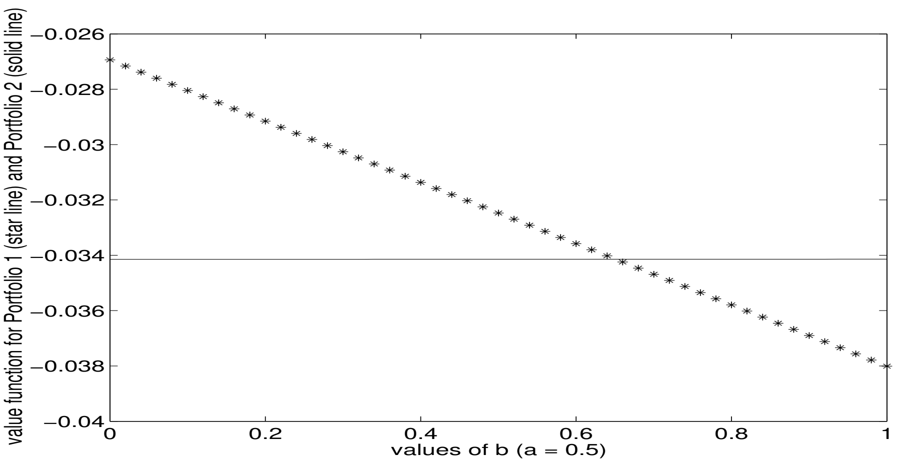

Figure 6.2. Expected Value Function for Portfolios 1 (star line) and 2 (solid line) when $a = 0.5$ and b varies from 0 to 1

average) and short-selling constraints of the form $\omega \geq \mathbf{0}$, and $\omega'\mathbf{1} \leq 1$. For *Portfolio 1* we set $\kappa = 1, \nu = 1$ (only one LMI) $\Lambda = \frac{1}{30}I$. For *Portfolio 2* we considered $\kappa = 2, \nu = 2$ (four LMIs), $\Lambda(1) = \Lambda$, $\Lambda(2)$ with $\lambda(2,t) = 0.94^{(T-t)}/c$ (where c is a normalizing factor). For *Portfolio 3* we considered $\kappa = 2, \nu = 2$ (four LMIs), $\Lambda(1) = \Lambda$, $\Lambda(2)$ with $\lambda(2,t) = 0.97^{(T-t)}/c$. In all cases we calculated the vector of $\mu(1)$ according to the mean average of the last 30 days, and for *Portfolio 2* and *Portfolio 3* we generated $\mu(2)$ using the matrix of weights $\Lambda(2)$ and the last 30 days. We used the weights of the selected stocks in the Bovespa index to set ω_B. We also considered r as the risk-free asset daily return, in our case the Brazilian Interbank Rate.

In figure 6.3 we have the 13 4-weeks returns of *Portfolios 1, 2, 3*, and Bovespa Index throughout the 52 weeks of January 2, 2000 to January 2, 2001. The period considered for the simulations was of high volatility, and the BOVESPA index lost about 2.7% over the period. We can see from the figure 6.3 that *Portfolios 1 and 3* followed the benchmark better than *Portfolio 2*, possibly due to a too conservative choice of the scenarios for *Portfolio 2*. Since the period considered was relatively short and of high volatility, it is hard to draw some conclusions from this numerical example. However the LMI formulation solved the problems very efficiently, being a valuable computational tool for robust portfolio selection problems.

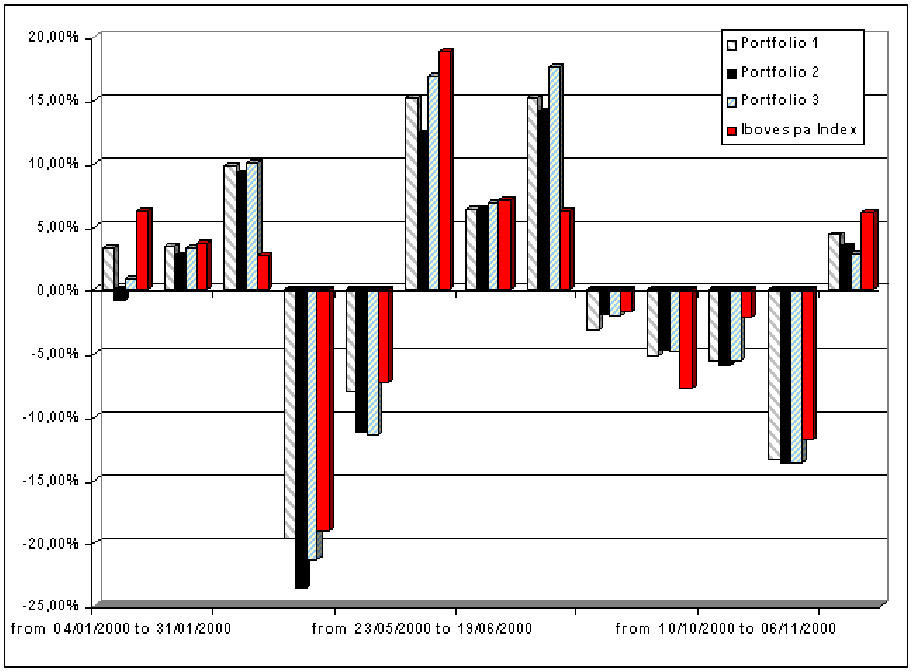

Figure 6.3. Monthly returns of the portfolios

5. Conclusions

In this chapter we have considered the problem of optimal portfolio selection using the semivariances as a measure for the risk function of the tracking error. We assumed that the expected return of the risky and risk-free assets as well as the weights for the calculation of the semivariances belong to a convex polytope defined by some known elements which form the vertices of this polytope. The value function is defined as a convex combination of the risk function minus the expected tracking error return. We showed that the problem of finding a portfolio of minimum worst case objective function is equivalent to solving an LMI (linear matrix inequalities) optimization problem, so that the powerful numerical packages available for this class of problems can be used. A numerical example in the São Paulo stock exchange (BOVESPA) was presented. We compared three portfolios, *Portfolio 1, Portfolio 2, Portfolio 3*, using real data collected from 02/01/2000 to 02/01/2001. The computational results suggest that the use of LMI algorithms can be very useful for robust portfolio optimization problems.

Appendix: Proofs

In this appendix we present the proofs of the Corollary 1, Propositions 2, 3, and Theorem 1.

Corollary 1

Proof. From Proposition 1 it follows that for each i

$$\begin{bmatrix} x(i)'Qx(i) & x(i)'Q^{1/2} \\ Q^{1/2}x(i) & I \end{bmatrix} \geq 0$$

and therefore

$$\sum_{i=1}^{k} \alpha(i) \begin{bmatrix} x(i)'Qx(i) & x(i)'Q^{1/2} \\ Q^{1/2}x(i) & I \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k} \alpha(i)x(i)'Qx(i) & \sum_{i=1}^{k} \alpha(i)x(i)'Q^{1/2} \\ \sum_{i=1}^{k} \alpha(i)Q^{1/2}x(i) & I \end{bmatrix} \geq 0.$$

The desired result follows by applying Proposition 1 to the above inequality. ∎

Proposition 2

Proof. Since $x = [x]_+ + [x]_-$ it follows that

$$\begin{aligned} ||\Upsilon(x,u)||^2 &= ||\Lambda^{\frac{1}{2}}((u-[x]_+)-[x]_-)||^2 \\ &= ||\Lambda^{\frac{1}{2}}(u-[x]_+)||^2 + ||\Lambda^{\frac{1}{2}}[x]_-||^2 - 2(u-[x]_+)'\Lambda[x]_-. \end{aligned} \tag{6.A.1}$$

But from (6.4)

$$-(u-[x]_+)'\Lambda[x]_- = -u'\Lambda[x]_- + [x]_+'\Lambda[x]_- = -u'\Lambda[x]_- \geq 0$$

since $u \geq \mathbf{0}$, $-[x]_- \geq \mathbf{0}$ and Λ is a diagonal matrix with positive components. Therefore from (6.A.1) for any $u \geq \mathbf{0}$

$$||\Upsilon(x,u)||^2 \geq ||\Lambda^{\frac{1}{2}}(u-[x]_+)||^2 + ||\Lambda^{\frac{1}{2}}[x]_-||^2 \tag{6.A.2}$$

and it is clear that the minimum of (6.A.2) is reached at $u = [x]_+$, with value $||\Lambda^{\frac{1}{2}}[x]_-||^2$. Similarly the minimum of $\Delta(y,v)$ over $v \in \mathbb{R}^n$, $v \geq \mathbf{0}$ is reached at $v = -[y]_-$ with value $||\Lambda^{\frac{1}{2}}[y]_+||^2$. This completes the proof. ∎

Proposition 3

Proof. For any $u \geq \mathbf{0}, v \geq \mathbf{0}, \varphi \in \Xi, \Lambda \in \Sigma$, with $\varphi = \sum_{i=1}^{\kappa} \alpha(i)\varphi(i), \sum_{i=1}^{\kappa} \alpha(i) = 1, \alpha(i) \geq 0$ and $\Lambda = \sum_{k=1}^{\nu} \beta(k)\Lambda(k), \sum_{k=1}^{\nu} \beta(k) = 1, \beta(k) \geq 0$, we have from Proposition 2 that

$$g_{\varphi,\Lambda}(\omega,u,v) \geq f_{\varphi,\Lambda}(\omega) \tag{6.A.3}$$

and the minimum on the left hand side of (6.A.3) over $u \geq 0, v \geq 0$ is $f_{\varphi,\Lambda}(\omega)$, reached at $u = [(\mathcal{A}-\mathbf{1}\mu')(\omega-\omega_B)]_+$, $v = -[(\mathcal{A}-\mathbf{1}\mu')(\omega-\omega_B)]_-$. From Corollary 1

$$\begin{aligned} \left\|\Upsilon_{\varphi,\Lambda}(\omega,u)\right\|^2 &= \left\|\Lambda^{\frac{1}{2}}\left(u - \left(\mathcal{A} - \mathbf{1}\sum_{i=1}^{\kappa}\alpha(i)\mu(i)'\right)\left(\omega-\omega_B\right)\right)\right\|^2 \\ &= \left\|\Lambda^{\frac{1}{2}}\left(\sum_{i=1}^{\kappa}\alpha(i)\left(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B)\right)\right)\right\|^2 \\ &\leq \sum_{i=1}^{\kappa}\alpha(i)\left\|\Lambda^{\frac{1}{2}}(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))\right\|^2 \end{aligned}$$

and since

$$\begin{aligned}
\|\Upsilon_{\varphi(i),\Lambda}(\omega,u))\|^2 &= \|\Lambda^{\frac{1}{2}}(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))\|^2 \\
= (u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))' &\textstyle\sum_{k=1}^{\nu}\beta(k)\Lambda(k)(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B)) \\
= \textstyle\sum_{k=1}^{\nu}\beta(k)(u-(\mathcal{A}-\mathbf{1}\mu(i)')&(\omega-\omega_B))'\Lambda(k)(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))
\end{aligned}$$

it follows that

$$\begin{aligned}
\|\Upsilon_{\varphi,\Lambda}(\omega,u)\|^2 &\leq \sum_{i=1}^{\kappa}\sum_{k=1}^{\nu}\beta(k)\alpha(i)\|\Lambda(k)^{\frac{1}{2}}(u-(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))\|^2 \\
&= \sum_{i=1}^{\kappa}\sum_{k=1}^{\nu}\beta(k)\alpha(i)\|\Upsilon_{\varphi(i),\Lambda(k)}(\omega,u)\|^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\|\Delta_{\varphi,\Lambda}(\omega,v)|^2 &\leq \sum_{i=1}^{\kappa}\sum_{k=1}^{\nu}\beta(k)\alpha(i)\|\Lambda(k)^{\frac{1}{2}}(v+(\mathcal{A}-\mathbf{1}\mu(i)')(\omega-\omega_B))\|^2 \\
&= \sum_{i=1}^{\kappa}\sum_{k=1}^{\nu}\beta(k)\alpha(i)\|\Delta_{\varphi(i),\Lambda(k)}(\omega,u)\|^2.
\end{aligned}$$

Since

$$(\omega-\omega_B)'(\mu-\mathbf{1}r)=\sum_{i=1}^{\kappa}\alpha(i)(\omega-\omega_B)'(\mu(i)-\mathbf{1}r(i))$$

we can conclude that

$$g_{\varphi,\Lambda}(\omega,u,v)\leq\sum_{i=1}^{\kappa}\sum_{k=1}^{\nu}\beta(k)\alpha(i)g_{\varphi(i),\Lambda(k)}(\omega,u,v).$$

Therefore,

$$\underset{\varphi\in\Xi,\Lambda\in\Sigma}{sup}\; g_{\varphi,\Lambda}(\omega,u,v)=\underset{i=1,\ldots,\kappa,k=1,\ldots,\nu}{max} g_{\varphi(i),\Lambda(k)}(\omega,u,v)$$

and the maximum is reached for some $i\in\{1,\ldots,\kappa\}, k\in\{1,\ldots,\nu\}$. Taking the minimum over $u\geq\mathbf{0}, v\geq\mathbf{0}$ we obtain from (6.A.3) the desired result. ■

Theorem 1

Proof. Fix $\omega\in\Gamma$. According to Proposition 1 the inequality (6.24) is equivalent to

$$z \quad \geq \quad g_{\varphi(i),\Lambda(k)}(\omega,u,v) \tag{6.A.4}$$

and must be satisfied for each $i=1,\ldots,\kappa, k=1,\ldots,\nu$. Therefore from (6.A.4)

$$z \quad \geq \quad \underset{i=1,\ldots,\kappa,k=1,\ldots,\nu}{max} g_{\varphi(i),\Lambda(k)}(\omega,u,v).$$

Since we want to minimize z, the best choice is to take the minimum over $u\geq\mathbf{0}, v\geq\mathbf{0}$. It follows from Proposition 3 that

$$\begin{aligned}
z \quad &\geq \quad \underset{u\geq\mathbf{0},v\geq\mathbf{0}}{min}\;\underset{k=1,\ldots,\kappa,\ell=1,\ldots,\nu}{max}\; g_{\varphi(k),\Lambda(\ell)}(\omega,u,v) \\
&= \quad \underset{\varphi\in\Xi,\Lambda\in\Sigma}{sup}\; f_{\varphi,\Lambda}(\omega). \qquad (6.A.5)
\end{aligned}$$

and the minimum is reached at u and v like in equations (6.26), (6.27), (6.28). From (6.A.5), minimize (6.23) under the restrictions (6.24) to (6.25) is equivalent to solving (6.20). ■

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