Necessary and Sufficient Condition for Robust Stability and Stabilizability of Continuous-time Linear Systems With Markovian Jumps[1]

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Abstract. In this paper we investigate the quadratic stability and quadratic stabilizability of the class of continuous-time linear systems with Markovian jumps and norm-bound uncertainties in the parameters. Under some appropriate assumptions a necessary and sufficient condition is established for mean square quadratic stability and mean square quadratic stabilizability of this class of systems. The quadratic guaranteed cost control problem is also addressed via LMI optimization problem.

Keywords. Continuous-time linear systems, Parameter uncertainties, Riccati equation, Linear systems with Markovian jumps.

1 Introduction

The concepts of quadratic stability and quadratic stabilizability have retained the attention of many researchers of the control community. Many results regarding the quadratic stability and the quadratic stabilizability for linear uncertain continuous-time or discrete-time systems have been established under some appropriate conditions. For more details regarding these concepts, we refer the reader to Khargonekar et al. Ref. 1, Petersen Ref. 2, Petersen and McFarlane Ref. 3 and the references therein.

In this paper we deal with the class of continuous-time linear systems with Markovian jumps. This class of systems has been used in different areas such as manufacturing systems, power systems, economic systems, to model systems subject to abrupt variations in its dynamics. For the details of such applications, we refer the reader to Boukas et al. Ref. 4, Mariton Ref. 5 and the references therein.

For the class of linear systems with Markovian jumps and no uncertainties on the parameters, many problems and concepts from the linear systems literature have been addressed such as the concepts of stochastic stability, the stochastic controlability, the stochastic observability, and the problems of the jump linear regulator, the $\mathcal{H}_\infty$ control problem etc. For more details on these problems and concepts, we refer the reader to Ji and Chizeck Ref. 6, Feng et al. Ref. 7, Costa Ref. 8, De Souza and Fragoso Ref. 9, Boukas and Yang Refs. 10, 11 and 12, Morozan Refs. 13, 14 and 15, Pan and Basar Refs. 16, 17 and 18, ElGhaoui and Ait Rami Ref. 19, Dufour and Elliot Ref. 20 and the references therein.

The class of uncertain linear systems with Markovian jumps has also recently been addressed and results regarding the robust stability and stabilizability, robust $\mathcal{H}_\infty$ control problem and the robust filtering problem have been established. The reader interested on these concepts can find some develpments in Boukas Ref. 10, Boukas and Yang Ref. 12, Boukas Shi and Boukas Ref. 21, De Souza and Fragoso Ref. 22 and the references therein.

The goal of this paper consists of deriving a necessary and sufficient condition for mean square quadratic stability and stabilizability for continuous-time linear systems with Markovian jumps. This condition can be seen as an extension of the necessary and sufficient condition obtained in Ref. 23 from the deterministic framework to the stochastic one. For this we shall use the concepts of mean square quadratic stability and stabilizability to the class of continuous-time linear systems with Markovian jumps, defined below. The quadratic guaranteed cost control problem will also be addressed, leading to a LMI optimization problem.

The paper is organized as follows. In section 2, we will give the problem statement and the assumptions that the uncertainties should satisfy. We also establish some preliminary results. In section 3, we will deal with the mean square quadratic stability of the class of uncertain linear continuous-time systems. A necessary and sufficient condition is established to check the robust mean square quadratic stability of the class of systems under consideration. In section 4, we will deal with the mean square quadratic stabilizability of the same class of systems and give a necessary and sufficient condition for checking whether the system at hand is mean square quadratic stabilizable or not. Section 5 deals with the quadratic guaranteed cost control problem, while section 6 presents a numerical example. The paper is concluded with some final remarks in section 7.

2 Problem statement and preliminary results

Let us consider the class of uncertain continuous-time linear systems with Markovian jumps with dynamics described by the following differential equation:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \left[\ \mathbf{A}(\mathbf{r}(t)) + \Delta\mathbf{A}(\mathbf{r}(t),t)\ \right]\mathbf{x}(t) + \left[\ \mathbf{B}(\mathbf{r}(t)) + \Delta\mathbf{B}(\mathbf{r}(t),t)\ \right]\mathbf{u}(t) && (1) \\ \mathbf{x}(0) &= \mathbf{x}_0,\ \mathbf{r}(0) = \mathbf{r}_0 && (2) \end{aligned}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\Delta\mathbf{A}(.)$ and $\Delta\mathbf{B}(.)$ are the system's uncertainties.

In Eq. (1) the term $\mathbf{r}(t)$ is a continuous-time stochastic Markovian process with finite discrete state space $\mathcal{B} = \{1, \ldots, N\}$ and transition rate matrix $\mathbf{Q} = \left[\, q_{\alpha\beta} \, \right]$ with $\sum_{\beta=1}^{N} q_{\alpha\beta} = 0$ and $q_{\alpha\beta} \geq 0$ for all $\alpha \neq \beta$.

The system uncertainties, $\Delta\mathbf{A}(\mathbf{r}(t), t)$ and $\Delta\mathbf{B}(\mathbf{r}(t), t)$, will be supposed in the rest of the paper to be described by the following expressions:

$$
\begin{aligned}
\Delta\mathbf{A}(\mathbf{r}(t), t) &= \mathbf{H}(\mathbf{r}(t))\Delta(\mathbf{r}(t), t)\mathbf{E}(\mathbf{r}(t)) \\
&= \sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{H}_k(\mathbf{r}(t))\Delta_{kl}(\mathbf{r}(t), t)\mathbf{E}_l(\mathbf{r}(t)) \\
\Delta\mathbf{B}(\mathbf{r}(t), t) &= \mathbf{F}(\mathbf{r}(t))\Delta(\mathbf{r}(t), t)\mathbf{G}(\mathbf{r}(t)) \\
&= \sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{F}_k(\mathbf{r}(t))\Delta_{kl}(\mathbf{r}(t), t)\mathbf{G}_l(\mathbf{r}(t))
\end{aligned}
$$

where $\mathbf{H}(\mathbf{r}(t))$, $\mathbf{F}(\mathbf{r}(t))$, $\mathbf{E}(\mathbf{r}(t))$ and $\mathbf{G}(\mathbf{r}(t))$ are known real matrices with appropriate dimensions defined as :

$$
\begin{aligned}
\mathbf{H}(\alpha) &= \begin{bmatrix} \mathbf{H}_1(\alpha) & \mathbf{H}_2(\alpha) & \ldots & \mathbf{H}_r(\alpha) \end{bmatrix} \\
\mathbf{F}(\alpha) &= \begin{bmatrix} \mathbf{F}_1(\alpha) & \mathbf{F}_2(\alpha) & \ldots & \mathbf{F}_r(\alpha) \end{bmatrix} \\
\mathbf{E}(\alpha) &= \begin{bmatrix} \mathbf{E}_1(\alpha) \\ \mathbf{E}_2(\alpha) \\ \vdots \\ \mathbf{E}_s(\alpha) \end{bmatrix} \\
\mathbf{G}(\alpha) &= \begin{bmatrix} \mathbf{G}_1(\alpha) \\ \mathbf{G}_2(\alpha) \\ \vdots \\ \mathbf{G}_s(\alpha) \end{bmatrix}
\end{aligned}
$$

$\Delta(\mathbf{r}(t), t)$ is an $(r \times s)$ real unknown matrix respresenting the system's uncertainties. The uncertainty will have the following form:

$$
\Delta(\mathbf{r}(t), t) = \begin{bmatrix} \Delta_{11}(\mathbf{r}(t), t) & \Delta_{12}(\mathbf{r}(t), t) & \ldots & \Delta_{1s}(\mathbf{r}(t), t) \\ \Delta_{21}(\mathbf{r}(t), t) & \Delta_{22}(\mathbf{r}(t), t) & \ldots & \Delta_{2s}(\mathbf{r}(t), t) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{r1}(\mathbf{r}(t), t) & \Delta_{r2}(\mathbf{r}(t), t) & \ldots & \Delta_{rs}(\mathbf{r}(t), t) \end{bmatrix}
$$

For the rest of the paper, we will use the following definitions for the matrix $\tilde{\Delta}_{kl}(t)$:

$$\tilde{\Delta}_{kl}(t) = \begin{bmatrix} \Delta_{kl}(1,t) & & 0 \\ & \ddots & \\ 0 & & \Delta_{kl}(N,t) \end{bmatrix}$$

with the following restriction;

$$\sum_{k=1}^{r}\sum_{l=1}^{s}\|\tilde{\Delta}_{kl}(t)\|^2 \leq 1 \tag{3}$$

or in matrix form:

$$\tilde{\Delta}(t) = \begin{bmatrix} \|\tilde{\Delta}_{11}(t)\| & \|\tilde{\Delta}_{12}(t)\| & \dots & \|\tilde{\Delta}_{1s}(t)\| \\ \|\tilde{\Delta}_{21}(t)\| & \|\tilde{\Delta}_{22}(t)\| & \dots & \|\tilde{\Delta}_{2s}(t)\| \\ \vdots & \vdots & \ddots & \vdots \\ \|\tilde{\Delta}_{r1}(t)\| & \|\tilde{\Delta}_{r2}(t)\| & \dots & \|\tilde{\Delta}_{rs}(t)\| \end{bmatrix}$$

with $\|\tilde{\Delta}(t)\|_F \leq 1$ (the subscript is used for denoting the Frobenius norm). We shall omit sometimes the argument t for simplicity of notation.

Let us define the sets $\mathcal{S}$ and $\mathcal{V}$ as follows:

$$\begin{aligned} \mathcal{S} &= \left\{ \Delta_{kl}(\alpha) : \|\tilde{\Delta}\|_F \leq 1 \right\} \\ \mathcal{V} &= \left\{ \Delta_{kl}(\alpha) : \|\Delta(\alpha)\|_F \leq 1, \alpha = 1, \dots, N \right\} \end{aligned}$$

where $\tilde{\Delta}_{kl} = \begin{bmatrix} \Delta_{kl}(1) & & 0 \\ & \ddots & \\ 0 & & \Delta_{kl}(N) \end{bmatrix}$, $\tilde{\Delta} = \begin{bmatrix} \|\tilde{\Delta}_{11}\| & \dots & \|\tilde{\Delta}_{1s}\| \\ \vdots & \ddots & \vdots \\ \|\tilde{\Delta}_{r1}\| & \dots & \|\tilde{\Delta}_{rN}\| \end{bmatrix}$ and
$\Delta(\alpha) = \begin{bmatrix} \|\Delta_{11}(\alpha)\| & \dots & \|\Delta_{1s}(\alpha)\| \\ \vdots & \ddots & \vdots \\ \|\Delta_{r1}(\alpha)\| & \dots & \|\Delta_{rN}(\alpha)\| \end{bmatrix}$ for $\alpha = 1, \dots, N$.
The following lemma states that $\mathcal{S}$ is included in $\mathcal{V}$.

Lemma 2.1: The set $\mathcal{S}$ is included in the set $\mathcal{V}$, i.e. $\mathcal{S} \subset \mathcal{V}$.

Proof: For any $\{\Delta_{kl}(\alpha), \alpha = 1, \dots, N, l = 1, \dots, s, k = 1, \dots, r\} \in \mathcal{S}$, we have

$$\|\Delta_{kl}(\alpha)\| \leq \|\tilde{\Delta}_{kl}\| = \max_{\alpha=1,\dots,N}\{\|\Delta_{kl}(\alpha)\|\}$$

and thus

$$\|\Delta(\alpha)\|_F^2 = \sum_{k=1}^{r}\sum_{l=1}^{s}\|\Delta_{kl}(\alpha)\|^2 \le \sum_{k=1}^{r}\sum_{l=1}^{s}\|\tilde{\Delta}_{kl}\|^2 = \|\tilde{\Delta}\|_F \le 1$$

which shows that $\{\Delta_{kl}(\alpha)\} \in \mathcal{V}$. This ends the proof of the lemma 2.1. □□

The following lemmas will be useful in the sequel.

Lemma 2.2: (Schur complement) Consider an arbitrary matrix $\mathbf{Q}$ such that:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}' & \mathbf{Q}_{22} \end{bmatrix}$$

where $\mathbf{Q}_{ij}$, $i,j = 1,2$ are square matrices. Then the matrix $\mathbf{Q}$ is positive definite if and only if either

$$\begin{cases} \mathbf{Q}_{22} > 0 \\ \mathbf{Q}_{11} - \mathbf{Q}_{12}\mathbf{Q}_{22}^{-1}\mathbf{Q}_{12}' > 0 \end{cases}$$

or

$$\begin{cases} \mathbf{Q}_{11} > 0 \\ \mathbf{Q}_{22} - \mathbf{Q}_{12}'\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} > 0. \end{cases}$$

For the proof of the next two lemmas, see Ref. 23 and Ref. 2 respectively.

Lemma 2.3: For any $\mathbf{x} \neq 0$

$$\max_{\|\Delta_N\|_F \le 1}\left\{\sum_{k=1}^{r}\sum_{l=1}^{s}\mathbf{x}'\mathbf{P}\mathbf{H}_k\Delta_{kl}\mathbf{E}_l\mathbf{x}\right\} = \sqrt{\mathbf{x}'\left(\sum_{k=1}^{r}\mathbf{P}\mathbf{H}_k\mathbf{H}_k'\mathbf{P}\right)\mathbf{x}\mathbf{x}'\left(\sum_{l=1}^{s}\mathbf{E}_l'\mathbf{E}_l\right)\mathbf{x}}$$

where $\Delta_N = \begin{bmatrix} \|\Delta_{11}\| & \dots & \|\Delta_{1s}\| \\ \vdots & \ddots & \vdots \\ \|\Delta_{r1}\| & \dots & \|\Delta_{rs}\| \end{bmatrix}$.

Lemma 2.4: Let us assume that $\mathbf{A} \ge 0$, $\mathbf{C} \ge 0$ and $\mathbf{B} < 0$. If

$$(\mathbf{w}'\mathbf{B}\mathbf{w})^2 - 4\mathbf{w}'\mathbf{A}\mathbf{w}\mathbf{w}'\mathbf{C}\mathbf{w} > 0$$

for all $\mathbf{w} \neq 0$, then for some $\lambda > 0$, we have

$$\lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C} < 0.$$

We need the following definitions.

Definition 2.1: The system (1)-(2) (with $\mathbf{u}(t) = 0$, for all $t \geq 0$) is mean square stable (MSS) if

$$\lim_{t\to\infty} E\left(\|x(t)\|^2\right) = 0$$

for any initial condition $(\mathbf{x}_0, \mathbf{r}_0)$ and for all admissible uncertainties $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$.

For notation simplicity, we define

$$\mathcal{L}(\mathbf{A},\mathbf{P},\mathbf{H},\mathbf{E},\Delta) = (\mathcal{L}(\mathbf{A},\mathbf{P},\mathbf{H},\mathbf{E},\Delta)(1), \ldots (\mathcal{L}(\mathbf{A},\mathbf{P},\mathbf{H},\mathbf{E},\Delta)(N))$$

as follows:

$$\begin{aligned}\mathcal{L}(\mathbf{A},\mathbf{P},\mathbf{H},\mathbf{E},\Delta)(\alpha) := \Big[\ \mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\ \Big]' \mathbf{P}(\alpha) + \\ \mathbf{P}(\alpha)\Big[\ \mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\ \Big] + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta).\end{aligned}$$

Definition 2.2: The system (1)-(2) is mean square quadratic stable (MSQS) if there exists $(\mathbf{P}(1), \ldots, \mathbf{P}(N)) > 0$ such that:

$$\mathcal{L}(\mathbf{A},\mathbf{P},\mathbf{H},\mathbf{E},\Delta)(\alpha) < 0, \qquad \text{for all } \alpha \in \mathcal{B}$$

for all $\{\Delta_{lk}(\alpha)\} \in \mathcal{S}$.

Proposition 2.1: If the system (1)-(2) (with $\mathbf{u}(t) = 0$ for all $t \geq 0$) is mean square quadratic stable then it is MSS.

Proof: Since the system is mean square quadratic stable, there exists $(\mathbf{P}(1), \ldots, \mathbf{P}(N)) > 0$ such that

$$\mathcal{L}(\mathbf{A},\mathbf{P},\mathbf{H},\mathbf{E},\Delta)(\alpha) < 0, \qquad \text{for all } \alpha \in \mathcal{B}.$$

Define a Lyapunov function as:

$$v(\mathbf{x},\alpha) = \mathbf{x}'\mathbf{P}(\alpha)\mathbf{x}, \qquad \text{for all } \alpha \in \mathcal{B}.$$

It can be shown that the weak infinitesimal operator $\mathcal{A}$ of the Markov process $\{(\mathbf{r}(t), \mathbf{x}(t)), t \geq 0\}$ applied to $v(\mathbf{x}, \alpha)$ is given by:

$$\begin{aligned}
\mathcal{A}v(\mathbf{x}(t), \mathbf{r}(t)) &= \mathbf{x}'(t)\left[\ \mathbf{A}(\mathbf{r}(t)) + \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{H}_k(\mathbf{r}(t))\Delta_{kl}(\mathbf{r}(t), t)\mathbf{E}_l(\mathbf{r}(t))\ \right]' \mathbf{P}(\mathbf{r}(t))\mathbf{x}(t) \\
&\quad + \mathbf{x}'(t)\mathbf{P}(\mathbf{r}(t))\left[\ \mathbf{A}(\mathbf{r}(t)) + \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{H}_k(\mathbf{r}(t))\Delta_{kl}(\mathbf{r}(t), t)\mathbf{E}_l(\mathbf{r}(t))\ \right]\mathbf{x}(t) \\
&\quad + \mathbf{x}'(t)\sum_{\beta=1}^{N} q_{\mathbf{r}(t)\beta}\mathbf{P}(\beta)\mathbf{x}(t)
\end{aligned}$$

which can be rewritten as:

$$\mathcal{A}v(\mathbf{x}, \alpha) = \mathbf{x}'\bar{\mathbf{A}}(\alpha)\mathbf{x} + 2\mathbf{x}'\sum_{k=1}^{r}\sum_{l=1}^{s}\mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha, t)\mathbf{E}_l(\alpha)\mathbf{x} \tag{4}$$

where $\bar{\mathbf{A}}(\alpha) = \mathbf{A}'(\alpha)\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathbf{A}(\alpha) + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta)$, $\mathbf{x}(t) = \mathbf{x}, \mathbf{r}(t) = \alpha$.

The Eq. (4) gives:

$$\mathcal{A}v(\mathbf{x}, \alpha) < \mathbf{x}'\mathcal{L}(\mathbf{A}, \mathbf{P}, \mathbf{H}, \mathbf{E}, \Delta)(\alpha)\mathbf{x}$$

where $\mathcal{L}(\mathbf{A}, \mathbf{P}, \mathbf{H}, \mathbf{E}, \Delta)(\alpha) < 0$ for all α in $\mathcal{B}$.

Therefore for any $\mathbf{x} \neq 0$, we have:

$$\begin{aligned}
\frac{\mathcal{A}v(\mathbf{x}, \alpha)}{v(\mathbf{x}, \alpha)} &< \frac{\mathbf{x}'\mathcal{L}(\mathbf{A}, \mathbf{P}, \mathbf{H}, \mathbf{E}, \Delta)(\alpha)\mathbf{x}}{\mathbf{x}'\mathbf{P}(\alpha)\mathbf{x}} \\
&< -\mu
\end{aligned}$$

where $\mu = \min_{\Delta\in\mathcal{S}, \alpha\in\mathcal{B}}\left[\frac{\lambda_{min}[-\mathcal{L}(\mathbf{A}, \mathbf{P}, \mathbf{H}, \mathbf{E}, \Delta)(\alpha)]}{\lambda_{max}[\mathbf{P}(\alpha)]}\right] > 0$.

Using now the Dynkin formula and the Gronwall-Bellmann lemma and the proof of Theorem 1 in Ji and Chizeck Ref. 6, we obtain the desired result. This ends the proof of the proposition 2.1. □□

3 Mean square quadratic robust stability

Let us restrict our study to the autonomous system given by Eqs. (1)-(2) with:

$$\dot{\mathbf{x}}(t) = \left[\ \mathbf{A}(\mathbf{r}(t)) + \Delta\mathbf{A}(\mathbf{r}(t), t)\ \right]\mathbf{x}(t) \tag{5}$$

$$\mathbf{x}(0) = \mathbf{x}_0, \mathbf{r}(0) = \mathbf{r}_0 \tag{6}$$

where the variables and the matrices have the same meaning and same dimensions as previously.

Theorem 3.1 System (5)-(6) with uncertainties of the type (3) is mean square quadratic stable if and only if there exists $(\mathbf{P}(1), \mathbf{P}(2), \ldots, \mathbf{P}(N)) > 0$ such that the inequality

$$\mathbf{A}'(\alpha)\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathbf{A}(\alpha) + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) + \sum_{k=1}^{r} \mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\mathbf{H}'_k(\alpha)\mathbf{P}(\alpha) + \sum_{l=1}^{s} \mathbf{E}'_l(\alpha)\mathbf{E}_l(\alpha) < 0$$

holds for each α in $\mathcal{B}$.

Proof: The proof of this theorem is based on the results in Ref. 23, and will be given in two parts. In part one, we will show the sufficient condition and in part two, we prove the necessary condition.

- sufficient condition: Let us show that for each $\alpha = 1, \ldots, N$, we have:

  $$\left[\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\right]' \mathbf{P}(\alpha) + \mathbf{P}(\alpha)\left[\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\right] + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) < 0$$

  whatever $\{\Delta_{kl}(\alpha,t)\} \in \mathcal{S}$ is. Indeed, we have that:

  $$\begin{aligned}
  &\left[\ \mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\ \right]' \mathbf{P}(\alpha) + \mathbf{P}(\alpha)\left[\ \mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\ \right] + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) \\
  &= \left[\ \mathbf{A}(\alpha) + \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\ \right]' \mathbf{P}(\alpha) \\
  &+\mathbf{P}(\alpha)\left[\ \mathbf{A}(\alpha) + \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\ \right] + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) \\
  &= \mathbf{A}'(\alpha)\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathbf{A}(\alpha) + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) + 2\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha). \qquad (7)
  \end{aligned}$$

  For any $\mathbf{x}_\alpha \neq 0$, using Lemma 2.1 and Lemma 2.3, we get:

  $$\begin{aligned}
  &\max_{\|\tilde{\Delta}(t)\|_F \leq 1} \left\{\ \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{x}_\alpha\mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha\ \right\} = \\
  &\max_{\{\Delta_{kl}(\alpha,t)\in\mathcal{S}\}} \left\{\ \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{x}_\alpha\mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha\ \right\} \leq \\
  &\max_{\{\Delta_{kl}(\alpha,t)\in\mathcal{V}\}} \left\{\ \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{x}_\alpha\mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha\ \right\} = \\
  &\max_{\|\Delta(\alpha,t)\|_F \leq 1} \left\{\ \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{x}_\alpha\mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha\ \right\} = \\
  &\sqrt{\mathbf{x}'_\alpha\mathbf{P}(\alpha)\left(\ \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{H}_k(\alpha)\mathbf{H}'_k(\alpha)\mathbf{P}(\alpha)\mathbf{x}_\alpha\ \right)\mathbf{x}'_\alpha\left(\ \mathbf{E}'_l(\alpha)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha\ \right)}. \qquad (8)
  \end{aligned}$$

But since

$$2\sqrt{\mathbf{x}'_\alpha \mathbf{P}(\alpha)\left(\textstyle\sum_{k=1}^r \mathbf{H}_k(\alpha)\mathbf{H}'_k(\alpha)\mathbf{P}(\alpha) \right)\mathbf{x}_\alpha \mathbf{x}'_\alpha \left(\textstyle\sum_{l=1}^s \mathbf{E}'_l(\alpha)\mathbf{E}_l(\alpha) \right)\mathbf{x}_\alpha} \leq$$

$$\mathbf{x}'_\alpha \mathbf{P}(\alpha)\left(\textstyle\sum_{k=1}^r \mathbf{H}_k(\alpha)\mathbf{H}'_k(\alpha)\mathbf{P}(\alpha) \right)\mathbf{x}_\alpha + \mathbf{x}'_\alpha \left(\textstyle\sum_{l=1}^s \mathbf{E}'_l(\alpha)\mathbf{E}_l(\alpha) \right)\mathbf{x}_\alpha \quad (9)$$

we conclude, from (7)-(9), that for any $\{\Delta_{kl}(\alpha,t)\} \in \mathcal{S}$, and $\mathbf{x}_\alpha \neq 0$,

$$\begin{aligned}
&\mathbf{x}'_\alpha \left[\left(\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha) \right)' \mathbf{P}(\alpha) + \mathbf{P}(\alpha)\left(\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha) \right) \right]\mathbf{x}_\alpha \\
&\quad + \mathbf{x}'_\alpha \sum_{\beta=1}^N q_{\alpha\beta}\mathbf{P}(\beta)\mathbf{x}_\alpha \leq \mathbf{x}_\alpha \left[\mathbf{A}'(\alpha)\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathbf{A}(\alpha) + \textstyle\sum_{\beta=1}^N q_{\alpha\beta}\mathbf{P}(\beta) \right]\mathbf{x}_\alpha \\
&\quad + \mathbf{x}'_\alpha \mathbf{P}(\alpha)\left(\textstyle\sum_{k=1}^r \mathbf{H}_k(\alpha)\mathbf{H}'_k(\alpha) \right)\mathbf{P}(\alpha)\mathbf{x}_\alpha + \mathbf{x}'_\alpha \sum_{l=1}^s \mathbf{E}'_l(\alpha)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha < 0
\end{aligned}$$

which proves the sufficient condition of the theorem.

- necessary condition:

If the autonomous uncertain system is mean square quadratic stable, then there exists $(\mathbf{Q}(1), \ldots, \mathbf{Q}(N)) > 0$ such that:

$$\begin{aligned}
&\mathbf{x}'_\alpha \left[\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha) \right]' \mathbf{Q}(\alpha)\mathbf{x}_\alpha + \\
&\quad \mathbf{x}'_\alpha \mathbf{Q}(\alpha)\left[\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha) \right]\mathbf{x}_\alpha + \mathbf{x}'_\alpha \sum_{\beta=1}^N q_{\alpha\beta}\mathbf{Q}(\beta)\mathbf{x}_\alpha < 0
\end{aligned}$$

for all $\mathbf{x}_\alpha \neq 0 \in \mathbb{R}^n$ and $\{\Delta_{kl}(\alpha,t)\} \in \mathcal{S}$. Therefore, for any $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} \neq 0$, $\mathbf{x} \in \mathbb{R}^{Nn}$, we have

$$\begin{aligned}
&\max_{\|\tilde{\Delta}(t)\|_F \leq 1} \left\{ \sum_{\alpha=1}^N \mathbf{x}'_\alpha \left[\left[\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha) \right]' \mathbf{Q}(\alpha) + \mathbf{Q}(\alpha)\left[\mathbf{A}(\alpha) + \mathbf{H}(\alpha)\Delta(\alpha)\mathbf{E}(\alpha) \right] \right]\mathbf{x}_\alpha \right. \\
&\quad \left. + \mathbf{x}'_\alpha \sum_{\beta=1}^N q_{\alpha\beta}\mathbf{Q}(\beta)\mathbf{x}_\alpha \right\} = \sum_{\alpha=1}^N \mathbf{x}'_\alpha \left[\mathbf{A}'(\alpha)\mathbf{Q}(\alpha) + \mathbf{Q}(\alpha)\mathbf{A}(\alpha) + \textstyle\sum_{\beta=1}^N q_{\alpha\beta}\mathbf{Q}(\beta) \right]\mathbf{x}_\alpha + \\
&\quad 2 \max_{\|\tilde{\Delta}(t)\|_F \leq 1} \left\{ \textstyle\sum_{\alpha=1}^N \mathbf{x}'_\alpha \mathbf{Q}(\alpha)\mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\mathbf{x}'_\alpha \right\} < 0. \qquad (10)
\end{aligned}$$

Note that

$$\sum_{\alpha=1}^{N} \mathbf{x}_\alpha' \mathbf{Q}(\alpha)\mathbf{H}(\alpha)\Delta(\alpha,t)\mathbf{E}(\alpha)\mathbf{x}_\alpha = \sum_{k=1}^{r}\sum_{l=1}^{s}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha' \mathbf{Q}(\alpha)\mathbf{H}_k(\alpha)\Delta_{kl}(\alpha,t)\mathbf{E}_l(\alpha)\mathbf{x}_\alpha = \sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{x}'\tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\Delta}_{kl}(t)\tilde{\mathbf{E}}_l\mathbf{x}$$

where $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{bmatrix}$, $\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}(1) & & 0 \\ & \ddots & \\ 0 & & \mathbf{Q}(N) \end{bmatrix}$, $\tilde{\mathbf{H}}_k = \begin{bmatrix} \mathbf{H}_k(1) & & 0 \\ & \ddots & \\ 0 & & \mathbf{H}_k(N) \end{bmatrix}$ and $\tilde{\mathbf{E}}_l = \begin{bmatrix} \mathbf{E}_l(1) & & 0 \\ & \ddots & \\ 0 & & \mathbf{E}_l(N) \end{bmatrix}$.

From Lemma 2.3, we get

$$\max_{\|\tilde{\Delta}\|_F \le 1} \left\{ \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{x}'\tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\Delta}_{kl}\tilde{\mathbf{E}}_l\mathbf{x} \right\} = \sqrt{\mathbf{x}'\sum_{k=1}^{r}\tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k'\tilde{\mathbf{Q}}\mathbf{x}\mathbf{x}'\sum_{l=1}^{s}\tilde{\mathbf{E}}_l'\tilde{\mathbf{E}}_l\mathbf{x}}.$$

By defining the $(nN \times nN)$ matrices $\tilde{\mathbf{A}}$ and $\Gamma(\mathbf{Q})$ as follows:

$$\begin{aligned} \tilde{\mathbf{A}} &= \begin{bmatrix} \mathbf{A}(1) & & 0 \\ & \ddots & \\ 0 & & \mathbf{A}(N) \end{bmatrix} \\ \Gamma(\mathbf{Q}) &= \begin{bmatrix} \sum_{\beta=1}^{N} q_{1\beta}\mathbf{Q}(\beta) & & 0 \\ & \ddots & \\ 0 & & \sum_{\beta=1}^{N} q_{N\beta}\mathbf{Q}(\beta) \end{bmatrix} \end{aligned}$$

we have that:

$$\sum_{\alpha=1}^{N} \mathbf{x}_\alpha' \left(\tilde{\mathbf{A}}'(\alpha)\tilde{\mathbf{Q}}(\alpha) + \tilde{\mathbf{Q}}(\alpha)\tilde{\mathbf{A}}(\alpha) + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{Q}(\alpha) \right) \mathbf{x}_\alpha = \mathbf{x}' \left(\tilde{\mathbf{A}}'\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}\tilde{\mathbf{A}} + \Gamma(\mathbf{Q}) \right) \mathbf{x}.$$

Thus for any $\mathbf{x} \neq 0$, we have:

$$\mathbf{x}' \left(\tilde{\mathbf{A}}'\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}\tilde{\mathbf{A}} + \Gamma(\mathbf{Q}) \right) \mathbf{x} + 2\sqrt{\mathbf{x}'\sum_{k=1}^{r}\tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k'\tilde{\mathbf{Q}}\mathbf{x}\mathbf{x}'\sum_{l=1}^{s}\tilde{\mathbf{E}}_l'\tilde{\mathbf{E}}_l\mathbf{x}} < 0.$$

Or, in other words, for any $\mathbf{x} \neq 0$ we get:

$$\left[\mathbf{x}' \left(\tilde{\mathbf{A}}'\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}\tilde{\mathbf{A}} + \Gamma(\mathbf{Q}) \right) \mathbf{x} \right]^2 - 4\mathbf{x}' \left(\sum_{k=1}^{r} \tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k'\tilde{\mathbf{Q}} \right) \mathbf{x}\mathbf{x}' \left(\sum_{l=1}^{s} \tilde{\mathbf{E}}_l'\tilde{\mathbf{E}}_l \right) \mathbf{x} > 0.$$

Since $\tilde{\mathbf{A}}'\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}\tilde{\mathbf{A}} + \Gamma(\mathbf{Q}) < 0$, $\sum_{k=1}^{r} \tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k'\tilde{\mathbf{Q}} \geq 0$ and $\sum_{l=1}^{s} \tilde{\mathbf{E}}_l'\tilde{\mathbf{E}}_l \geq 0$, we have from Lemma 2.4 that for some $\lambda > 0$,

$$\lambda^2 \left[\sum_{k=1}^{r} \tilde{\mathbf{Q}}\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k'\tilde{\mathbf{Q}}\right] + \lambda \left[\tilde{\mathbf{A}}'\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}\tilde{\mathbf{A}} + \Gamma(\mathbf{Q})\right] + \sum_{l=1}^{s} \tilde{\mathbf{E}}_l'\tilde{\mathbf{E}}_l(\alpha) < 0.$$

Setting $\mathbf{P}(\alpha) = \lambda\mathbf{Q}(\alpha)$, for $\alpha = 1, \ldots, N$, we obtain:

$$\mathbf{A}'(\alpha)\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathbf{A}(\alpha) + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) + \sum_{k=1}^{r} \mathbf{P}(\alpha)\mathbf{H}_k(\alpha)\mathbf{H}_k'(\alpha)\mathbf{P}(\alpha)$$
$$+ \sum_{l=1}^{s} \mathbf{E}_l'(\alpha)\mathbf{E}_l(\alpha) < 0.$$

This ends the proof of the necessary condition and completes the proof of Theorem 3.1. □□

4 Mean square quadratic robust stabilizability

Let us get back to the system (1)-(2) and consider a feedback control, $\mathbf{u}(t) = \mathbf{K}(\mathbf{r}(t))\mathbf{x}(t)$, so that system (1)-(2) can be rewritten as:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \left[\ \mathbf{A}(\mathbf{r}(t)) + \mathbf{B}(\mathbf{r}(t))\mathbf{K}(\mathbf{r}(t)) + \mathbf{L}(\mathbf{r}(t))\Delta(\mathbf{r}(t), t)\mathbf{M}(\mathbf{r}(t)) \ \right]\mathbf{x}(t) \\
&= \left[\ \mathbf{A}(\mathbf{r}(t)) + \mathbf{B}(\mathbf{r}(t))\mathbf{K}(\mathbf{r}(t)) + \textstyle\sum_{k=1}^{r}\sum_{l=1}^{s} \mathbf{L}_k(\mathbf{r}(t))\Delta_{kl}(\mathbf{r}(t), t)\mathbf{M}_l(\mathbf{r}(t)) \ \right]\mathbf{x}(t) \\
\mathbf{x}(0) &= \mathbf{x}_0, \mathbf{r}(0) = \mathbf{r}_0
\end{aligned}$$

where $\mathbf{L}(\alpha) = \begin{bmatrix} \mathbf{L}_1(\alpha) & \ldots & \mathbf{L}_r(\alpha) \end{bmatrix}$, $\mathbf{L}_k(\alpha) = \begin{bmatrix} \mathbf{H}_k(\alpha) & \mathbf{F}_k(\alpha) \end{bmatrix}$, $\mathbf{M}(\alpha) = \begin{bmatrix} \mathbf{M}_1(\alpha) \\ \vdots \\ \mathbf{M}_s(\alpha) \end{bmatrix}$, $\mathbf{M}_l(\alpha) = \begin{bmatrix} \mathbf{E}_l(\alpha) \\ \mathbf{G}_l(\alpha)\mathbf{K}_l(\alpha) \end{bmatrix}$, $\mathbf{E}(\alpha) = \begin{bmatrix} \mathbf{E}_1(\alpha) \\ \vdots \\ \mathbf{E}_s(\alpha) \end{bmatrix}$, and $\mathbf{G}(\alpha) = \begin{bmatrix} \mathbf{G}_1(\alpha) \\ \vdots \\ \mathbf{G}_s(\alpha) \end{bmatrix}$.

According to Theorem 3.1, system (1)-(2) is mean square quadratic stablizable if and only if for some $\mathbf{K} = (\mathbf{K}(1), \ldots, \mathbf{K}(N))$ and $\mathbf{P} = (\mathbf{P}(1), \ldots, \mathbf{P}(N)) > 0$, the following inequality

$$\left[\ \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\mathbf{K}(\alpha) \ \right]'\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\left[\ \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\mathbf{K}(\alpha) \ \right] +$$

$$\sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) + \sum_{k=1}^{r} \mathbf{P}(\alpha)\mathbf{L}_k(\alpha)\mathbf{L}_k'(\alpha)\mathbf{P}(\alpha) +$$
$$\sum_{l=1}^{s} \mathbf{E}_l'(\alpha)\mathbf{E}_l(\alpha) + \sum_{l=1}^{s} \mathbf{K}'(\alpha)\mathbf{G}_l'(\alpha)\mathbf{G}_l(\alpha)\mathbf{K}(\alpha) < 0 \tag{11}$$

holds for every $\alpha \in \mathcal{B}$.

Let the matrix $\mathbf{C}(\alpha)$ for each $\alpha \in \mathcal{B}$ be defined as follows:

$$\mathbf{C}(\alpha) := \ \mathbf{A}(\alpha)\mathbf{W}(\alpha) + \mathbf{W}(\alpha)\mathbf{A}'(\alpha) + \mathbf{B}(\alpha)\mathbf{R}(\alpha) + \mathbf{R}'(\alpha)\mathbf{B}(\alpha) + \mathbf{L}(\alpha)\mathbf{L}'(\alpha) + q_{\alpha\alpha}\mathbf{W}(\alpha).$$

Let the set $\mathcal{C}$ be defined as follows:

$$\begin{aligned} \mathcal{C} \ &= \ \Big\{\mathbf{W} = (\mathbf{W}(1), \ldots, \mathbf{W}(N)) > 0, \mathbf{R} = (\mathbf{R}(1), \ldots, \mathbf{R}(N)); \\ &\qquad \begin{bmatrix} \mathbf{X}(\alpha) & \mathbf{Y}(\alpha) \\ \mathbf{Y}'(\alpha) & \mathbf{Z}(\alpha) \end{bmatrix} > 0 \quad \text{for } \alpha = 1, \ldots, N\Big\} \end{aligned}$$

where

$$\mathbf{X}(\alpha) = \begin{bmatrix} -\mathbf{C}(\alpha) & \begin{bmatrix} \mathbf{W}(\alpha)\mathbf{E}'(\alpha) & \mathbf{R}'(\alpha)\mathbf{G}'(\alpha) \end{bmatrix} \\ \begin{bmatrix} \mathbf{E}(\alpha)\mathbf{W}(\alpha) \\ \mathbf{G}(\alpha)\mathbf{R}(\alpha) \end{bmatrix} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{Y}(\alpha) = \begin{bmatrix} q_{\alpha 1}^{\frac{1}{2}}\mathbf{W}(\alpha) & \ldots & q_{\alpha\ \alpha-1}^{\frac{1}{2}}\mathbf{W}(\alpha) & q_{\alpha\ \alpha+1}^{\frac{1}{2}}\mathbf{W}(\alpha) & \ldots & q_{\alpha N}^{\frac{1}{2}}\mathbf{W}(\alpha) \\ & & & 0 & & \end{bmatrix} \text{ and}$$

$$\mathbf{Z}(\alpha) = \ \mathrm{diag}\Big\{ \ \mathbf{W}(1) \quad \ldots \quad \mathbf{W}(\alpha-1) \quad \mathbf{W}(\alpha+1) \quad \mathbf{W}(N) \ \Big\} \ .$$

From this, we have the following proposition.

Proposition 4.1: System (1)-(2) is mean square quadratic stabilizable if and only if the convex set $\mathcal{C}$ is not empty. Morever, if $\mathbf{W} = (\mathbf{W}(1), \ldots, \mathbf{W}(N))$, $\mathbf{R} = (\mathbf{R}(1), \ldots, \mathbf{R}(N))$ belong to the set $\mathcal{C}$, then $\mathbf{P}(\alpha) = \mathbf{W}^{-1}(\alpha)$, $\mathbf{K}(\alpha) = \mathbf{R}(\alpha)\mathbf{W}^{-1}(\alpha)$ will satisfy (11).

Proof: Suppose $\mathbf{W} = (\mathbf{W}(1), \ldots, \mathbf{W}(N))$, $\mathbf{R} = (\mathbf{R}(1), \ldots, \mathbf{R}(N))$ belong to $\mathcal{C}$. Then, from Schur complement, we get that

$$\begin{bmatrix} \mathbf{X}(\alpha) & \mathbf{Y}(\alpha) \\ \mathbf{Y}'(\alpha) & \mathbf{Z}(\alpha) \end{bmatrix} > 0$$

implies:

$$\mathbf{X}(\alpha) > \mathbf{Y}(\alpha)\mathbf{Z}^{-1}(\alpha)\mathbf{Y}'(\alpha)$$

which implies in turn that:

$$\begin{bmatrix} -\mathbf{N}(\alpha) & \begin{bmatrix} \mathbf{W}(\alpha)\mathbf{E}'(\alpha) & \mathbf{R}'(\alpha)\mathbf{G}'(\alpha) \end{bmatrix} \\ \begin{bmatrix} \mathbf{E}(\alpha)\mathbf{W}(\alpha) \\ \mathbf{G}(\alpha)\mathbf{R}(\alpha) \end{bmatrix} & \mathbf{I} \end{bmatrix} > 0$$

with

$$\begin{aligned} \mathbf{N}(\alpha) \;:=\; & \mathbf{A}(\alpha)\mathbf{W}(\alpha) + \mathbf{W}(\alpha)\mathbf{A}'(\alpha) + \mathbf{B}(\alpha)\mathbf{R}(\alpha) + \mathbf{R}'(\alpha)\mathbf{B}'(\alpha) + \mathbf{L}(\alpha)\mathbf{L}'(\alpha) + \\ & \mathbf{W}(\alpha)\left(\sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{W}^{-1}(\beta)\right)\mathbf{W}(\alpha). \end{aligned}$$

From the Schur complement again, we get that:

$$\begin{aligned} & \mathbf{A}(\alpha)\mathbf{W}(\alpha) + \mathbf{W}(\alpha)\mathbf{A}'(\alpha) + \mathbf{B}(\alpha)\mathbf{R}(\alpha) + \mathbf{R}'(\alpha)\mathbf{B}'(\alpha) + \mathbf{L}(\alpha)\mathbf{L}'(\alpha) \\ & +\mathbf{W}(\alpha)\big(\sum_{\beta=1}^{N} q_{\alpha\alpha}\mathbf{W}^{-1}(\beta)\big)\mathbf{W}(\alpha) + \mathbf{W}(\alpha)\mathbf{E}'(\alpha)\mathbf{E}(\alpha)\mathbf{W}(\alpha) \\ & +\mathbf{R}'(\alpha)\mathbf{G}'(\alpha)\mathbf{G}(\alpha)\mathbf{R}(\alpha) < 0 \end{aligned} \tag{12}$$

Defining $\mathbf{P}(\alpha) = \mathbf{W}^{-1}(\alpha)$ and $\mathbf{K}(\alpha) = \mathbf{R}(\alpha)\mathbf{W}^{-1}(\alpha)$, we get from (12) that (11) holds.

On the other hand, if (11) holds, then by defining $\mathbf{W}(\alpha) = \mathbf{P}^{-1}(\alpha)$ and $\mathbf{R}(\alpha) = \mathbf{K}(\alpha)\mathbf{P}^{-1}(\alpha)$ we get, from Schur complement, that $\mathbf{W} = (\mathbf{W}(1), \ldots, \mathbf{W}(N))$, $\mathbf{R} = (\mathbf{R}(1), \ldots, \mathbf{R}(N))$ belong to $\mathcal{C}$. □□

5 Quadratic guaranteed cost control problem

In this section, we consider system (1)-(2) with $\mathbf{u}(t) = \mathbf{K}(\mathbf{r}(t))\mathbf{x}(t)$ and $\mathbf{K} = (\mathbf{K}(1), \ldots, \mathbf{K}(N))$ such that it stabilizes system (1)-(2) in the mean square quadratic sense. Associated to $\mathbf{K}$, we define the cost $J_{\mathbf{K}}(\mathbf{x}_0, \mathbf{r}_0)$ as follows:

$$J_{\mathbf{K}}(\mathbf{x}_0, \mathbf{r}_0) = E\left\{\int_0^{\infty} (\mathbf{x}'(t)\mathbf{Q}(\mathbf{r}(t))\mathbf{x}(t) + \mathbf{u}'(t)\mathbf{U}(\mathbf{r}(t))\mathbf{x}(t))\, dt\right\}$$

where $\mathbf{Q} = (\mathbf{Q}(1), \ldots, \mathbf{Q}(N)) \geq 0$, $\mathbf{U} = (\mathbf{U}(1), \ldots, \mathbf{U}(N)) > 0$.

It is desired to find a $\mathbf{K} = (\mathbf{K}(1), \ldots, \mathbf{K}(N))$ such that stabilizes system (1)-(2) in the mean square quadratic sense and guarantees that the cost $J_{\mathbf{K}}(\mathbf{x}_0, \mathbf{r}_0)$ satisfies the following constraint for some $\delta > 0$:

$$J_{\mathbf{K}}(\mathbf{x}_0, \mathbf{r}_0) \leq \delta$$

for all $(\mathbf{x}_0, \mathbf{r}_0)$, with $E\left(\|\mathbf{x}_0\|^2\right) \leq 1$ and $\{\Delta_{kl}(\alpha, t) \in \mathcal{S}; t \geq 0\}$.

This will be defined as our quadratic guaranteed cost control problem.

Proposition 5.1: Suppose there exist $\mathbf{K} = (\mathbf{K}(1), \ldots, \mathbf{K}(N))$ and $\mathbf{P} = (\mathbf{P}(1), \ldots, \mathbf{P}(N)) > 0$ such that

$$\left[\ \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\mathbf{K}(\alpha)\ \right]' \mathbf{P}(\alpha) + \mathbf{P}(\alpha)\left[\ \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\mathbf{K}(\alpha)\ \right] + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) +$$
$$\sum_{k=1}^{r} \mathbf{P}(\alpha)\mathbf{L}_k(\alpha)\mathbf{L}_k'(\alpha)\mathbf{P}(\alpha) + \sum_{l=1}^{s} \mathbf{M}_l'(\alpha)\mathbf{M}_l(\alpha) + \mathbf{Q}(\alpha) + \mathbf{K}'(\alpha)\mathbf{U}(\alpha)\mathbf{K}(\alpha) < 0 \quad (13)$$

Then $\mathbf{K} = (\mathbf{K}(1), \ldots, \mathbf{K}(N))$ stabilizes system (1)-(2) in the mean square quadratic sense and if $E\left(\|\mathbf{x}_0\|^2\right) \leq 1$, the cost $J_{\mathbf{K}}(\mathbf{x}_0, \mathbf{r}_0)$ satisfies the following inequality:

$$J_{\mathbf{K}}(\mathbf{x}_0, \mathbf{r}_0) \leq \delta, \quad \text{for } \delta = \textstyle\sum_{\alpha=1}^{N} \operatorname{tr}(\mathbf{P}(\alpha)).$$

Proof: Let us first prove that if (13) is satisfied, the system (1)-(2) is mean square quadratic stable. Since $\mathbf{Q}(\alpha) \geq 0$ and $\mathbf{U}(\alpha) > 0$, it results that

$$\mathbf{Q}(\alpha) + \mathbf{K}'(\alpha)\mathbf{U}(\alpha)\mathbf{K}(\alpha) \geq 0.$$

Therefore, we obtain from (13) that

$$\left[\ \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\mathbf{K}(\alpha)\ \right]' \mathbf{P}(\alpha) + \mathbf{P}(\alpha)\left[\ \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\mathbf{K}(\alpha)\ \right] + \sum_{\beta=1}^{N} q_{\alpha\beta}\mathbf{P}(\beta) +$$
$$\sum_{k=1}^{r} \mathbf{P}(\alpha)\mathbf{L}_k(\alpha)\mathbf{L}_k'(\alpha)\mathbf{P}(\alpha) + \sum_{l=1}^{s} \mathbf{M}_l'(\alpha)\mathbf{M}_l(\alpha) < 0$$

which proves, from Theorem 3.1, that the closed-loop system is mean square quadratic stable.

Let us define a Lyapunov function $v(\mathbf{x},\alpha)$ by

$$v(\mathbf{x},\alpha)=\mathbf{x}'\mathbf{P}(\alpha)\mathbf{x}, \quad \text{for all } \alpha\in\mathcal{B}$$

By Dynkin formula and the result in the proof of Proposition 2.1, we have:

$$\begin{aligned} &E\left(v(\mathbf{x}(t),\mathbf{r}(t))-v(\mathbf{x}_0,\mathbf{r}_0)\right)=E\left(\int_0^t \mathcal{A}v(\mathbf{x}(\tau),\mathbf{r}(\tau))d\tau\right)< \\ &-E\left(\int_0^t\left[\mathbf{x}'(\tau)\mathbf{Q}(\mathbf{r}(\tau))\mathbf{x}(\tau)+\mathbf{u}'(\tau)\mathbf{U}(\mathbf{r}(\tau))\mathbf{u}(\tau)\right]d\tau\right) \end{aligned} \tag{14}$$

whatever $\{\Delta_{kl}(\alpha,t)\in\mathcal{S}, t\geq 0\}$ is. Recalling that $\lim_{t\to\infty} E\left(\|\mathbf{x}(t)\|^2\right)=0$, we obtain that:

$$J_{\mathbf{K}}(\mathbf{x}_0,\mathbf{r}_0)<E\left(\mathbf{x}_0'\mathbf{P}\left(\mathbf{r}_0\right)\mathbf{x}_0\right)=\sum_{\alpha=1}^{N}\operatorname{tr}\left(\mathbf{P}(\alpha)\mathbf{X}_0(\alpha)\right) \tag{15}$$

where $\mathbf{X}_0(\alpha)=E\left(\mathbf{x}_0\mathbf{x}_0' 1_{\{\mathbf{r}_0=\alpha\}}\right)$.

For $\mathbf{x}_0$ such that $E\left(\|\mathbf{x}_0\|^2\right)=\sum_{\alpha=1}^{N}\operatorname{tr}\left(\mathbf{X}_0(\alpha)\right)\leq 1$ we get

$$\sum_{\alpha=1}^{N}\operatorname{tr}\left(\mathbf{P}(\alpha)\mathbf{X}_0(\alpha)\right)\leq\sum_{\alpha=1}^{N}\operatorname{tr}\left(\mathbf{P}(\alpha)\right)=\delta$$

and this completes the proof of the proposition. □□

We can now establish the following LMI optimization problem, $\mathcal{P}$, which provides a solution for the quadratic guaranteed cost control problem.

$$\mathcal{P}: \qquad \min\sum_{\alpha=1}^{N}\operatorname{tr}\left(\mathbf{V}(\alpha)\right)$$

subject to

$$\begin{bmatrix}\mathbf{X}(\alpha) & \mathbf{Y}(\alpha)\\ \mathbf{Y}'(\alpha) & \mathbf{Z}(\alpha)\end{bmatrix}>0, \qquad \begin{bmatrix}\mathbf{V}(\alpha) & \mathbf{I}\\ \mathbf{I} & \mathbf{W}(\alpha)\end{bmatrix}\geq 0,$$

$$\mathbf{W}(\alpha)>0, \mathbf{V}(\alpha)>0, \quad \text{for all } \alpha=1,\ldots,N$$

$$\mathbf{X}(\alpha) \;=\; \begin{bmatrix} -\mathbf{C}(\alpha) & \mathbf{W}(\alpha)\begin{bmatrix}\mathbf{E}'(\alpha) & \mathbf{Q}^{\frac{1}{2}}(\alpha)\end{bmatrix} & \mathbf{R}'(\alpha)\begin{bmatrix}\mathbf{G}'(\alpha) & \mathbf{U}^{\frac{1}{2}}(\alpha)\end{bmatrix} \\ \begin{bmatrix}\mathbf{E}(\alpha)\\ \mathbf{Q}^{\frac{1}{2}}(\alpha)\end{bmatrix}\mathbf{W}(\alpha) & \mathbf{I} & \\ \begin{bmatrix}\mathbf{G}(\alpha)\\ \mathbf{U}^{\frac{1}{2}}(\alpha)\end{bmatrix}\mathbf{R}(\alpha) & & \mathbf{I} \end{bmatrix}$$

$\mathbf{Y}(\alpha)$ and $\mathbf{Z}(\alpha)$ as in Proposition 4.1 (with appropriate dimension), with $\mathbf{C}(\alpha)$ defined as follows:

$$\begin{aligned}\mathbf{C}(\alpha) \;&:=\; \big(\mathbf{A}(\alpha)\mathbf{W}(\alpha) + \mathbf{W}(\alpha)\mathbf{A}'(\alpha) + \mathbf{B}(\alpha)\mathbf{R}(\alpha) + \mathbf{R}'(\alpha)\mathbf{B}'(\alpha) + q_{\alpha\alpha}\mathbf{W}(\alpha) + \\ &\qquad \mathbf{L}(\alpha)\mathbf{L}'(\alpha)\big).\end{aligned}$$

Proposition 5.2: Suppose that the LMI optimization problem, $\mathcal{P}$, has a solution $\mathbf{W} = (\mathbf{W}(1), \ldots, \mathbf{W}(N)) > 0$, $\mathbf{R} = (\mathbf{R}(1), \ldots, \mathbf{R}(N))$, $\mathbf{V} = (\mathbf{V}(1), \ldots, \mathbf{V}(N)) > 0$. Then by defining $\mathbf{K}(\alpha) = \mathbf{R}(\alpha)\mathbf{W}^{-1}(\alpha)$ and $\mathbf{P}(\alpha) = \mathbf{W}^{-1}(\alpha)$, we have that $\mathbf{V}(\alpha) = \mathbf{P}(\alpha)$ and $\mathbf{P} = (\mathbf{P}(1), \ldots, \mathbf{P}(N))$, $\mathbf{K} = (\mathbf{K}(1), \ldots, \mathbf{K}(N))$ satisfy Eq (13).

Proof: Similar to the proof of Proposition 4.1. Notice only that, from Schur complement, $\mathbf{V}(\alpha) \geq \mathbf{W}^{-1}(\alpha)$ and since we are minimizing $\sum_{\alpha=1}^{N} \operatorname{tr}(\mathbf{V}(\alpha))$, the optimal solution will be $\mathbf{V}(\alpha) = \mathbf{W}^{-1}(\alpha) = \mathbf{P}(\alpha)$. □□

6 Numerical Example

To illustrate the usefulness of the proposed theoretical results, let us consider an uncertain continuous-time linear system with Markovian jumps with three modes and the continuous vector state $\mathbf{x}$ is in $\mathbb{R}^3$. Let the dynamics of the nominal system in each mode be described as follows:

Mode 1:

$$\mathbf{A}(1) = \begin{bmatrix} -2.5 & 0.3 & 0.8 \\ 1 & -3 & 0.2 \\ 0 & 0.5 & -2 \end{bmatrix}, \qquad \mathbf{B}(1) = \begin{bmatrix} 0.707 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mode 2:

$$\mathbf{A}(2) = \begin{bmatrix} -2.5 & 1.2 & 0.3 \\ -0.5 & 5 & -1 \\ 0.25 & 1.2 & 5 \end{bmatrix}, \qquad \mathbf{B}(2) = \begin{bmatrix} 0.707 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.707 \end{bmatrix}$$

Mode 3:

$$\mathbf{A}(3) = \begin{bmatrix} 2 & 1.5 & -0.4 \\ 2.2 & 3 & 0.7 \\ 1.1 & 0.9 & -2 \end{bmatrix}, \qquad \mathbf{B}(3) = \begin{bmatrix} 0.707 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let also the transition rates matrix Π describing the evolution between the three modes be given by:

$$\Pi = \begin{bmatrix} -3 & 0.5 & 2.5 \\ 1 & -2 & 1 \\ 1.7 & 2.3 & -4 \end{bmatrix}$$

Let the initial state be given as: $\mathbf{x}_0 = [1\ 0\ 0]'$, $\mathbf{P}r\{\mathbf{r}_0 = \alpha\} = \frac{1}{3}$ for each mode $\alpha \in \{1, 2, 3\}$, and the uncertainty matrices be given as follows:$\Delta\mathbf{B}(\alpha) = 0$ and

$$\Delta\mathbf{A}(\alpha) \;=\; 2\sigma(t) \begin{bmatrix} 0.18 & 0.2 & 0.18 \\ 0.18 & 0.2 & 0.16 \\ 0.2 & 0.2 & 0.18 \end{bmatrix} \quad \text{for each } \alpha \in \{1, 2, 3\}.$$

Based on what we have presented in section 3, we get the following norm bounded uncertainty:

$$\begin{aligned} \Delta(\alpha, t) &= \sigma(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{H}(\alpha) &= 0.1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \\ \mathbf{E}(\alpha) &= 2 \begin{bmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 0.8 \\ 1 & 1 & 0.8 \end{bmatrix}, \quad \text{for each } \alpha \in \{1, 2, 3\} \end{aligned}$$

where $\sigma(t) \in \mathbb{R}$ is an uncertain parameter satisfying $|\sigma(t)| < 1$ and $\Delta(\alpha, t)$ verifies the condition given by Eq. (3). We considered the quadratic guaranteed cost control problem as defined in section 5, with matrix $\mathbf{Q}(\alpha) = \mathbf{I}$ and $\mathbf{U}(\alpha)$ given by

$$\mathbf{U}(\alpha) = \begin{bmatrix} 1.734 & 0.2484 & 0.7814 \\ 0.2484 & 1.1515 & 0.3059 \\ 0.7814 & 0.3059 & 1.9128 \end{bmatrix}$$

for $\alpha \in \{1, 2, 3\}$. We have obtained the following solution for this problem. $\delta = 90.56$ and the controllers are:

Mode 1:

$$\mathbf{K}(1) = \begin{bmatrix} -1.512 & -1.918 & 0.110 \\ -0.7213 & -2.995 & -0.2482 \\ -0.0023 & -1.339 & -2.1748 \end{bmatrix}$$

Mode 2:

$$\mathbf{K}(2) = \begin{bmatrix} -0.6685 & -1.222 & -0.51 \\ 1.3875 & -11.6513 & -2.1457 \\ 5.2 & -3.713 & -13.0126 \end{bmatrix}$$

Mode 3:

$$\mathbf{K}(3) = \begin{bmatrix} -4.4062 & -5.6729 & 1.6432 \\ -2.3946 & -9.0647 & 0.6995 \\ -0.0260 & -2.5967 & -3.3551 \end{bmatrix}$$

7 Conclusion

In this paper, we dealt with the class of linear systems with Marovian jumps and uncertain parameters. We have extended the concepts of quadratic stability and quadratic stabilizability to this class of systems and established a necessary and sufficient condition for checking it. The quadratic guaranteed cost control problem has been also investigated.

References

1. Khargonegar P. P., Petersen I. R., and Zhou, K., *Robust Stabilizability and $\mathcal{H}_\infty$ Control of Uncertain Linear Systems*, IEEE, Transations on Automatic Control, Vol. 35, No. 4, pp. 356–361, 1990.

2. Petersen, I. R., *A stabilization algorithm for a class of uncertain linear systems.*, System & Control Letters, Vol. 8, No. 4, pp. 351–357, 1987.

3. Petersen I. R., and McFarlane, D.C., *Optimal Guaranteed Cost and Filtering for Uncertain Linear Systems*, IEEE, Transactions on Automatic Control, Vol. 39, No. 9, pp. 1971–1977, 1994.

4. Boukas, E. K., Zhang, Q., and Yin, G., *Robust production and maintenance planning in stochastic manufacturing systems*, IEEE Transactions on Automatic Control, Vol. 40, No. 6, 1098–1102, 1995.

5. Mariton, M., *Jump Linear Systems in Automatic Control*, New York and Basel, Marcel Dekker, 1990.

6. Ji, Y., and Chizech, H.J., *Controllability and Stabilizability and Continuous-time Jump Linear Quadratic Control*, IEEE, Transactions on Automatic Control, Vol. 35, No. 7, pp. 777–788, 1990.

7. Feng, X., Loparo, K.A., Ji, Y., and Chizech, H.J., *Stochastic Stability Properties of Jump Linear Systems*, IEEE, Transactions on Automatic Control, Vol. 37, No. 1, pp. 38–53, 1992.

8. Costa, O.L.V., do Val, J.B.R., and Geromel, J.L., *Continuous-Time $\mathcal{H}_2$ Control of Markovian Jump Linear Systems via Convex Analysis*, to appear, 1997.

9. De Souza, C.E., and Fragoso, M.D., $\mathcal{H}^\infty$ *Control for linear systems with Markovian Jumping parameters*, Control Theory and Advanced Technology, Vol. 9, No. 2, pp. 457–466, 1993.

10. Boukas, E.K., *Robust Stability of Linear Piecewise Deterministic Systems Under Matching Conditions*, Control Theory and Advanced Technology, Vol. 10, No. 4, Part 3, pp. 1541–1549, 1995.

11. Boukas, E.K., and Yang, H., *Stability of Discrete-Time Linear Systems with Markovian Jumping Parameters*, Mathematics of Control, Signals and Systems, Vol. 8, pp. 390–402, 1995.

12. Boukas, E.K., and Yang, H., *Stability of Stochastic Systems with Jumps*, Mathematical Probems in Engineering, Vol. 3, pp. 173–185, 1997.

13. Morozan, T., *Optimal stationary control for dynamic systems with markov perturbations*, Stochastic Analysis and Applications, Vol. 1, No. 3, pp. 299–325, 1983.

14. Morozan, T., *Stability and control for linear systems with jump markov perturbations*, Stochastic Analysis and Applications, Vol. 13, No. 1, pp. 91–110, 1995.

15. Morozan, T., *Parametrized Riccati equations for controlled linear differential systems with jump markov perturbations*, Technical report, Institute of Math. of Romanian Academy, 20/1996, 1996.

16. Pan, Z., and Basar, T., *Zero-sum differential games with random structures and applications in H^∞-control of jump linear systems*, Proceedings of the Sixth International Symposium on Dynamic Games and Applications, pp. 466–474, St-Jovite, Quebec, Canada, 1994.

17. Pan, Z., and Basar, T., *H^∞ control of large scale jump linear systems via averaging and aggregation*, Proceedings of the 34th IEEE Conference on Decision & Control, pp. 2574–2579, New Orleans, LA, 1995.

18. Pan, Z., and Basar, T., *Random evolutionary time-scale decomposition in robust control of jump linear systems*, Proceedings of the 35th IEEE Conference on Decision & Control, pp. 505–510, Kobe, Japan, 1996.

19. ElGhaoui L., and Ait Rami, M., *Solving Non-Standard Riccati Equations unsing*

LMI Optimization, Proceedings of the 34th conference on Decision & Control, Vol. 4, pp. 3855–3856, New Orleans, LA, December 1995.

20. Dufour, F., Elliot, R.J., and Tsoi, A., *Asymptotic Study of Estimation in Filtering for Linear Systems with Jumps Parameters*, Proceedings of the 34th conference on Decision & Control, Vol. 4, pp. 3349–3353, New Orleans, LA, December 1995.

21. Shi, P., and Boukas, E.K., *H_∞ control for Markovian jumping linear systems with parametric uncertainty*, Journal of Optimization Theory and Applications, Vol. 95, No. 2, 1997.

22. De Souza, C.E., and Fragoso, M.D., *H_∞ filtering for Markovian jump linear systems*, Proceedings 35th IEEE Conference on Decision & Control, pp. 4814–4818, Kobe, Japan, 1996.

23. Lee, J.H., Kwon, W.H., and Lee, J.W., *Quadratic Stability and Stabilizability of Linear Systems with Frobenius Norme-Bounded Uncertainties*, IEEE, Transactions on Automatic Control, Vol. 41, No. 3, pp. 453–456, 1996.