OUTPUT FEEDBACK CONTROL OF MARKOV JUMP LINEAR SYSTEMS IN CONTINUOUS-TIME[1]

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ABSTRACT

This paper addresses the dynamic output feedback control problem of continuous-time Markovian jump linear systems. The fundamental point in the analysis is a LMI characterization, comprising all dynamical compensators that stabilizes the closed-loop system in the mean square sense. The H_2 and H_∞ norm control problems are studied and previous results on LQG with output observations are revisited. Applying these results into a specific control problem, formulated as the H_2 or H_∞ norm problem of the estimation error, the corresponding H_2 and H_∞ filters are obtained by a simple matter of reduction. The control and the filtering solutions are also expressed by two sets of coupled Riccati equations, showing when recognized, that a separation principle for the present formulation holds.

Keywords: Jump parameter systems, H_2 and H_∞ control, H_2 and H_∞ filtering, separation principle, linear matrix inequalities.

1 Introduction

There have been nowadays several examples showing the importance of dynamic systems subject to abrupt variations in their structures. This is in part due to the fact that very often dynamic systems are inherently vulnerable to component failures or repairs, sudden environmental disturbances, changing subsystem interconnections, abrupt variations of the operating point of a non-linear plant, etc. Markovian jump linear systems (MJLS) comprise an important class of stochastic dynamic systems which can, in several situations, model the above problems. The theory of stability, optimal control and H_∞-control, as well as some important applications of such systems, can be found in several papers in the current literature like, for instance, in [1], [3], [7], [8], [10], [11], [13], [15], [18], [20], [22], [23], [27].

Continuous-time LQ control problems for MJLS have been studied in the current literature usually under the assumption of perfect information of the jump variable of the system as well as of the state variable. By using standard arguments on dynamic programming it has been shown that this formulation leads to a set of coupled differential Riccati equations (CDRE) for the solution of the problem ([3][22]). Taking the limit as time goes to infinity we have, under some appropriate conditions, that the CDRE tends to coupled algebraic Riccati equations (CARE), and numerical methods have been designed for solving it in [1], [7], [11], [15], [27]. The state feedback H_∞ control problem has also been analyzed, and generalized coupled algebraic Riccati equations have been derived for this problem in [10]. The LQG control problem with no direct knowledge of the state variable, but assuming that the jump variable is available, was addressed in [18] (named JLQG). In this article it was shown that the separation principle holds for the optimal control problem, leading to a sample path dependent optimal state estimator (the Kalman filter) combined with the CDRE obtained for the state feedback LQ control problem. As pointed out in [18], this optimal solution is not satisfactory since the filter portion of the controller for the optimal infinite time horizon JLQG is not invariant, which makes any asymptotic and stability analysis unfeasible. Bearing this fact in mind, a suboptimal filter was proposed in [18], which leads to a time-invariant controller. The idea for deriving this filter is based on throwing away information on past values of the jump variable and

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replacing them with estimated values based on a posteriori probabilities. By doing this the sub-optimal filter is written in terms of a CARE with a fictitious transition rate obtained by the estimated a posteriori probabilities. Note however that no guarantee of closed loop stability (in the mean square sense) for this sub-optimal compensator is provided in [18].

More recently, another line of investigation to the partial state observation of MJLS has appeared in the literature. It involves the output observation of the state variable and noisy observation of the Markov state, see for instance, [5], [12], [25]. JLQG control, filtering and adaptive control has been studied in finite time, using optimal and suboptimal stochastic solutions. For the case with perfect observation of the Markov state, a separation principle involving CDRE is shown to hold in [12].

In this paper the output feedback control of continuous-time MJLS with perfect measurement of the Markov state is revisited under a convex programming approach. The H_2 and the H_∞ norm control problem is solved employing dynamic compensators that can change structure when a jump of the Markov state occurs. The formulation provides a suitable framework for implementations when compared with the optimal stochastic filter, and in addition, a separation principle is shown to be valid in this restricted complexity setting.

Convex analysis has shown to be a powerful tool to derive numerical algorithms for several important control problems like, for instance, H_2 guaranteed cost control for uncertain systems, H_∞ control problems, and mixed H_2/H_∞ control problems and has been widely studied in the international literature (see, for instance, [4], [6], [28]). For the state feedback MJLS case, convex analysis had been previously considered in [7], [8] and [27]. In [7] the state-feedback H_2-control problem of continuous-time MJLS was defined and a convex programming approach was used to study it. It was shown that the existence of a solution to the convex programming problem is completely equivalent to the existence of a mean square stabilizing solution to CARE, and one solution can be recovered from the other, showing the equivalence between these two formulations. In [27] the authors present a different LMI optimization problem which gives the maximal solution of the CARE. In [8] it is desired to find a robust controller for a continuous-time MJLS in the sense of having mean square stability even in the presence of parameters uncertainties of the system. In all these cases it was assumed that the state variable, as well as the jump variable, were available to the controller.

As in [18] we assume in this paper that the jump variable is perfectly observable to the controller but not the state variable, which is observed only through an output variable. The first important problem we consider in this paper is to find necessary and sufficient conditions for the existence of output feedback dynamic controllers, dependent on the jump variable, such that the closed loop system is stable in the mean square sense. It is proven that the necessary and sufficient conditions can be written in terms of an LMI problem, providing thus a parametrization for the mean square stabilizing compensators in terms of the elements of a convex set. Moreover it is shown how to get a compensator that achieves the desired closed loop mean square stability from the solution of the LMI problem. Having obtained this result we can move on the control problems.

For the H_2-norm control problem, it is shown that it can be written in terms of an LMI optimization problem and, moreover, the optimal solution can be obtained by means of two CARE's, one associated to the state feedback control problem and the other associated to a filtering problem. This result can be seen as a separation principle for the H_2-norm control of continuous-time MJLS since it parallels the solution of the output feedback problems in the deterministic case and in some LQG stochastic problems. The CARE for the control problem derived here and in [18] are the same. But it is important to point out that the CARE derived in this paper for the filtering is different from the one presented in [18] for the sub-optimal filtering. Whereas in [18] the CARE is written in terms of the transition rate obtained by the estimated a posteriori probabilities, in this paper the CARE associated to the filtering equation can be seen as dual to the one for the control problem, with the duality concept being defined taking into account the transition rate of the system. Most important of all, under our approach we can guarantee stochastic stability (in the mean square sense) of the closed loop system, while in [18] stability is of no avail. In fact we present a numerical example in which our approach leads to a stable closed loop system, while the controller presented in [18] is not stable.

The next problem we consider is the output feedback H_∞-norm control problem of continuous-time MJLS. An LMI optimization problem which leads to a positive definite solution of the generalized CARE presented in [10] is obtained, and therefore solves the problem. As before, the optimal solution of the LMI optimization problem leads to a closed loop controller which stabilizes the system in the mean square sense. By making some simple changes in the objective function of the LMI optimization problem we can also consider an upper bound for the mixed H_2/H_∞ control problem.

As a byproduct of the output feedback control problems presented above, we can also consider the

H_2 and H_∞ filtering problems for the continuous-time MJLS. The LMI optimization problems, as before, provides mean square stabilizing controllers for the error estimator equation.

Lastly, but not the least, it is important to emphasize that, besides providing important theoretical results for the H_2 and H_∞ norm control problems of continuous-time MJLS, the convex approach naturally leads to powerful numerical methods, as illustrated by an example of a multiple mode VTOL helicopter. The LMI method parametrizes all dynamic output feedback controllers that are mean square stable and have bounded H_2-norm or specified H_∞-norm, allowing the optimization problem to account for extra LMI restrictions in the design of control, whenever necessary. If no extra restrictions are included, the problem can also be solved directly using the two CARE's formulation, which is more efficient numerically, see [11] for a comparison.

The paper is organized in the following way. The main notation to be employed throughout is presented in section 2 as well as some basic results concerning stability in the mean square sense. The definition of the H_2 and the H_∞ norms for continuous-time MJLS are also presented, and the connection between the H_2-norm and the observability and controllability gramians is made. The formulation of the control and filtering problems closes section 2. Section 3 shows that the existence of a compensator that stabilizes in the mean square sense an output feedback continuous-time MJLS is equivalent to a convex set described in terms of LMI's not being empty and, moreover, we show how to obtain a compensator with this property from an element of this convex set. Section 4 considers the H_2 control problem and shows that the solution of the LMI optimization problem can be obtained from two CARE's, a results that can be seen as the principle of separation for output feedback MJLS. Section 5 considers the H_∞ and H_2/H_∞ control problems for the output feedback system via LMI optimization problems, and section 6 analyses the filtering problems. Some numerical examples are presented in section 7, and the paper is concluded in section 8 with an application and some final remarks.

2 Problem Formulation and Basic Results

A Markov jump linear system (MJLS) is described as follows. Let $\mathcal{S} = \{1, \ldots, N\}$ be an index set, and consider the collections of real matrices: $A = (A_1, \ldots, A_N)$, $\dim(A_i) = n \times n$; $E = (E_1, \ldots, E_N)$, $\dim(E_i) = n \times m$; $C_1 = (C_{11}, \ldots, C_{1N})$, $\dim(C_{1i}) = p \times n$; $C_2 = (C_{21}, \ldots, C_{2N})$, $\dim(C_{2i}) = q \times n$; and $D_2 = (D_{21}, \ldots, D_{2N})$, $\dim(D_{2i}) = q \times m$; $i = 1, \ldots, N$.

Let us consider a continuous-time homogeneous Markov chain, $\Theta = \{\theta_t;\ t \geq 0\}$ having $\mathcal{S}$ as state space and $\Lambda = [\lambda_{ij},\ i \in \mathcal{S},\ j \in \mathcal{S}]$ as the transition rate matrix. The probability distribution of the Markov chain at the initial time is given by $\mu = (\mu_1, \ldots, \mu_N)$ in such a way that $P(\theta_0 = i) = \mu_i$.

We denote by L_2^m the Hilbert space formed by the stochastic processes $z = \{z_t;\ t \geq 0\}$ such that, for each $t \geq 0$, z_t is a second order, real valued, m-dimensional random variable, and

$$\| z \|_2^2 \ := \ \int_0^\infty E(|z_t|^2)\, dt < \infty$$

Consider a fundamental probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. The MJLS is the stochastic system:

$$\mathcal{G}_0 : \begin{cases} \dot{x}_t = A(\theta_t)x_t + E(\theta_t)w_t, \\ z_t = C_1(\theta_t)x_t, \\ y_t = C_2(\theta_t)x_t + D_2(\theta_t)w_t, \quad t \geq 0, \\ w \in L_2^m, \quad E(|x_0|^2) < \infty, \quad \theta_0 \sim \mu. \end{cases} \tag{2.1}$$

Thus, whenever $\theta_t = i \in \mathcal{S}$, one has that $A(\theta_t) = A_i$, $E(\theta_t) = E_i$, $C_1(\theta_t) = C_{1i}$, $C_2(\theta_t) = C_{2i}$ and $D_2(\theta_t) = D_{2i}$. We assume that the process w is either a Markovian process or w is $\mathcal{F}_0$-measurable. The processes $x = \{x_t, t \geq 0\}$, $z = \{z_t, t \geq 0\}$ and $y = \{y_t,\ t \geq 0\}$ are respectively, the state, the output and the measured output for system $\mathcal{G}_0$. The definitions below generalize of the concept of stable systems for stochastic systems.

Definition 2.1 *We say that model $\mathcal{G}_0$ with $w \equiv 0$, is*

i) mean square stable (MSS) if $E(|x_t|^2) \to 0$, as $t \to \infty$,

ii) is stochastically stable (SS) if $x \in L_2^n$,

for any initial condition x_0 and initial distribution for θ_0.

The following proposition connects the main results on stochastic stability for MJLS. They can be derived from [17] and [21]:

Proposition 2.1 *The assertions below are equivalent:*

a) System $\mathcal{G}_0$ is MSS.

b) System $\mathcal{G}_0$ is SS.

c) The linear inequalities

$$A_i'P_i + P_iA_i + \sum_{j=1}^{N} \lambda_{ij}P_j < 0, \quad i = 1, \ldots, N.$$

are feasible for some $P = (P_1, \ldots, P_N)$ with $dim(P_i) = n \times n$, $P_i > 0$ for all $i = 1, \ldots, N$.

d) (coupled Lyapunov equations) For any given $U = (U_1, \ldots, U_N)$ with $dim(U_i) = n \times n$, $U_i > 0$, there exists a unique $P = (P_1, \ldots, P_N)$, $P_i > 0$ $\forall i$ satisfying

$$A_i'P_i + P_iA_i + \sum_{j=1}^{N} \lambda_{ij}P_j + U_i = 0, \quad i = 1, \ldots, N. \tag{2.2}$$

If we set $U_i \geq 0$, the uniqueness stated also holds for $P_i \geq 0$, $i = 1, \ldots, N$.

Proposition 2.2 *If $\mathcal{G}_0$ is MSS, for every $w = \{w_t; t \geq 0\} \in L_2^m$, then $x = \{x_t; t \geq 0\} \in L_2^n$ for any initial condition x_0 and initial distribution for θ_0.*

Proof: See Appendix A. □

The next definition is a generalization of the H_2 norm from continuous-time deterministic systems to the stochastic Markovian jump case.

Definition 2.2 *We define the H_2-norm of a MSS system $\mathcal{G}_0$ as*

$$\| \mathcal{G}_0 \|_2^2 = \sum_{s=1}^{r} \sum_{i=1}^{N} \mu_i \| z_{s,i} \|_2^2$$

where $z_{s,i}$ represents the output $\{z_t; t \geq 0\}$ when:

a) the input is given by $w = \{w_t; t \geq 0\}$, $w_t = e_s\delta(t)$, $\delta(t)$ the unitary impulse, and e_s the r-dimensional unitary vector formed by 1 at the s-th position and zero elsewhere,

b) $x_0 = 0$ and $\theta_0 = i$.

For the deterministic case ($N = 1$) the above definition reduces to the usual H_2 norm. The following proposition states that the H_2 norm defined above can be calculated as the solution of the coupled observability and controllability gramians, a result that mirrors its deterministic counterpart. Suppose that $\mathcal{G}_0$ is MSS and consider

$$A_i'W_i + W_iA_i + \sum_{j=1}^{N} \lambda_{ij}W_j + C_{1i}'C_{1i} = 0, \quad i = 1, \ldots, N, \tag{2.3}$$

and

$$A_iV_i + V_iA_i' + \sum_{j=1}^{N} \lambda_{ji}V_j + \mu_iE_iE_i' = 0, \quad i = 1, \ldots, N.$$

with $W_i \geq 0$ and $V_i \geq 0$, $i = 1, \ldots, N$ the unique solution of these equations.

Proposition 2.3 ([7]) $\| \mathcal{G}_0 \|_2^2 = \sum_{i=1}^{N} \mu_i \, Tr(E_i'W_iE_i) = \sum_{i=1}^{N} Tr(C_{1i}V_iC_{1i}')$

Remark 2.1 *A very interesting property obtained from Proposition 2.1 d) is the increasing behavior for the solutions $P_i > 0$ of*

$$A_i' P_i + P_i A_i + \sum_{j=1}^{N} \lambda_{ij} P_j + C_{1i}' C_{1i} < 0, \quad i = 1, \ldots, N, \tag{2.4}$$

Indeed, $P_i > W_i \geq 0$, $i = 1, \ldots, N$ and the consequence is that $||\mathcal{G}_0||_2^2 < \sum_{i=1}^N \mu_i \, Tr(E_i' P_i E_i)$. In other words, feasible solutions of the above linear matrix inequalities provide upper bounds to the H_2 norm. Besides, we can find P_i arbitrarily close to W_i. Indeed, suppose that (2.4) has a feasible solution $\bar{P}_i$. Then subtracting (2.3) from (2.4) and multiplying the resulting expression by $\epsilon > 0$, we get that

$$A_i' \epsilon(\bar{P}_i - W_i) + \epsilon(\bar{P}_i - W_i) A_i + \sum_{j=1}^{N} \lambda_{ij} \epsilon(\bar{P}_i - W_i) < 0$$

Since the system has to be stochastically stable for (2.4) to have a solution, then $\bar{P}_i - W_i > 0$. Defining $P_i = W_i + \epsilon(\bar{P}_i - W_i)$, it follows that each P_i is positive definite, it again satisfies (2.4), and can be made arbitrarily close to W_i. Hence the H_2 norm can be found by the following optimization problem:

$$\begin{array}{ll} \inf & \displaystyle\sum_{i=1}^{N} \mathrm{Tr}(E_i' P_i E_i) \\ \text{s.t.} & A_i' P_i + P_i A_i + \displaystyle\sum_{j=1}^{N} \lambda_{ij} P_j + C_{1i}' C_{1i} < 0, \quad i = 1, \ldots, N \end{array}$$

If the Markov chain process Θ is irreducible and recurrent then $P(\theta_t = i) \to \eta_i$ as $t \to \infty$ exponentially fast for some probability measure η with $\eta_i > 0$, $i = 1, \ldots, N$. In this case, if $\mu_i = \eta_i$, we get that the definition of the H_2 norm will coincide with the expected value of the stationary variance of the output z, when the input is the unit variance white noise. Indeed, consider the stochastic differential equation

$$dx_t = A(\theta_t) x_t + E(\theta_t) \, d\nu_t$$

where $\nu = \{\nu_t, \, t \geq 0\}$ is a Wiener process with incremental covariance equal to the identity. Let us define $Q(t) = (Q_1(t), \ldots, Q_N(t))$, with

$$Q_i(t) := E(x_t x_t' 1_{\{\theta_t = i\}}) \tag{2.5}$$

where $1_{\{.\}}$ stands for the Dirac measure. The following proposition is a simple extension of the results in [21], [22] and we shall omit the proof.

Proposition 2.4 *The stationary covariance of x satisfies*

$$A_i Q_i + Q_i A_i' + \sum_{j=1}^{N} \lambda_{ji} Q_j + \eta_i E_i E_i' = 0, \quad i = 1, \ldots, N.$$

If the model $\mathcal{G}_0$ is MSS and $w \in L_2^m$, then from Proposition 2.2 we have that $z \in L_2^p$. The next definition is a generalization of the H_∞ norm for MJLS's.

Definition 2.3 *The H_∞-norm of a MSS system $\mathcal{G}_0$, $\| \mathcal{G}_0 \|_\infty$ is the smallest $\gamma > 0$, such that*

$$\| z \|_2 < \gamma \| w \|_2$$

holds for all $w \in L_2^m$.

The next result was proven in [10].

Proposition 2.5 *If $P = (P_1, \ldots, P_N), P_i > 0$ satisfies:*

$$A_i' P_i + P_i A_i + \sum_{j=1}^{N} \lambda_{ij} P_j + \gamma^{-2} P_i E_i E_i' P_i + C_{1i}' C_{1i} < 0, \quad i = 1, \ldots, N,$$

then $\| \mathcal{G}_0 \|_\infty < \gamma$.

2.1 The filtering problem

One of the main objectives of this paper is to study the output filtering problem of system $\mathcal{G}_0$. Consider $A_c = (A_{c1}, \ldots, A_{cN})$, $\dim(A_{ci}) = n \times n$; $B_c = (B_{c1}, \ldots, B_{cN})$, $\dim(B_{ci}) = n \times q$; and $C_c = (C_{c1}, \ldots, C_{cN})$, $\dim(C_{ci}) = p \times n$; and let the output filter be of form

$$\mathcal{G}_f : \begin{cases} \dot{v}_t = A_c(\theta_t)v_t + B_c(\theta_t)y_t \\ \widehat{z}_t = C_c(\theta_t)v_t, \quad t \geq 0 \end{cases}$$

The H_2 filtering problem is to find (A_c, B_c, C_c) such that the H_2 norm of system $\mathcal{G}_e$ is minimized, where $\mathcal{G}_e$ stands for a system obtained from the connection of $\mathcal{G}_0$ and $\mathcal{G}_f$ with output $e_t = z_t - \widehat{z}_t$. Similarly, the H_∞ filtering problem is to find (A_c, B_c, C_c) such that for some $\gamma > 0$, $\|\mathcal{G}_e\|_\infty < \gamma$.

2.2 The control problem

The second objective of this paper is to study controlled MJLS with output observations. Let $B = (B_1, \ldots, B_N)$, $\dim(B_i) = n \times r$ and $D_1 = (D_{1i}, \ldots, D_{Ni})$, $\dim(D_{1i}) = p \times r$, for each $i = 1, \ldots, N$, be some associated set of matrices, and consider the stochastic system:

$$\mathcal{G} : \begin{cases} \dot{x}_t = A(\theta_t)x_t + E(\theta_t)w_t + B(\theta_t)u_t, \\ z_t = C_1(\theta_t)x_t + D_1(\theta_t)u_t, \\ y_t = C_2(\theta_t)x_t + D_2(\theta_t)w_t, \quad t \geq 0, \\ w \in L_2^m, \quad E(|x_0|^2) < \infty, \quad \theta_0 \sim \mu. \end{cases}$$

For the synthesis of the control action u_t, we introduce a dynamical compensator of the form

$$\mathcal{G}_c : \begin{cases} \dot{v}_t = A_c(\theta_t)v_t + B_c(\theta_t)y_t, \\ u_t = C_c(\theta_t)v_t, \quad t \geq 0, \end{cases}$$

with the same dimensions as specified to the filter $\mathcal{G}_f$. The H_2 control problem is to find a compensator (A_c, B_c, C_c) such that the norm $\|\mathcal{G}\|_2$ is minimized. Similarly, the H_∞ control problem is to find (A_c, B_c, C_c) such that the norm $\|\mathcal{G}\|_\infty$ is smaller than γ.

3 Stochastic Stability

We shall address the problem of finding all dynamic compensators $\mathcal{G}_c$ such that the system $\mathcal{G}$ coupled with $\mathcal{G}_c$ becomes stochastically stable. To this end, we apply Proposition 2.1 to get the following conditions

$$\tilde{A}_i' P_i + P_i \tilde{A}_i + \sum_{j=1}^{N} \lambda_{ij} P_j < 0, \quad i = 1, \ldots, N, \tag{3.1}$$

where $\tilde{A}_i, i = 1, ..., N$ are given by

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i C_{ci} \\ B_{ci} C_{2i} & A_{ci} \end{bmatrix}, \tag{3.2}$$

$i = 1, \ldots, N$, and $P_i > 0, i = 1, \ldots, N$.

Let diag$(\cdot)$ indicates a block diagonal matrix whose entries are given by $(\cdot)$. We are ready to present the following theorem.

Theorem 3.1 *There exist $P_i > 0$, A_{ci}, B_{ci} and $C_{ci}, i = 1, \ldots, N$ such that (3.1) hold iff*

$$\begin{bmatrix} A_i Y_i + Y_i A_i' + B_i F_i + F_i' B_i' + \lambda_{ii} Y_i & R_i(Y) \\ R_i'(Y) & S_i(Y) \end{bmatrix} < 0 \tag{3.3}$$

$$A_i' X_i + X_i A_i + L_i C_{2i} + C_{2i}' L_i' + \sum_{j=1}^{N} \lambda_{ij} X_j < 0 \tag{3.4}$$

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} > 0 \tag{3.5}$$

with

$$
\begin{aligned}
R_i(Y) &= \left[\sqrt{\lambda_{1i}}Y_i \cdots \sqrt{\lambda_{(i-1)i}}, Y_i \ \sqrt{\lambda_{(i+1)i}}Y_i \cdots \sqrt{\lambda_{Ni}}Y_i\right], \\
S_i(Y) &= -\mathrm{diag}(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_N),
\end{aligned} \tag{3.6}
$$

$i = 1, \ldots, N$, *has a feasible solution* $X_i = X_i'$, $Y_i = Y_i'$, L_i *and* F_i.

Proof: For the necessity, let us consider the following partition for P_i:

$$
P_i = \begin{bmatrix} P_{1i} & P_{2i} \\ P_{2i}' & P_{3i} \end{bmatrix} \tag{3.7}
$$

We shall assume, with no loss of generality, that P_{2i} is non-singular [28]. Defining matrices

$$
\mathcal{Y}_i = (P_{1i} - P_{2i}P_{3i}^{-1}P_{2i}')^{-1} > 0 \tag{3.8}
$$

$$
T_i = \begin{bmatrix} \mathcal{Y}_i & I \\ \mathcal{Y}_i & 0 \end{bmatrix} \tag{3.9}
$$

$$
J_i = \begin{bmatrix} I & 0 \\ 0 & -P_{3i}^{-1}P_{2i}' \end{bmatrix} \tag{3.10}
$$

and multiplying (3.1) to the left by $T_i'J_i'$ and to the right by J_iT_i we get

$$
T_i'(J_i^{-1}\tilde{A}_iJ_i)'(J_i'P_iJ_i)T_i + T_i'(J_i'P_iJ_i)(J_i^{-1}\tilde{A}_iJ_i)T_i + \sum_{j=1}^{N}\lambda_{ij}T_i'J_i'P_jJ_iT_i < 0 \tag{3.11}
$$

which, after performing the indicated calculations, yields

$$
\begin{aligned}
&\begin{bmatrix} A_i\mathcal{Y}_i + \mathcal{Y}_iA_i' + B_i\mathcal{F}_i + \mathcal{F}_i'B_i' & \mathcal{M}_i' \\ \mathcal{M}_i & A_i'P_{1i} + P_{1i}A_i + \mathcal{L}_iC_{2i} + C_{2i}'\mathcal{L}_i' \end{bmatrix} + \\
&+\sum_{j=1}^{N}\lambda_{ij}\begin{bmatrix} \mathcal{Y}_i\left[\mathcal{Y}_j^{-1} + (P_{2i}P_{3i}^{-1}P_{3j} - P_{2j})P_{3j}^{-1}(P_{2i}P_{3i}^{-1}P_{3j} - P_{2j})'\right]\mathcal{Y}_i & 0 \\ 0 & P_{1j} \end{bmatrix} < 0
\end{aligned} \tag{3.12}
$$

where

$$
\begin{aligned}
\mathcal{F}_i &= -C_{ci}P_{3i}^{-1}P_{2i}'\mathcal{Y}_i \\
\mathcal{L}_i &= P_{2i}B_{ci} \\
\mathcal{M}_i &= A_i' + P_{1i}A_i\mathcal{Y}_i + P_{1i}B_i\mathcal{F}_i + \mathcal{L}_iC_{2i}\mathcal{Y}_i - P_{2i}A_{ci}P_{3i}^{-1}P_{2i}'\mathcal{Y}_i + \sum_{j=1}^{N}\lambda_{ij}(P_{1j} - P_{2j}P_{3i}^{-1}P_{2i}')\mathcal{Y}_i
\end{aligned} \tag{3.13}
$$

Besides, the conditions $P_i > 0, i = 1, \ldots, N$ are equivalent to

$$
T_i'J_i'P_iJ_iT_i = \begin{bmatrix} \mathcal{Y}_i & I \\ I & P_{1i} \end{bmatrix} > 0. \tag{3.14}
$$

Since

$$
\begin{aligned}
&\sum_{j=1}^{N}\lambda_{ij}(P_{2i}P_{3i}^{-1}P_{3j} - P_{2j})P_{3j}^{-1}(P_{2i}P_{3i}^{-1}P_{3j} - P_{2j})' \\
&= \sum_{j=1, j\neq i}^{N}\lambda_{ij}(P_{2i}P_{3i}^{-1}P_{3j} - P_{2j})P_{3j}^{-1}(P_{2i}P_{3i}^{-1}P_{3j} - P_{2j})' \\
&\geq 0
\end{aligned} \tag{3.15}
$$

it follows that (3.3), (3.4) and (3.5) hold for $Y_i = \mathcal{Y}_i$, $X_i = P_{1i}$, $F_i = \mathcal{F}_i$ and $L_i = \mathcal{L}_i, i = 1, \ldots, N$.

For the sufficiency part, let us suppose that $X_i = X_i'$, $Y_i = Y_i'$, F_i and L_i, $i = 1, \ldots, N$ are feasible solutions of (3.3), (3.4) and (3.5). Then if we set for each i

$$A_{ci} = (X_i - Y_i^{-1})(A_i' + X_i A_i Y_i + X_i B_i F_i + L_i C_{2i} Y_i + \sum_{j=1}^{N} \lambda_{ij} Y_j^{-1} Y_i) Y_i^{-1} \tag{3.16}$$

$$B_{ci} = (Y_i^{-1} - X_i) L_i \tag{3.17}$$

$$C_{ci} = F_i Y_i^{-1} \tag{3.18}$$

$$P_i = \begin{bmatrix} X_i & Y_i^{-1} - X_i \\ Y_i^{-1} - X_i & X_i - Y_i^{-1} \end{bmatrix} > 0 \tag{3.19}$$

$$T_i = \begin{bmatrix} Y_i & I \\ Y_i & 0 \end{bmatrix} \tag{3.20}$$

it follows that

$$\begin{aligned} & T_i'(\tilde{A}_i' P_i + P_i \tilde{A}_i + \sum_{j=1}^{N} \lambda_{ij} P_j) T_i' \\ &= \begin{bmatrix} A_i Y_i + Y_i A_i' + B_i F_i + F_i' B_i' + \sum_{j=1}^{N} \lambda_{ij} Y_i Y_j^{-1} Y_i & 0 \\ 0 & A_i' X_i + X_i A_i + L_i C_{2i} + C_{2i}' L_i' + \sum_{j=1}^{N} \lambda_{ij} X_j \end{bmatrix} \\ &< 0 \end{aligned} \tag{3.21}$$

hence (3.1) hold and the proof is completed. □

One important aspect refers to the parameterization of the compensators. Notice that Theorem 3.1 selects a particular parametrization, but it also establishes the relations which are satisfied by any compensator $\mathcal{G}_c$ that stabilizes the closed-loop system in the stochastic sense. To generate each possible stabilizing compensator, we first find a set of values $X_i, Y_i, F_i, L_i, i = 1, \ldots, N$, in such a way that (3.3), (3.4) and (3.5) are satisfied. Then we need to find the corresponding P_{2i}, P_{3i} and $\mathcal{M}_i$ for each i, which satisfy (3.8) and (3.12). This may not be an easy task, since these constraints are not LMI's. However, one can always choose the trivial solution $P_{2i} = -P_{3i} = Y_i^{-1} - X_i$ and $\mathcal{M}_i = 0$, yielding the compensator adopted in (3.16), (3.17) and (3.18). As we shall see in the next sections, the H_2 and H_∞ norm problems are not affected by the particular parametrization of $\mathcal{G}_c$, and the trivial solution in terms of the choice made in Theorem 3.1 will be adopted throughout the paper. Consequently, the matrices P_i's assume the particular structure in (3.19).

4 $\mathbf{H_2}$ control

We consider the problem of finding a compensator $\mathcal{G}_c$ such that the H_2-norm $||\mathcal{G}||_2$ of the closed-loop system is minimized. From the discussion in Remark 2.1, this problem can be written as

$$\inf \quad \sum_{i=1}^{N} \mu_i \mathrm{Tr}(\tilde{E}_i{}' P_i \tilde{E}_i) \tag{4.1}$$

$$\text{s.t.} \quad \tilde{A}_i{}' P_i + P_i \tilde{A}_i + \tilde{C}_i{}' \tilde{C}_i + \sum_{j=1}^{N} \lambda_{ij} P_j < 0 \tag{4.2}$$

$$P_i = P_i' > 0 \quad i = 1, \ldots, N, \tag{4.3}$$

where the matrices $\tilde{A}_i$, $\tilde{E}_i$ and $\tilde{C}_i$ are given by

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i C_{ci} \\ B_{ci} C_{2i} & A_{ci} \end{bmatrix}, \; \tilde{E}_i = \begin{bmatrix} E_i \\ B_{ci} D_{2i} \end{bmatrix}, \; \tilde{C}_i = \begin{bmatrix} C_{1i} & D_{1i} C_{ci} \end{bmatrix}, \quad i = 1, \ldots, N. \tag{4.4}$$

Notice that (4.2) ensures stochastic stability for the closed-loop system of approaching solutions to the problem of minimizing $||\mathcal{G}||_2$. The next theorem casts the solution in terms of LMI's and in the sequel a separation principle for the problem is provided, by means of two independent sets of interconnected Riccati equations.

Theorem 4.1 *The H_2 output feedback control problem is solved by the following LMI problem:*

$$\inf \sum_{i=1}^{N} \mu_i \mathrm{Tr}(Z_i) \tag{4.5}$$

$$\text{s.t.} \quad \begin{bmatrix} Y_i & I & E_i \\ I & X_i & X_iE_i + L_iD_{2i} \\ E_i' & E_i'X_i + D_{2i}'L_i' & Z_i \end{bmatrix} > 0 \tag{4.6}$$

$$\begin{bmatrix} A_iY_i + Y_iA_i' + B_iF_i + F_i'B_i' + \lambda_{ii}Y_i & Y_iC_{1i}' + F_i'D_{1i}' & R_i(Y) \\ C_{1i}Y_i + D_{1i}F_i & -I & 0 \\ R_i'(Y) & 0 & S_i(Y) \end{bmatrix} < 0 \tag{4.7}$$

$$A_i'X_i + X_iA_i + L_iC_{2i} + C_{2i}'L_i' + C_{1i}'C_{1i} + \sum_{j=1}^{N} \lambda_{ij}X_j < 0, \quad i = 1, \ldots, N. \tag{4.8}$$

The corresponding parameters of $\mathcal{G}_c$ are

$$\begin{aligned} C_{ci} &= F_iY_i^{-1} \\ B_{ci} &= (Y_i^{-1} - X_i)^{-1}L_i \\ A_{ci} &= (Y_i^{-1} - X_i)^{-1}M_iY_i^{-1} \end{aligned} \tag{4.9}$$

where $M_i = -A_i' - X_iA_iY_i - X_iB_iF_i - L_iC_{2i}Y_i - C_{1i}'(C_{1i}Y_i + D_{1i}F_i) - \sum_{j=1}^{N} \lambda_{ij}Y_j^{-1}Y_i$.

Proof: By adopting similar steps to those in Theorem 3.1, it straightforward to verify that (4.2) and (4.3) are equivalent to (4.7), (4.8) and

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} > 0 \tag{4.10}$$

Let us turn our attention to the objective function. Defining P_i, T_i, J_i and $\mathcal{L}_i$ as in (3.7), (3.9), (3.10) and (3.13), we can write

$$\begin{aligned} \tilde{E}_i'P_i\tilde{E}_i &= \tilde{E}_i'(J_i^{-1})'(J_i'P_iJ_i)T_i(T_i'J_i'P_iJ_iT_i)^{-1}T_i'(J_i'P_iJ_i)J_i^{-1}\tilde{E}_i \\ &= \begin{bmatrix} E_i \\ P_{1i}E_i + \mathcal{L}_iD_{2i} \end{bmatrix}' \begin{bmatrix} \mathcal{Y}_i & I \\ I & P_{1i} \end{bmatrix}^{-1} \begin{bmatrix} E_i \\ P_{1i}E_i + \mathcal{L}_iD_{2i} \end{bmatrix}' \\ &= \begin{bmatrix} E_i \\ X_iE_i + L_iD_{2i} \end{bmatrix}' \begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix}^{-1} \begin{bmatrix} E_i \\ X_iE_i + L_iD_{2i} \end{bmatrix}' \end{aligned} \tag{4.11}$$

where the last expression is obtained from the correspondence between $P_{1i}, \mathcal{L}_i, \mathcal{Y}_i$ and X_i, L_i, Y_i. To make the objective function linear, we introduce auxiliary variables $Z_i, i = 1, ..., N$, such that $Z_i > \tilde{E}_i'P_i\tilde{E}_i$. By using Schur's complement we get that this constraint is equivalent to (4.6). Writing $Z_i = \tilde{E}_i'P_i\tilde{E}_i + \mathcal{Z}_i$, where $\mathcal{Z}_i$ is an arbitrary positive definite matrix, it follows that

$$\begin{aligned} \inf \sum_{i=1}^{N} \mu_i \mathrm{Tr}(Z_i) &= \inf \sum_{i=1}^{N} \mu_i \mathrm{Tr}(\tilde{E}_i'P_i\tilde{E}_i + \mathcal{Z}_i) \\ &= \inf \sum_{i=1}^{N} \mu_i \mathrm{Tr}(\tilde{E}_i'P_i\tilde{E}_i) \end{aligned}$$

We can readily conclude that problem (4.1)–(4.3) can be rewritten as (4.5)–(4.8). In addition, we adopt the trivial solution in Theorem 3.1, since for fixed $P_{1i}, \mathcal{L}_i, \mathcal{Y}_i$ the H_2-norm of the closed-loop system is not affected by the particular choice of P_{2i}, P_{3i} and $\mathcal{M}_i$. The compensator $\mathcal{G}_c$ is therefore, calculated exactly as (3.16), (3.17) and (3.18), with the additional term $C_{1i}'(C_{1i}Y_i + D_{1i}F_i)$ in the former, for the identity $\mathcal{M}_i = 0$ to hold. □

The H_2-norm problem can be equally solved by means of a CARE formulation, as we shall see. Let us assume that $D_{1i}'D_{1i}$ and $D_{2i}D_{2i}'$ are invertible matrices for each $i = 1, \ldots, N$, and denote:

$$\begin{aligned} A_{qi} &= A_i - B_i(D_{1i}'D_{1i})^{-1}D_{1i}'C_{1i} \\ C_{qi} &= (I - D_{1i}(D_{1i}'D_{1i})^{-1}D_{1i}')C_{1i}, \quad i = 1, \ldots, N, \end{aligned} \tag{4.12}$$

and also,

$$A_{pi} = A_i - E_i D_{2i}'(D_{2i}D_{2i}')^{-1}C_{2i}$$
$$B_{pi} = E_i(I - D_{2i}'(D_{2i}D_{2i}')^{-1}D_{2i}), \quad i = 1,\ldots,N. \tag{4.13}$$

Consider the following CARE's:

$$A_{qi}'U_i + U_iA_{qi} - U_iB_i(D_{1i}'D_{1i})^{-1}B_i'U_i + C_{qi}'C_{qi} + \sum_{j=1}^{N}\lambda_{ij}U_j = 0, \tag{4.14}$$

and

$$A_{pi}Q_i + Q_iA_{pi}' - \mu_i^{-1}Q_iC_{2i}'(D_{2i}D_{2i}')^{-1}C_{2i}Q_i + \mu_iB_{pi}B_{pi}' + \sum_{j=1}^{N}\lambda_{ji}Q_j = 0, \tag{4.15}$$

$i = 1,\ldots,N$, and the correspondent Riccati gains:

$$K_i^R = -(D_{1i}'D_{1i})^{-1}(B_i'U_i + D_{1i}'C_{1i}) \tag{4.16}$$
$$G_i^R = (\mu_i^{-1}Q_iC_{2i}' + E_iD_{2i}')(D_{2i}D_{2i}')^{-1}, \quad i = 1,\ldots,N. \tag{4.17}$$

Definition 4.1 *We say that $U_i \geq 0, i = 1,\ldots,N$ is the stochastic stabilizing solution of (4.14) if it satisfies (4.14) and the closed-loop system $\mathcal{G}$ with $u_t = K^R(\theta_t)x_t, t \geq 0$ is stochastically stable.*

Analogously, we say that $Q_i \geq 0, i = 1,\ldots,N$ is the stochastic stabilizing solution of (4.15) if it satisfies (4.15) and the MJLS given by $\dot{\zeta}_t = (A(\theta_t) - G^R(\theta_t)C_2(\theta_t))\zeta_t, t \geq 0$ is stochastically stable.

Theorem 4.2 *Suppose that the CARE's (4.14) and (4.15) have stochastic stabilizing solutions $\bar{U}_i > 0$ and $\bar{Q}_i > 0,\ i = 1,\ldots,N$. Then the optimal $\mathcal{G}_c$ for the H_2-norm problem is given by*

$$\bar{C}_{ci} = K_i^R \tag{4.18}$$
$$\bar{B}_{ci} = G_i^R \tag{4.19}$$
$$\bar{A}_{ci} = A_i - \bar{B}_{ci}C_{2i} + B_i\bar{C}_{ci}, \quad i = 1,\ldots,N. \tag{4.20}$$

Proof: First, let us rewrite the problem (4.5)–(4.8). Multiplying (4.6) to the left and to the right by an appropriate permutation matrix, we get that (4.6) is equivalent to

$$\begin{bmatrix} X_i & I & X_iE_i + L_iD_{2i} \\ I & Y_i & E_i \\ E_i'X_i + D_{2i}'L_i' & E_i' & Z_i \end{bmatrix} > 0. \tag{4.21}$$

We can eliminate the auxiliary variables Z_i's by applying the Schur's complement to (4.21), yielding the equivalent form of the objective function (4.5):

$$f = \sum_{i=1}^{N}\mu_i\mathrm{Tr}((E_i' + D_{2i}'L_i'X_i^{-1})(E_i + X_i^{-1}L_iD_{2i}) + D_{2i}'L_i'X_i^{-1}(Y_i - X_i^{-1})^{-1}X_i^{-1}L_iD_{2i})$$

Besides, we can write (4.7) and (4.8) respectively as

$$A_iY_i + Y_iA_i' + B_iF_i + F_i'B_i' + (C_{1i}Y_i + D_{1i}F_i)'(C_{1i}Y_i + D_{1i}F_i) + \sum_{j=1}^{N}\lambda_{ij}Y_iY_j^{-1}Y_i < 0 \tag{4.22}$$

$$A_i'X_i + X_iA_i + L_iC_{2i} + C_{2i}'L_i' + C_{1i}'C_{1i} + \sum_{j=1}^{N}\lambda_{ij}X_j < 0 \tag{4.23}$$

The LMI in (4.6) also provides

$$X_i > Y_i^{-1} > 0. \tag{4.24}$$

With a view of finding the optimal values of Y_i and $F_i, i = 1,\ldots,N$, let us consider fixed values of X_i and L_i and the problem of minimizing f subject to (4.22). Let us assume that (4.24) holds for some feasible X_i's. We have that $(Y_i - X_i^{-1}) > (\tilde{Y}_i - X_i^{-1}) \iff Y_i > \tilde{Y}_i$, hence the minimization of f is equivalent to that of $\bar{f}$, a new objective function defined by

$$\bar{f} = \sum_{i=1}^{N}\mu_i\mathrm{Tr}(D_{2i}'L_i'X_i^{-1}Y_i^{-1}X_i^{-1}L_iD_{2i})$$

We introduce a new variable $U_i = Y_i^{-1}$. Multiplying (4.22) by U_i to the left and to the right, we obtain the following optimization problem:

$$\begin{aligned} \inf \quad & \sum_{i=1}^{N} \mu_i \mathrm{Tr}(D_{2i}' L_i' X_i^{-1} U_i X_i^{-1} L_i D_{2i}) \\ \text{s.t.} \quad & (A_i + B_i C_{ci})' U_i + U_i (A_i + B_i C_{ci}) + (C_{1i} + D_{1i} C_{ci})'(C_{1i} + D_{1i} C_{ci}) + \sum_{j=1}^{N} \lambda_{ij} U_j < 0 \end{aligned}$$

Notice that the problem above is similar to that in (B.3) in the Appendix B, and by simple associations one can verify that the optimal gain here is given by

$$\bar{C}_{ci} = -(D_{1i}' D_{1i})^{-1} (B_i' \bar{U}_i + D_{1i}' C_{1i}) \tag{4.25}$$

where $\bar{U}_i$ is the positive definite, stabilizing solution of (4.14).

The reasoning above makes sense if only the fixed value of X_i is such that there exists some feasible Y_i. Indeed, let us assume for a moment that $Y_i^{-1} < \bar{U}_i$. In this case, $f(Y_i) < f(\bar{U}_i^{-1})$, which is a contradiction since we know that $\bar{U}_i$ is the optimal solution. Hence, $\bar{U}_i < Y_i^{-1}$ and together with (4.24) one has that $X_i > \bar{U}_i$. In order to find another CARE to determine B_{ci}, let us denote $W_i = X_i - \bar{U}_i > 0$. Subtracting (4.14) from (4.23), we get that

$$(A_i - B_{ci} C_{2i})' W_i + W_i (A_i - B_{ci} C_{2i}) + C_{ci}' D_{1i}' D_{1i} C_{ci} + \sum_{j=1}^{N} \lambda_{ij} W_j < 0 \tag{4.26}$$

Applying the Schur's complement to (4.6) to eliminate the auxiliary variables $Z_i, i = 1, \ldots, N$, we get that the objective function can alternatively be written as

$$f = \sum_{i=1}^{N} \mu_i \mathrm{Tr}(E_i' Y_i^{-1} E_i + (E_i' - D_{2i}' B_{ci}')(X_i - Y_i^{-1})(E_i - B_{ci} D_{2i})) \tag{4.27}$$

Bearing in mind that now Y_i^{-1} has a fixed value, namely the solution of the CARE in (4.14), $\bar{U}_i$, the objective function f can be redefined as

$$\tilde{f} = \sum_{i=1}^{N} \mu_i \mathrm{Tr}((E_i' - D_{2i}' B_{ci}') W_i (E_i - B_{ci} D_{2i})) \tag{4.28}$$

Hence we have a new problem involving the variables W_i and B_{ci}. According to Appendix B, we know that the optimal solution of this problem is given by

$$B_{ci} = (\mu_i^{-1} \bar{Q}_i C_{2i}' + E_i D_{2i}')(D_{2i} D_{2i}')^{-1} \tag{4.29}$$

where $\bar{Q}_i$ is the solution of the CARE in (4.15).

Finally, we calculate A_{ci} as in (4.9):

$$\begin{aligned} A_{ci} =& W_i^{-1} \big[A_i' + (W_i + \bar{U}_i) A_i Y_i + (W_i + \bar{U}_i) B_i F_i - W_i B_{ci} C_{2i} Y_i + C_{1i}' C_{1i} Y_i + \\ & + C_{1i}' D_{1i} F_i + \sum_{j=1}^{N} \lambda_{ij} \bar{U}_j Y_i \big] \bar{U}_i \\ =& A_i - B_{ci} C_{2i} + B_i C_{ci} + \\ & + W_i^{-1} \big[A_{qi}' \bar{U}_i + \bar{U}_i A_{qi} - \bar{U}_i B_i (D_{1i}' D_{1i})^{-1} B_i' \bar{U}_i + C_{qi}' C_{qi} + \sum_{j=1}^{N} \lambda_{ij} \bar{U}_j \big] \\ =& A_i - B_{ci} C_{2i} + B_i C_{ci} \end{aligned}$$

¿From Appendix C we know that if $\bar{U}_i$ and $\bar{Q}_i$ are stabilizing solutions of (4.14) and (4.15), respectively, then the controller given by (4.18)-(4.20) stochastically stabilizes the closed-loop system and the proof is completed. □

Remark 4.1 *Some conditions for the CARE (4.14) to have an stochastic stabilizing solution with $U_i > 0, i = 1, \ldots, N$ is given as follows:*

- *The pairs $(A_i, B_i), i = 1, \ldots, N$ of $\mathcal{G}$ are stochastically stabilizable in the sense that there exists $K_i, i = 1, \ldots, N$ that makes $\mathcal{G}$ with $u_t = K(\theta_t)x_t, t \geq 0$ stochastic stable; and*
  - *The pairs (C_{qi}, A_{qi}) are observable for each i; or,*
  - *The pairs $(C_{qi}, A_{qi}), i = 1, \ldots, N$ are stochastically detectable, in the sense that there exists $F_i, i = 1, \ldots, N$ such that the MJLS $\dot{\zeta}_t = (A_q(\theta_t) - F(\theta_t)C_q(\theta_t))\zeta_t, t \geq 0$ is stochastically stable; or,*
  - *(C_{qi}, A_{qi}) has no unobservable mode with real part greater than $-\lambda_{ii}/2$.*

For details, see [11]. Similar dual conditions apply to the CARE (4.15).

Remark 4.2 *An interpretation one can give, regarding the LMI method is that it approaches the H_2-norm solution by means of stabilizing solutions that can be arbitrarily near the optimal. However, the optimal may fail to be stable and assumptions such as in Remark 4.1 allow us to bridge the gap between the two methods of solution. It is important to emphasize that, whereas the two CARE's method gives a single compensator, the LMI method parameterize all compensators that provide closed-loop systems with bounded H_2-norm.*

Comparing the result of Theorem 4.2 with that of state feedback control (see [11]), one can verify that C_c in (4.18), determined by (4.14), coincides with the control gain and the CARE obtained in the state feedback problem, a clear indication that a Separation Principle holds for the class of problems that we are dealing with. As we shall see in the filtering problem, the dynamic part of the controller is indeed a state estimator filter, ensuring that this principle can be applied.

In [18] a dynamic compensator is introduced to solve the H_2 output control problem, without knowing if separation holds. Although the CARE related to the gain C_c coincides with (4.14), the CARE associated to the filter differs from (4.15). In the notation here,

$$A_{pi}Q_i + Q_iA'_{pi} - \mu_i^{-1}Q_iC'_{2i}(D_{2i}D'_{2i})^{-1}C_{2i}Q_i + \mu_iB_{pi}B'_{pi} + \sum_{j=1}^{N}\lambda^p_{ij}Q_j = 0 \tag{4.30}$$

where $\lambda^p_{ij} = \lambda_{ji}, i \neq j$ and $\sum_{j=1}^{N}\lambda^p_{ij} = 0$. Besides, the probability distribution of θ_0 is considered uniform, i.e., $\mu_i = 1/N$, for all i. Later, in Example 7.1, we show that this compensator fails to ensure the stochastic stability of the closed-loop system.

5 H$_\infty$ control

Let us turn our attention to the H_∞ control problem with output observations. We wish to write the H_∞ constraints

$$\tilde{A}'_iP_i + P_i\tilde{A}_i + \tilde{C}'_i\tilde{C}_i + \gamma^{-2}P_i\tilde{B}_i\tilde{B}'_iP_i + \sum_{j=1}^{N}\lambda_{ij}P_j < 0, \tag{5.1}$$

$i = 1, \ldots, N$, as LMI's.

Lemma 5.1 *The H_∞ constraints (5.1) are equivalent to*

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} > 0 \tag{5.2}$$

$$\begin{bmatrix} A_iY_i + Y_iA'_i + B_iF_i + F'_iB'_i + \lambda_{ii}Y_i + \gamma^{-2}E_iE'_i & (C_{1i}Y_i + D_{1i}F_i)' & R_i(Y) \\ C_{1i}Y_i + D_{1i}F_i & -I & 0 \\ R'_i(Y) & 0 & S_i(Y) \end{bmatrix} < 0 \tag{5.3}$$

$$\begin{bmatrix} A'_iX_i + X_iA_i + L_iC_{2i} + C'_{2i}L'_i + C'_{1i}C_{1i} + \sum_{j=1}^{N}\lambda_{ij}X_j & X_iE_i + L_iD_{2i} \\ E'_iX_i + D'_{2i}L'_i & -\gamma^2 I \end{bmatrix} < 0 \tag{5.4}$$

Proof : The proof is similar to that of Theorem 3.1. □

Remark 5.1 *It is important to acknowledge once again that different choices of matrices P_{2i}, P_{3i} and $\mathcal{M}_i$ do not affect the set of values of γ that can be attained by feasible compensators $\mathcal{G}_c$, and the trivial solution in Theorem 3.1 can be adopted without loss. The choice provides the compensator $\mathcal{G}_c$:*

$$\begin{aligned} C_{ci} &= F_i Y_i^{-1} \\ B_{ci} &= (Y_i^{-1} - X_i)^{-1} L_i \\ A_{ci} &= (Y_i^{-1} - X_i)^{-1} M_i Y_i^{-1} \end{aligned} \tag{5.5}$$

where,

$$\begin{aligned} M_i = -A_i' - X_i A_i Y_i - X_i B_i F_i - L_i C_{2i} Y_i - C_{1i}'(C_{1i} Y_i + D_{1i} F_i) - \\ -\gamma^{-2}(X_i E_i + L_i D_{2i}) E_i' - \sum_{j=1}^{N} \lambda_{ij} Y_j^{-1} Y_i. \end{aligned}$$

Notice that the M_i's matrices of the H_∞-norm control problem differ from those of the H_2-norm problem by a term depending on γ, that goes to zero as γ tends to infinity. □

Remark 5.2 *The result above allow us to deal with the mixed H_2/H_∞ problem, by adopting the objective function:*

$$\sum_{i=1}^{N} \mu_i \operatorname{Tr}(\tilde{E}_i' P_i \tilde{E}_i) \tag{5.6}$$

with the constraints (5.2)–(5.4). This furnishes an upper bound to the H_2-norm of the closed-loop system. As we have seen before, (5.6) can be converted into the linear objective function (4.5), with the introduction of the auxiliary variables $Z_i, i = 1, \ldots, N$ and the additional constraints (4.6).

We can produce a CARE solution for the H_∞-norm problem that parallels the one obtained for the H_2-norm problem.

Definition 5.1 (Central controller) *We name central controller the compensator $\mathcal{G}_c$ for system $\mathcal{G}$ with the structure defined by the following CARE's*

$$A_{qi}' U_i + U_i A_{qi} - U_i [B_i (D_{1i}' D_{1i})^{-1} B_i' - \gamma^{-2} E_i E_i'] U_i + C_{qi}' C_{qi} + \sum_{j=1}^{N} \lambda_{ij} U_j = 0, \tag{5.7}$$

$$A_{pi} Q_i + Q_i A_{pi}' - Q_i [C_{2i}' (D_{2i} D_{2i}')^{-1} C_{2i} - \gamma^{-2} C_{1i}' C_{1i}] Q_i + B_{pi} B_{pi}' + \sum_{j=1}^{N} \lambda_{ij} Q_i Q_j^{-1} Q_i = 0, \tag{5.8}$$

$$\begin{aligned} C_{ci} =& -(D_{1i}' D_{1i})^{-1} (B_i' U_i + D_{1i}' C_{1i}), \\ B_{ci} =& (I - \gamma^{-2} Q_i U_i)^{-1} (Q_i C_{2i} + E_i D_{2i}') (D_{2i} D_{2i}')^{-1}, \\ A_{ci} =& A_i + B_i C_{ci} + \gamma^{-2} E_i E_i' U_i - B_{ci} (C_{2i} + \gamma^{-2} D_{2i} E_i' U_i), \end{aligned}$$

$i = 1, \ldots, N$, whenever the constraints $U_i^{-1} > \gamma^{-2} Q_i, i = 1, \ldots, N$ are satisfied, and the CARE's have stochastic stabilizing solutions.

Theorem 5.1 *Suppose that (5.2)–(5.4) have a feasible solution. Then the central controller ensures that the H_∞-norm constraint holds for the closed-loop system.*

Proof: Let us consider an auxiliary optimization problem involving the minimization of an objective function which does not involve the variables F_i's and L_i's, such as $f = \sum_{i=1}^{N} \operatorname{Tr}(Y_i^{-1} + X_i)$, subject to the H_∞ constraints (5.2)–(5.4).

Like in the H_2-norm problem, we introduce an augmented closed feasible set, which contains the original open sets defined by (5.3) and (5.4) and includes the corresponding boundary sets. We can calculate the optimal value of the F_i gains by applying the necessary optimality conditions, analogously to the proof of Theorem 4.2. We obtain here

$$F_i = -(D_{1i}' D_{1i})^{-1} (B_i' + D_{1i}' C_{1i} Y_i)$$

which implies that

$$C_{ci} = F_i U_i = -(D'_{1i} D_{1i})^{-1} (B'_i U_i + D'_{1i} C_{1i}) \tag{5.9}$$

where $U_i, i = 1, \ldots, N$ is the solution of the CARE in (5.7).

Since the auxiliary objective function does not depend on the gains L_i's either, the optimal value can be calculated using the necessary optimality conditions, and it yields

$$L_i = -(\gamma^2 C'_{2i} + X_i E_i D'_{2i})(D_{2i} D'_{2i})^{-1}$$

We introduce the relation $Q_i = \gamma^2 X_i^{-1}$, leading to

$$\begin{aligned} B_{ci} &= (U_i - X_i)^{-1} L_i = -(U_i - \gamma^2 Q_i^{-1})^{-1} L_i \\ &= (I - \gamma^{-2} Q_i U_i)^{-1} (Q_i C_{2i} + E_i D'_{2i})(D_{2i} D'_{2i})^{-1} \end{aligned} \tag{5.10}$$

where Q_i is the solution of the CARE in (5.8).

Finally, we calculate A_{ci} as follows:

$$\begin{aligned} A_{ci} = {} & (X_i - U_i)^{-1} (A'_i + X_i A_i Y_i + X_i B_i F_i + L_i C_{2i} Y_i + C'_{1i}(C_{1i} Y_i + D_{1i} F_i) + \\ & + \gamma^{-2} (X_i E_i + L_i D_{2i}) E'_i + \sum_{j=1}^{N} U_j Y_i) U_i \\ = {} & A_i + B_i C_{ci} + \gamma^{-2} E_i E'_i U_i - B_{ci}(C_{2i} + \gamma^{-2} D_{2i} E'_i U_i) + (X_i - U_i)^{-1} \times \\ & \times \left\{ A'_{qi} U_i + U_i A_{qi} - U_i [B_i (D'_{1i} D_{1i})^{-1} B'_i - \gamma^{-2} E_i E'_i] U_i + C'_{qi} C_{qi} + \sum_{j=1}^{N} \lambda_{ij} U_j \right\} \\ = {} & A_i + B_i C_{ci} + \gamma^{-2} E_i E'_i U_i - B_{ci}(C_{2i} + \gamma^{-2} D_{2i} E'_i U_i) \end{aligned} \tag{5.11}$$

Notice that due to the conditions $U_i^{-1} > \gamma^{-2} Q_i, \ i = 1, \ldots, N$, the central controller is feasible, since from this relation, we have that

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} > 0, \ i = 1, \ldots, N \tag{5.12}$$

are satisfied. □

Remark 5.3 *The requirement on stochastic stabilizing solutions for the central controller is equivalent to impose that the MJLS's $\dot{\zeta}_t = \tilde{A}_q(\theta_t)\zeta_t$, and $\dot{\xi}_t = \tilde{A}_p(\theta_t)\xi_t, t \geq 0$ with*

$$\tilde{A}_{qi} = A_{qi} - [B_i (D'_{1i} D_{1i})^{-1} B'_i - \gamma^{-2} E_i E'_i] U_i, \tag{5.13}$$

and

$$\tilde{A}_{pi} = A_{pi} - Q_i [C'_{2i} (D_{2i} D'_{2i})^{-1} C_{2i} - \gamma^{-2} C'_{1i} C_{1i}], \tag{5.14}$$

$i = 1, \ldots, N$, are stochastically stable systems. In addition, we should point out that the compensator $\mathcal{G}_c$ does not depend on the initial probability distribution μ.

Noticed once again that the central controller approach will provide a single controller, whereas the LMI's (5.2)–(5.4) parameterize all feasible controllers.

6 The Filtering Problem

In this section, we present the solution for the H_2 and H_∞ filtering problems as stated in section 2.

The augmented MJLS made up by system $\mathcal{G}_0$ and the filter $\mathcal{G}_f$ can be written as

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ B_{ci} C_{2i} & A_{ci} \end{bmatrix}, \ \tilde{E}_i = \begin{bmatrix} E_i \\ B_{ci} D_{2i} \end{bmatrix}, \ \tilde{C}_i = \begin{bmatrix} C_{1i} & -C_{ci} \end{bmatrix} \tag{6.1}$$

If we compare the above parameters with those of the output feedback control problem (4.4), we verify that the problem here is a particular case of that one, with $B_i = 0$ and $D_{1i} = -I$. Therefore, we can find the H_2 and H_∞ filters in terms of LMI's or coupled Riccati equations in a straightforward manner, from the corresponding control results.

6.1 $\mathbf{H_2}$ filtering

In the H_2-norm filtering problem, one wishes to minimize $||\mathcal{G}_e||_2^2$. Notice that a particular filter structure is imposed by $\mathcal{G}_f$, which comprises all linear n-dimensional filters that can vary on time only when the Θ state jumps. The Kalman filter for MJLS also aims at the minimum mean square error. But as we mentioned, it is intrinsically a time variant filter, no steady state solution can be found, contrary to what occurs in the case of linear systems embed in additive Gaussian noise.

The next theorem relies on the results for the H_2-norm control problem.

Theorem 6.1 *The H_2 filtering problem can be written as*

$$\begin{aligned} \inf \quad & \sum_{i=1}^{N} \mu_i \mathrm{Tr}(Z_i) & (6.2) \\ \text{s.t.} \quad & \begin{bmatrix} X_i & X_i E_i + L_i D_{2i} \\ E_i' X_i + D_{2i}' L_i' & Z_i \end{bmatrix} > 0 & \\ & A_i' X_i + X_i A_i + L_i C_{2i} + C_{2i}' L_i' + C_{1i}' C_{1i} + \sum_{j=1}^{N} \lambda_{ij} X_j < 0, \quad i = 1, \ldots, N, & (6.3) \end{aligned}$$

and the filter parameters are

$$\begin{aligned} C_{ci} &= C_{1i} \\ B_{ci} &= -X_i^{-1} L_i \\ A_{ci} &= A_i - B_{ci} C_{2i}, \quad i = 1, \ldots, N. \end{aligned} \tag{6.4}$$

Proof: Let us retrieve the results of Theorem 4.1 to get here the following problem:

$$\min \sum_{i=1}^{N} \mu_i \mathrm{Tr}(Z_i) \tag{6.5}$$

$$\begin{aligned} \text{s.t} & \begin{bmatrix} Y_i & I & E_i \\ I & X_i & X_i E_i + L_i D_{2i} \\ E_i' & E_i' X_i + D_{2i}' L_i' & Z_i \end{bmatrix} > 0 \\ & \begin{bmatrix} A_i Y_i + Y_i A_i' + \lambda_{ii} Y_i & (C_{1i} Y_i - F_i)' & R_i(Y) \\ C_{1i} Y_i - F_i & -I & 0 \\ R_i'(Y) & 0 & S_i(Y) \end{bmatrix} < 0 \qquad (6.6) \\ & A_i' X_i + X_i A_i + L_i C_{2i} + C_{2i}' L_i' + C_{1i}' C_{1i} + \sum_{j=1}^{N} \lambda_{ij} X_j < 0, \quad i = 1, \ldots, N. \end{aligned}$$

For the optimal values of the gains F_i's and the corresponding Y_i's notice that the first LMI in (6.6) is equivalent to the constraints

$$Y_i > \begin{bmatrix} I & E_i \end{bmatrix} \begin{bmatrix} X_i & X_i E_i + L_i D_{2i} \\ E_i' X_i + D_{2i}' L_i' & Z_i \end{bmatrix}^{-1} \begin{bmatrix} I \\ E_i' \end{bmatrix}, \tag{6.7}$$

$$\begin{bmatrix} X_i & X_i E_i + L_i D_{2i} \\ E_i' X_i D_{2i}' L_i' & Z_i \end{bmatrix} > 0, \tag{6.8}$$

and (6.7) indicates that the absolute minimum of the cost functional is attained, regarding the matrices $Y_i > 0$ when $\lim Y_i^{-1} = 0, i = 1, \ldots, N$. We shall demonstrate that in the present situation, we can adopt this absolute minimum beforehand, and eliminate any restriction involving the matrices Y_i's. Indeed, if we choose

$$F_i = C_{1i} Y_i \tag{6.9}$$

which implies

$$C_{ci} = C_{1i} \tag{6.10}$$

we can rewrite the constraint on Y_i in (6.6) as

$$A_i Y_i + Y_i A_i' + \sum_{i=1}^{N} \lambda_{ij} Y_i Y_j^{-1} Y_i < 0 \tag{6.11}$$

or, setting $U_i = Y_i^{-1}$,

$$A_i'U_i + U_iA_i + \sum_{j=1}^{N} \lambda_{ij}U_j < 0 \tag{6.12}$$

Now, recall the increasing property of the U_i's that satisfy inequalities such as (6.12), as we discussed in Remark 2.1. We can conclude that $U_i = 0, i = 1, \dots, N$ is optimal in the present problem, which allow us to write the filtering problem as in (6.2) and (6.3).

The filter $\mathcal{G}_f$ can be calculated as

$$\begin{aligned} B_{ci} &= (Y_i^{-1} - X_i)^{-1}L_i = -X_i^{-1}L_i \\ A_{ci} &= (Y_i^{-1} - X_i)^{-1}M_iY_i^{-1} \\ &= X_i^{-1}(A_i + X_iA_iY_i + L_iC_{2i}Y_i + C_{1i}'(C_{1i}Y_i - F_i) + \sum_{j=1}^{N} \lambda_{ij}Y_j^{-1}Y_i)Y_i \\ &= A_i - B_{ci}C_{2i} \end{aligned}$$

Hence, the proof is completed. □

One should be aware of the duality between the filter problem above and the H_2 state feedback control as it appears for instance in [27]. Another correspondence between these problems was also indicated in [22], considering the change of variables $A_i = A_i'$, $C_{2i} = B_i'$, $L_i = F_i'$, $C_{1i} = E_i'$, $E_i = C_{1i}'$, $D_{2i} = D_{1i}'$ and $\Lambda = \Lambda'$. In [22] a filter similar to that in Lemma 6.1 below, was thought to be the Kalman filter for MJLS.

Here we find the H_2 optimal filter using a CARE.

Lemma 6.1 *The H_2 optimal filter is given by*

$$A_{pi}Q_i + Q_iA_{pi}' - \mu_i^{-1}Q_iC_{2i}'(D_{2i}D_{2i}')^{-1}C_{2i}Q_i + \mu_iB_{p1}B_{p1}' + \sum_{j=1}^{N} \lambda_{ji}Q_j = 0 \tag{6.13}$$

$$\begin{aligned} A_{ci} =& A_i - B_{ci}C_{2i} \\ B_{ci} =& (\mu_i^{-1}Q_iC_{2i}' + E_iD_{2i}')(D_{2i}D_{2i}')^{-1} \\ C_{ci} =& C_{1i}, \quad i = 1, \dots, N, \end{aligned}$$

provided that (6.13) has stochastic stabilizing solution.

Proof: To obtain the filter using CARE, we just need to consider the solution of the output feedback problem again. We get for each i,

$$A_i'U_i + U_iA_i + \sum_{j=1}^{N} \lambda_{ij}U_j = 0$$

$$A_{pi}Q_i + Q_iA_{pi}' - \mu_i^{-1}Q_iC_{2i}'(D_{2i}D_{2i}')^{-1}C_{2i}Q_i + \mu_iB_{p1}B_{p1}' + \sum_{j=1}^{N} \lambda_{ji}Q_j = 0$$

$$\begin{aligned} A_{ci} =& A_i - B_{ci}C_{2i} \\ B_{ci} =& (\mu_i^{-1}Q_iC_{2i}' + E_iD_{2i}')(D_{2i}D_{2i}')^{-1} \\ C_{ci} =& C_{1i} \end{aligned}$$

The same considerations made in Theorem 6.1, allow us to set $U_i = 0, i = 1, \dots, N$ here, and the proof is completed. □

Remark 6.1 *In the situation when the Markov chain Θ is irreducible and recurrent with invariant measure η on $\mathcal{S}$, we can make use of Proposition 2.4. By setting the initial measure $\mu = \eta$, the H_2-norm will coincide with the mean square error when the input variable $w_t = \nu_t$ is a m-dimensional Wiener process with incremental covariance equal to the identity.*

Remark 6.2 *Notice that the gains $B_{ci}, i = 1, \dots, N$ coincide with those obtained for the H_2 output feedback control in (4.19), and $A_{ci}, i = 1...N$ are also similar to (4.20), except for the extra term B_iC_{ci} appearing in the control problem. The conclusion is that the H_2-norm compensator $\mathcal{G}_c$ comprises a state estimator filter, and the Separation Principle is valid for the H_2-norm output feedback problem, a novelty for MJLS.*

6.2 H_∞ filtering

We deal in this section with the H_∞ filtering problem, i.e., we seek a filter of form $\mathcal{G}_f$, such that $||\mathcal{G}_e||_\infty < \gamma$ holds for some specified $\gamma > 0$. Similarly to the H_2 filtering problem, one can observe that the solution can be expressed by a reduction from the H_∞-norm problem of output feedback control, and the H_∞-norm filter is obtained in terms of LMI's or a CARE in a straightforward manner. We omit the proofs, since they are similar to those in the previous section.

Lemma 6.2 *Each feasible H_∞-norm filter is parameterized by the LMI's*

$$\begin{bmatrix} A_i'X_i + X_iA_i + L_iC_{2i} + C_{2i}'L_i' + C_{1i}'C_{1i} + \sum_{j=1}^N \lambda_{ij}X_j & X_iE_i + L_iD_{2i} \\ E_i'X_i + D_{2i}'L_i' & -\gamma^2 I \end{bmatrix} < 0$$

$i = 1, \ldots, N$, and the filter $\mathcal{G}_f$ is given by

$$\begin{aligned} B_{ci} =& (Y_i^{-1} - X_i)^{-1}L_i = -X_i^{-1}L_i \\ C_{ci} =& F_iY_i^{-1} = C_{1i} \\ A_{ci} =& (Y_i^{-1} - X_i)^{-1}M_iY_i^{-1} \\ =& X_i^{-1}(A_i + X_iA_iY_i + L_iC_{2i}Y_i + \sum_{j=1}^N \lambda_{ij}Y_j^{-1}Y_i + C_{1i}'(C_{1i}Y_i - F_i) + \\ & + \gamma^{-2}(X_iE_i + L_iD_{2i})E_i')Y_i^{-1} \\ =& A_i - B_{ci}C_{2i} \end{aligned}$$

Although the same LMI's in the variables X_i's appears for the filtering and control problems, we observe that the matrices A_{ci}'s for the filter will depend upon γ only through B_{ci}, whereas in the control problem, these matrices will be directly dependent upon the parameter γ.

The H_∞ central filter can be obtained from the CARE's of the central controller.

Lemma 6.3 *The H_∞ central filter is given by*

$$A_{pi}Q_i + Q_iA_{pi}' - Q_i[C_{2i}'(D_{2i}D_{2i}')^{-1}C_{2i} - \gamma^{-2}C_{1i}'C_{1i}]Q_i + B_{pi}B_{pi}' + \sum_{j=1}^N \lambda_{ij}Q_iQ_j^{-1}Q_i = 0 \tag{6.14}$$

$$\begin{aligned} A_{ci} &= A_i - B_{ci}D_{2i} \\ B_{ci} &= (Q_iC_{2i} + E_iD_{2i}')(D_{2i}D_{2i}')^{-1} \\ C_{ci} &= C_{1i} \end{aligned}$$

$i = 1, \ldots, N$, and provided that (6.14) has stochastic stabilizing solution, it ensures that the H_∞ constraint is satisfied.

Remark 6.3 *For the H_∞ problems notice that a Separation Principle is no longer valid. Although the CARE in (6.14) and in (5.8) are identical, the gains $B_{ci}, i = 1, \ldots, N$ are no longer the same. They could be made identical if $\gamma \to \infty$, but afterall one can check that with this limit one retrieves the H_2-norm control and filtering problems.*

7 Illustrative examples

In the first example, we present a system for which the H_2 controller proposed by [18] does not ensure stochastic stability.

Example 7.1 *Consider the following parameters for $\mathcal{G}$, having four-state Markov chain*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} -1 & 2 \\ 2 & 5 \end{bmatrix}, \; A_4 = \begin{bmatrix} 5 & 2 \\ 2 & -1 \end{bmatrix}$$

$$B_{11} = B_{12} = B_{13} = B_{14} = \begin{bmatrix} 0.4 & 0 & 1 \\ 0 & 0.4 & 1 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; B_{23} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_{24} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_{11} = C_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; C_{13} = C_{14} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$C_{21} = C_{22} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; C_{23} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; C_{24} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D_{11} = D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; D_{13} = D_{14} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D_{21} = D_{22} = D_{23} = D_{24} = \begin{bmatrix} 0 & 0 & 0.5 \end{bmatrix}$$

and the transition matrix

$$\Lambda = \begin{bmatrix} -5 & 2 & 3 & 0 \\ 1 & -4 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The compensator $\mathcal{G}_c$ obtained from the analysis here is given by

$$A_{c1} = \begin{bmatrix} -0.9407 & -0.9407 \\ -3.9571 & -8.0545 \end{bmatrix}, \; A_{c2} = \begin{bmatrix} -9.7479 & -5.5930 \\ 0.7432 & 0.7432 \end{bmatrix}$$

$$A_{c3} = \begin{bmatrix} -1.0000 & -0.0716 \\ -1.7138 & -17.0037 \end{bmatrix}, \; A_{c4} = \begin{bmatrix} -16.6459 & -1.7138 \\ -1.2872 & -1.0000 \end{bmatrix}$$

$$B_{c1} = \begin{bmatrix} 1.9407 \\ 0.6375 \end{bmatrix}, \; B_{c2} = \begin{bmatrix} 2.2953 \\ 0.2568 \end{bmatrix}, \; B_{c3} = \begin{bmatrix} 2.0716 \\ 10.3814 \end{bmatrix}, \; B_{c4} = \begin{bmatrix} 10.0235 \\ 3.2872 \end{bmatrix}$$

$$C_{c1} = \begin{bmatrix} -1.6598 & -4.7085 \end{bmatrix}, \; C_{c2} = \begin{bmatrix} -4.7263 & -1.6488 \end{bmatrix}$$

$$C_{c3} = \begin{bmatrix} -3.7138 & -11.6223 \end{bmatrix}, \; C_{c4} = \begin{bmatrix} -11.6223 & -3.7138 \end{bmatrix}$$

and the H_2-norm of the closed-loop system is 375.7984. *However, if we try to find a compensator $\mathcal{G}_c$ using the CARE's (4.14) and (4.30) as proposed by [18], the corresponding closed-loop system is not stochastically stable* □

In the next example, the performance of the central and the mixed controllers regarding the H_2-norm of the closed-loop system are evaluated.

Example 7.2 *We employ the same system in Example 7.1, and we produce a second example with a different transition matrix:*

$$\Lambda_1 = \begin{bmatrix} -5 & 2 & 3 & 0 \\ 1 & -4 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \Lambda_2 = \begin{bmatrix} -10 & 7 & 3 & 0 \\ 1 & -4 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The results are given in Table 1 for $\gamma = 10$. We observe that minimizing an upper bound of the H_2-norm may not correspond to the minimization of the real norm. In fact, it can be seen that the upper bound the mixed controller is far better than that for the central one. However, in the case $\Lambda = \Lambda_2$, we found that there is no sound difference between them when we obtain the real H_2-norm. Therefore, minimizing an upper bound for the H_2-norm does not really ensure an optimal performance; the central controller, which does not take into account any performance index at all, may give similar or even better results. □

	$\Lambda = \Lambda_1$		$\Lambda = \Lambda_2$	
	upper bound	H_2-norm	upper bound	H_2-norm
Central	9738.1850	630.6982	6613.0104	387.3138
Mixed	445.1108	386.7865	435.0471	378.8073

Table 1: H_∞ mixed and central controllers ($\gamma = 10$).

8 Application

We applied the H_2 and H_∞ norm control techniques to a VTOL helicopter model presented in [24]. The dynamics can be written as

$$\begin{aligned} \dot{x} &= A(\theta_t)x + B(\theta_t)u + Ew \\ z &= C_1 x + D_1 u \\ y &= C_2 x + D_2 w \end{aligned} \tag{8.1}$$

where θ_t indicates the airspeed and the state variables are the horizontal velocity (x_1), the vertical velocity (x_2), the pitch rate (x_3) and the pitch angle (x_4). The parameters are given by

$$\begin{aligned} A(\theta_t) &= \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & a_{32}(\theta_t) & -0.707 & a_{34}(\theta_t) \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ B(\theta_t) &= \begin{bmatrix} 0.4422 & 0.1761 \\ b_{21}(\theta_t) & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} \\ E &= \begin{bmatrix} I_{4\times 4} & 0_{4\times 1} \end{bmatrix} \\ C_1 &= \begin{bmatrix} I_{4\times 4} \\ 0_{2\times 4} \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0_{4\times 2} \\ I_{2\times 2} \end{bmatrix} \\ C_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \\ D_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{8.2}$$

We modeled the behavior of θ_t as a Markov chain with three different states, corresponding to airspeeds of 135 (nominal value), 60 and 170 knots. The values of parameters a_{32}, a_{34} and b_{32} are shown in Table 2. Evidently, other operation points could be added to provide a more refined model. Three different transition matrices are considered:

$$\begin{aligned} \Lambda_1 &= \begin{bmatrix} -0.0894 & 0.0671 & 0.0236 \\ 0.0671 & -0.0671 & 0 \\ 0.0236 & 0 & -0.0236 \end{bmatrix} \\ \Lambda_2 &= \begin{bmatrix} -0.0171 & 0.0007 & 0.0164 \\ 0.0013 & -0.0013 & 0 \\ 0.0986 & 0 & -0.0986 \end{bmatrix} \\ \Lambda_3 &= \begin{bmatrix} -0.0450 & 0.0002 & 0.0448 \\ 0.0171 & -0.0171 & 0 \\ 0.0894 & 0 & -0.0894 \end{bmatrix} \end{aligned}$$

and the initial distributions we adopt are $(0.3333, 0.3333, 0.3333)$, $(0.6, 0.3, 0.1)$ and $(0.6, 0.1, 0.3)$, which are stationary.

We solved the H_2 and H_∞ control problems. In both cases, the corresponding norms have been minimized. The results can be seen in Table 3.

The advantage of using Markov jump systems can be clearly seen. In this example, a refined description of the airspeed is included in the model, if we compare with that used in guaranteed control [16]. For instance, one would only be able to impose that the parameters a_{32},a_{34} and b_{2i} lie in a certain convex set, without knowing how they would change with time. Taking into account the information on how the

Airspeed (knots)	a_{32}	a_{34}	b_{21}
135	0.3681	1.4200	3.5446
60	0.0664	0.1198	0.9775
170	0.5047	2.5460	5.1120

Table 2: Parameters

	H_2 control	H_∞ control	
	optimal H_2-norm	optimal γ	H_2-norm
Λ_1	14.7376	2.6891	152.5675
Λ_2	13.5837	2.6829	84.6732
Λ_3	15.4574	2.6858	144.2702

Table 3: Application results

airspeed can vary, we can provide less restrictive conditions for stability, which leads to controllers with better performance. One should notice that in the guaranteed cost approach, the system must be stable for *all* values of a_{32},a_{34} and b_{2i} in the convex set in consideration, whereas in the stochastic stability framework, stability of all operation modes is not even required.

A Proof of Proposition 2.2

Note that the processes (θ_t, x_t, z_t), $t \geq 0$ is Markovian, since we assume that the processes w is either a Markovian processes or w is $\mathcal{F}_0$-measurable. The infinitesimal generator applied to a generic differentiable function $g(t,x,i)$ satisfies [10]:

$$\mathcal{U}g(t,x_t,\theta_t) = g_t(t,x_t,\theta_t) + [x_t' A(\theta_t)' + w_t' E(\theta_t)'] g_x(t,x_t,\theta_t) + \sum_{j=1}^{N} \lambda_{\theta_t,j} g(t,x_t,j)$$

where g_t and g_x are the corresponding derivatives of g.

According to Proposition 2.1 d), we set $U_1 > 0, \ldots, U_N > 0$ and find $P = (P_1, \ldots, P_N)$ $P_i > 0 \forall i$ satisfying (2.2). For the purpose here let $\epsilon > 0$ be such that $\epsilon^2 P_i^2 < U_i$, $\forall i$. We wish to evaluate $E(|P_{\theta_t}^{1/2} x_t|^2)$. The infinitesimal generator associated to $\mathcal{G}_0$ applied to function $g(t,x,i) = x' P_i x$ is given by

$$\mathcal{U}g(t,x_t,\theta_t) = (A_{\theta_t} x_t + E_{\theta_t} w_t)' P_{\theta_t} x_t + x_t' P_{\theta_t} (A_{\theta_t} x_t + E_{\theta_t} w_t) + \sum_{j=1}^{N} \lambda_{\theta_t j} x_t' P_j x_t$$

and since for any $\epsilon > 0$, we get that $(E_{\theta_t} w_t)' P_{\theta_t} x_t + x_t' P_{\theta_t} E_{\theta_t} w_t \leq \epsilon^2 x_t' P_{\theta_t}^2 x_t + \frac{1}{\epsilon^2} w_t' E_{\theta_t}' E_{\theta_t} w_t$, we can evaluate:

$$\begin{aligned}\mathcal{U}g(t,x_t,\theta_t) &\leq x_t' [A_{\theta_t}' P_{\theta_t} + P_{\theta_t} A_{\theta_t} + \sum_{j=1}^{N} \lambda_{\theta_t j} P_j] x_t + \epsilon^2 x_t' P_{\theta_t}^2 x_t + \frac{1}{\epsilon^2} w_t' E_{\theta_t}' E_{\theta_t} w_t \\ &\leq -x_t' Q_{\theta_t} x_t + \frac{1}{\epsilon^2} w_t' E_{\theta_t}' E_{\theta_t} w_t\end{aligned}$$

where $Q_i := U_i - \epsilon^2 P_i^2 > 0$ for $i = 1, \ldots, N$. Applying Dynkin's formula, we have that

$$\begin{aligned}E(|P_{\theta_t}^{1/2} x_t|^2) - E(|P_{\theta_0}^{1/2} x_0|^2) &= E\Big(\int_0^t \mathcal{U}g(s,x_s,\theta_s)\, ds\Big) \\ &= \leq -E\Big(\int_0^t |Q_{\theta_s}^{1/2} x_s|^2\, ds\Big) + \frac{1}{\epsilon^2} E\Big(\int_0^t |E_{\theta_s} w_s|^2\, ds\Big)\end{aligned}$$

and thus,

$$\begin{aligned}\lim_{t\to\infty} \Big(E(|P_{\theta_t}^{1/2} x_t|^2) + E\Big(\int_0^t |Q_{\theta_s}^{1/2} x_s|^2\, ds\Big)\Big) &\leq E(|P_{\theta_0}^{1/2} x_0|^2) + \frac{1}{\epsilon^2} E\Big(\int_0^\infty |E_{\theta_s} w_s|^2\, ds\Big) \\ &< \infty\end{aligned}$$

which implies that $\|x\|_2^2 = E\big(\int_0^\infty |x_s|^2\, ds\big) \le \frac{1}{c_0} E\big(\int_0^\infty |Q_{\theta_s}^{1/2} x_s|^2\, ds\big) < \infty$, with $c_0 = \min\{\lambda_{\min}(Q_i); i = 1, \ldots, N\}$. □

B A complement of the proof of Theorem 4.2

Let us consider the following problem

$$
\begin{aligned}
\inf \quad & \sum_{i=1}^{N} \mu_i \mathrm{Tr}((E_i' - D_{2i}' B_{ci}') W_i (E_i - B_{ci} D_{2i})) \\
\text{s.t.} \quad & (A_i - B_{ci} C_{2i})' W_i + W_i (A_i - B_{ci} C_{2i}) + C_{ci}' D_{1i}' D_{1i} C_{ci} + \sum_{j=1}^{N} \lambda_{ij} W_j < 0 \qquad \text{(B.1)} \\
& W_i = W_i' > 0, \quad i = 1, \ldots, N.
\end{aligned}
$$

where W_i and $B_{ci}, i = 1, \ldots, N$ are the problem variables. From the increasing property of the W_i's that satisfy (B.1), as discussed in Remark 2.1, we know that the optimal solution attains the equality in (B.1), and the problem can be written equivalently as

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} \mu_i \mathrm{Tr}((E_i' - D_{2i}' B_{ci}') W_i (E_i - B_{ci} D_{2i})) \\
\text{s.t.} \quad & (A_i - B_{ci} C_{2i})' W_i + W_i (A_i - B_{ci} C_{2i}) + C_{ci}' D_{1i}' D_{1i} C_{ci} + \sum_{j=1}^{N} \lambda_{ij} W_j = 0 \qquad \text{(B.2)} \\
& W_i = W_i' > 0, \quad i = 1, \ldots, N.
\end{aligned}
$$

According with Proposition 2.3, the objective function above is equivalent to the H_2-norm of a MJLS defined by

$$
\begin{cases}
\dot{x}_t = (A(\theta_t) - B_c(\theta_t) C_2(\theta_t)) x_t + (E(\theta_t) - B_c(\theta_t) D_2(\theta_t) w_t, \\
z_t = D_1(\theta_t) C_c(\theta_t) x_t, \quad t \ge 0.
\end{cases}
$$

It also follows from Proposition 2.3 that the problem in (B.2) is equivalent to

$$
\begin{aligned}
\min \quad & g = \sum_{i=1}^{N} \mathrm{Tr}(D_{1i} C_{ci} Q_i C_{ci}' D_{1i}') \qquad \text{(B.3)} \\
\text{s.t.} \quad & Q_i (A_i - B_{ci} C_{2i})' + (A_i - B_{ci} C_{2i}) Q_i + \mu_i (E_i - B_{ci} D_{2i})(E_i - B_{ci} D_{2i})' + \sum_{j=1}^{N} \lambda_{ji} Q_j \le 0 \qquad \text{(B.4)} \\
& Q_i = Q_i' > 0, \quad i = 1, \ldots, N.
\end{aligned}
$$

For the problem in (B.4) we can show that

$$
\bar{B}_{ci} = (\mu_i^{-1} Q_i C_{2i}' + E_i D_{2i}')(D_{2i} D_{2i}')^{-1}, \quad i = 1, \ldots, N. \qquad \text{(B.5)}
$$

is optimal. Indeed, if we denote by $\bar{Q}_i, i = 1, \ldots, N$, the solution of the equation:

$$
\bar{Q}_i (A_i - \bar{B}_{ci} C_{2i})' + (A_i - \bar{B}_{ci} C_{2i}) \bar{Q}_i + \mu_i (E_i - \bar{B}_{ci} D_{2i})(E_i - \bar{B}_{ci} D_{2i})' + \sum_{j=1}^{N} \lambda_{ji} \bar{Q}_j = 0, \qquad \text{(B.6)}
$$

$i = 1, \ldots, N$, we can write it equivalently as

$$
A_{pi} \bar{Q}_i + \bar{Q}_i A_{pi}' - \mu_i^{-1} \bar{Q}_i C_{2i}' (D_{2i} D_{2i}')^{-1} C_{2i} \bar{Q}_i + \mu_i B_{pi} B_{pi}' + \sum_{j=1}^{N} \lambda_{ji} \bar{Q}_j = 0, \qquad \text{(B.7)}
$$

$i = 1, \ldots, N$, where

$$
\begin{aligned}
A_{pi} &= A_i - E_i D_{2i}' (D_{2i} D_{2i}')^{-1} C_{2i} \\
B_{pi} &= E_i (I - D_{2i}' (D_{2i} D_{2i}')^{-1} D_{2i})
\end{aligned} \qquad \text{(B.8)}
$$

Thus, $\bar{Q}_i$ is the solution of the CARE in (4.15) and $\bar{B}_{ci} = G_i^R$ is the correspondent Riccati gain. If now we subtract (B.6) from the inequality in (B.4), we obtain for each i that

$$(Q_i - \bar{Q}_i)(A_i - \bar{B}_{ci}C_{2i})' + (A_i - \bar{B}_{ci}C_{2i})(Q_i - \bar{Q}_i)+ \\ + \sum_{j=1}^{N} \lambda_{ji}(Q_j - \bar{Q}_j) + \mu_i(\bar{B}_{ci} - B_{ci})D_{2i}D'_{2i}(\bar{B}_{ci} - B_{ci})' \leq 0 \tag{B.9}$$

¿From the assumption in Theorem 4.2, $\bar{B}_{ci}, i = 1, \ldots, N$ is a stabilizing gain; from Proposition 2.1 c) and d) and (B.9), it follows that $Q_i - \bar{Q}_i \geq 0, i = 1, \ldots, N$. In particular, this establishes that $g(\bar{Q}) \leq g(Q)$ for any $Q = (Q_1, \ldots, Q_N), Q_i > 0$, that satisfies (B.4), and hence the optimality of $\bar{Q}_i$ and $\bar{B}_{ci}$ for the problem (B.1).

C Stochastic stability of optimal controllers

Let us consider the controller given by (4.18)–(4.20) for the H_2-norm problem. We shall show that it stabilizes the closed-loop system in the stochastic sense.

Since U_i and Q_i are stabilizing solutions of (4.14) and (4.15), according to Proposition 2.1 for every $\mathcal{Q}_i > 0$ and $\mathcal{R}_i > 0, i = 1, \ldots, N$, the coupled Lyapunov equations

$$(A_i + B_i\bar{C}_{ci})'H_i + H_i(A_i + B_i\bar{C}_{ci}) + \sum_{j=1}^{N} \lambda_{ij}H_j + \mathcal{Q}_i = 0$$

$$(A_i - \bar{B}_{ci}C_{2i})'W_i + W_i(A_i - \bar{B}_{ci}C_{2i}) + \sum_{j=1}^{N} \lambda_{ij}W_j + \mathcal{R}_i = 0, \ i = 1, \ldots, N$$

have a positive definite solution $\widehat{H}_i, \widehat{W}_i, i = 1, \ldots, N$. This is particularly true for

$$\mathcal{R}_i = (\mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci})'\mathcal{Q}_i^{-1}(\mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci}) - \widehat{H}_iB_i\bar{C}_{ci} - \bar{C}'_{ci}B'_i\widehat{H}_i + \alpha_i I \tag{C.1}$$

where each $\alpha_i > 0$ is such that $\mathcal{R}_i > 0$. For this choice, if we set

$$P_i = \begin{bmatrix} \widehat{W}_i + \widehat{H}_i & -\widehat{W}_i \\ -\widehat{W}_i & \widehat{W}_i \end{bmatrix} \tag{C.2}$$

then $P_i > 0$ for each i, and we also define

$$T_i = \begin{bmatrix} \widehat{H}_i^{-1} & I \\ \widehat{H}_i^{-1} & 0 \end{bmatrix}. \tag{C.3}$$

It follows that the overall closed-loop system is stochastically stable if

$$T'_i(\tilde{A}'_iP_i + P_i\tilde{A}_i + \sum_{j=1}^{N} \lambda_{ij}P_j)T_i = \\ \begin{bmatrix} -\widehat{H}_i^{-1}\mathcal{Q}_i\widehat{H}_i^{-1} & -\widehat{H}_i^{-1}(\mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci}) \\ (\mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci})\widehat{H}_i^{-1} & -\mathcal{R}_i - \mathcal{Q}_i - \widehat{H}_iB_i\bar{C}_{ci} - \bar{C}'_{ci}B'_i\widehat{H}_i \end{bmatrix} < 0, \quad i = 1, \ldots, N \tag{C.4}$$

The above holds true iff

$$\mathcal{R}_i + \mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci} + \bar{C}'_{ci}B'_i\widehat{H}_i > (\mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci})'\mathcal{Q}_i^{-1}(\mathcal{Q}_i + \widehat{H}_iB_i\bar{C}_{ci}), \ i = 1, \ldots, N \tag{C.5}$$

which is satisfied for $\mathcal{R}_i$ given by (C.1).

Note that a completely analogous result holds for the solution of the H_∞-norm control problem.

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