Invariant probability measures for a class of Feller Markov chains *

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Abstract

In this paper we consider a Markov chain defined on a locally compact separable metric space which satisfies the Feller property. We introduce a new assumption which generalizes T-chain and irreducibility assumptions, well known in the literature of Markov chains. Under this new assumption, the Foster's criterion is shown to be equivalent to the existence of an invariant probability measure for Feller Markov chains, which is also equivalent to the existence of a non-singular invariant probability measure.

Keywords: Markov chain, Feller chain, invariant measures
AMS 1991 subject classification: Primary 60J10
Secondary 93E15

*This work has been supported by a USP/COFECUB Franco-Brazilian grant (number UC 40/97).
†This author received financial support from FAPESP (Research Council of the State of São Paulo) and CNPq (Brazilian National Research Council)
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1 Introduction

A problem of great importance in the theory of Markov chains that has been in evidence for over the last decades is to obtain conditions for the existence of an invariant probability measure. The existence of such a measure corresponds to an important form of stochastic stability and it is the first step towards an ergodic analysis of the process. In general it is not a simple matter to determine whether a given Markov chain on a general state space has an invariant probability measure. For discrete-time Markov chains there is available in [4] a rather complete discussion of this subject.
In this paper, we focus our attention on Feller Markov chains. The associated transition probability kernel (labeled P) maps the space of continuous bounded functions into itself. Such chains are relatively general to formulate many problems in several areas of applications. For example, it has been shown in [1] that the stability of a general class of continuous-time process (piecewise-deterministic Markov processes) is related to a Feller Markov kernel (under weak assumptions). Here we assume that the state space X of the Feller chain P is a locally compact separable metric space.
We first present two known results (this quick overview does not claim to be exhaustive) in order to present our work with respect to the literature.
In [3], Lasserre gave the following **necessary and sufficient condition** without any assumption on the chain:

Theorem 1.1 (Lasserre 97) *Let f_0 be an arbitrary strictly positive continuous function that vanishes at infinity. The following statements are equivalent.*
(i) There exists an invariant probability measure π for P.
(ii) There exists $x_0 \in X$ such that $\lim_n \frac{1}{n} \sum_{k=0}^{n-1} P^k f_0(x_0) \neq 0$

Although this condition is necessary and sufficient, it is difficult to check in practice. Indeed it needs the calculation of the infinite sum of iterations of the associated kernel.

On the other hand, in Theorem 12.3.4 of [4], the authors developed a Foster-type criterion to guarantee the existence of an invariant measure for Feller chains. This condition is only shown to be **sufficient** (contrary to the previous result). However this criterion is **very powerful** in practice since it is easier to check.

Theorem 1.2 (Meyn and Tweedie 93)
If there exist a real $\alpha > 0$, a compact set C and a function $V : X \longrightarrow [0, \infty]$ which is finite at least at one $x_0 \in X$, such that

$$PV(x) \quad \leq \quad V(x) - \alpha + I_C(x), \tag{1}$$

is satisfied for all x in X then there exists an invariant probability measure π for P.

A natural question to ask is if the criterion developed by Meyn and Tweedie [4] (see Theorem 1.2) is necessary. Under some ergodicity assumptions a result related to this question has been derived in Theorem 13.0.1 of [4]. However the assumption of ergodicity (aperiodicity, Harris recurrence and in particular irreducibility) is essential for the development of this result and is rather strong to provide a criterion for the existence of an invariant measure.

The main goal of this paper is to prove the necessity of condition (1) under a new assumption. Indeed, we shall show, by using this new assumption and an ergodic decomposition of the state space

[6] that this Foster's type criterion is **not only sufficient but also necessary** for the existence of an invariant probability measure, a result that generalizes Theorem 1.2. It is interesting to note that the proof of the necessity of this result highlights the advantage of allowing the test function $V(x)$ to be ∞. As a by product we also get that the existence of an invariant probability measure is equivalent to the existence of a non-singular invariant probability measure. In particular we show that this equivalence holds for Feller Markov T-chains

2 Definitions and main result

Let $(X, \mathcal{B}(X))$ be a locally compact metric space equipped with its borel σ-field.
Define $\mathbb{N}_* \doteq \mathbb{N} - \{0\}$.
For any $A \in \mathcal{B}(X)$, $A^c \doteq X - A$ and $I_A(x)$ is the indicator function associated to A.
Let P be a stochastic kernel defined on $(X, \mathcal{B}(X))$ satisfying the Feller hypothesis (i.e. P maps the space of continuous bounded functions into itself). We shall denote by $\{Y_n\}_{n\in\mathbb{N}}$ the Markov chain generated by the stochastic kernel P (i.e. $P(x, A) = P_x(Y_1 \in A)$ for all $x \in X$ and $A \in \mathcal{B}(X)$). We shall write $P^n(x, A) = P_x(Y_n \in A)$ for all $n \geq 1$.

We recall now some classical definitions related to Markov chains. For a complete exposition on the subject the reader is referred to the book of S. Meyn and R. Tweedie [4].
Let us define by σ_A (respectively τ_A) the first hitting time of the set $A \in \mathcal{B}(X)$, including time 0 (respectively not including time 0):

$$\begin{aligned} \sigma_A &\doteq \inf\{n \geq 0 : Y_n \in A\} \\ \tau_A &\doteq \inf\{n \geq 1 : Y_n \in A\}. \end{aligned}$$

Definition 2.1 *$\{Y_n\}$ is φ-irreducible if there exists a measure φ defined on $(X, \mathcal{B}(X))$ such that, whenever $\varphi(A) > 0$, it follows that $P_x(\tau_A < \infty) > 0$ for all $x \in X$.*

If $\{Y_n\}$ is φ-irreducible then there exists a maximal irreducibility measure (see Proposition 4.2.2 in [4, p. 88]). In the following, we will use ψ to denote a maximal irreducibility measure for $\{Y_n\}$. Associated to a maximal irreducibility measure for $\{Y_n\}$, we define:

$$\mathcal{B}(X)^+ \doteq \{A \in \mathcal{B}(X) : \psi(A) > 0\}.$$

Definition 2.2 *Assume $\{Y_n\}$ ψ-irreducible. Then a set $A \in \mathcal{B}(X)$ is called regular if*

$$\sup_{x\in A} E_x[\tau_B] < \infty \text{ for all } B \in \mathcal{B}(X)^+.$$

Definition 2.3 *A set $A \in \mathcal{B}(X)$ is called absorbing if $P(x, A) = 1$ for all $x \in A$.*

If $a = \{a_k\}_{k=0}^{\infty}$ is a sampling distribution, then the stochastic kernel K_a is defined on $X \times \mathcal{B}(X)$ by

$$(\forall x \in X), \quad (\forall A \in \mathcal{B}(X)), \quad K_a(x, A) \doteq \sum_{k=0}^{\infty} a_k P^k(x, A). \tag{2}$$

For B a fixed set in $\mathcal{B}(X)$, the kernel $U_B(.,.)$ is defined on $X \times \mathcal{B}(X)$ by

$$U_B(x, A) \doteq \sum_{k=1}^{\infty} [(P I_{B^c})^{k-1} P](x, A). \tag{3}$$

Definition 2.4 *T is a continuous component of K_a, if T is a substochastic kernel satisfying $K_a(x, A) \geq T(x, A)$ for all $x \in X$ and $A \in \mathcal{B}(X)$ and such that $T(., A)$ is a lower semicontinuous function for any $A \in \mathcal{B}(X)$.*

Definition 2.5 *A probability measure μ is said to be non-singular if there exists an integer k such that the Radon-Nikodym derivative of $P^k(x, .)$ with respect to μ: $\dfrac{dP^k(x,.)}{d\mu} \doteq p^k_\mu(x, y)$ (note that $p^k_\mu(., .)$ can be chosen to be measurable on $X \times X$) satisfies $\displaystyle\int_X \int_X p^k_\mu(x, y)\mu(dx)\mu(dy) > 0$.*

Now let us introduce the main hypotheses related to the stochastic kernel P:

Condition T': There exists a substochastic transition kernel T defined on $X \times \mathcal{B}(X)$ such that
$\mathbf{T'_i}$ For all $x \in X$, $T(x, .) << K_a(x, .)$.
$\mathbf{T'_{ii}}$ For all $x \in X$, $0 < T(x, .) \leq 1$.
$\mathbf{T'_{iii}}$ There exists a countable sequence of sets $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{B}(X)$ such that for each absorbing set E of P, there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that

$$\{z \in X \,:\, T(z, \overline{E}) > 0\} \quad = \quad \bigcup_{i=1}^{\infty} B_{n_i}$$

where

$$\overline{A} \doteq [A^\circ]^c, \tag{4}$$
$$A^\circ \doteq \{y \in X \,:\, P^k(y, A) = 0 \text{ for all } k \in \mathbb{N}_*\}. \tag{5}$$

The main result of our paper, presented below, shows that under condition T', the existence of an invariant probability measure is equivalent to a Foster's type criterion. In this case at least one non-singular invariant probability measure will exist. Consequently, we generalize Theorem 1.2 by showing that this criterion is also a necessary condition for existence of an invariant probability measure for a Feller chain satisfying condition T', and necessary and sufficient for the existence of a non-singular invariant probability measure.

Theorem 2.6 *The following assertions are equivalent for a Feller Markov chain satisfying condition T'.*
(i) There exists an invariant probability measure for P.
(ii) There exists a real $\alpha > 0$, a function $V : X \longrightarrow [0, \infty]$ which is finite at least at one point $x_0 \in X$, and a compact set C such that

$$PV(x) \quad \leq \quad V(x) - \alpha + I_C(x) \tag{6}$$

is satisfied for all $x \in X$.
(iii) There exists a non-singular invariant probability measure for P.

3 Preliminary results

The purpose of this section is to present the recurrent decomposition of the Markov chain (see Theorem 3.9) that will be required to obtain the main result of the paper.

In this section, we assume that there exists an invariant probability measure for the

weak Feller kernel P satisfying condition T'.

From [2], we have the following theorem:

Theorem 3.1 *There exists a set $S \in \mathcal{B}(X)$ such that:*
i) S is absorbing,
ii) there exists a unique invariant probability measure λ satisfying $\lambda(S) = 1$.

Proof: For a detailed proof, see [2] pages 111-115. □

Now according to the Lebesgue Decomposition, [4, page 107], the transition probability kernel P^n admits a decomposition into its absolutely continuous and singular parts with respect to λ. Therefore, for a fixed $y \in X$, we have for all $B \in \mathcal{B}(X)$

$$P^n(y,B) = P^n_\lambda(y,B) + P^n_\perp(y,B) \tag{7}$$

with

$$P^n_\lambda(y,B) = \int_B p^n(x,y)\lambda(dy) \tag{8}$$

where the density p^n is a measurable function on $X \times X$ for each n and $P^n_\perp(y,.)$ is orthogonal to λ. Consequently, for each $y \in X$ and $n \in \mathbb{N}_*$, there exists a set $L_{n,y}$ such that $\lambda(L_{n,y}) = 1$ and $P^n_\perp(y,B) = P^n_\perp(y, B \cap L^c_{n,y})$ for all $B \in \mathcal{B}(X)$.
Let us define $L_y \doteq \bigcap_{n=1}^{\infty} L_{n,y}$. Then $\lambda(L_y) = 1$ and $P^n_\perp(y,B) = P^n_\perp(y, B \cap L^c_y)$ for all $B \in \mathcal{B}(X)$. Moreover, there is no loss of generality in assuming that for each $y \in X$, L_y is absorbing. Indeed, if it is not the case, we can take the sequence defined by

$$F_{k+1,y} \doteq \{z \in F_{k,y} \,:\, P(z, F_{k,y}) = 1\} \tag{9}$$

with $F_{0,y} \doteq L_y$. Now, it can be shown by induction that the set $F_{\infty,y} \doteq \bigcap_{k=0}^{\infty} F_{k,y}$ is an absorbing set such that $\lambda(F_{\infty,y}) = 1$. Moreover, for all $B \in \mathcal{B}(X)$, $P^n_\perp(y,B) = P^n_\perp(y, B \cap F_{\infty,y})$. Then, we could re-define L_y as being $F_{\infty,y}$.

For $A \in \mathcal{B}(X)$, define (see condition T' for the definition of T)

$$A^* \doteq \{y \in X \,:\, T(y,A) = 0\}, \tag{10}$$

$$\widetilde{A} \doteq [A^*]^c. \tag{11}$$

Before going further, we introduce the following simple remarks:

Remark 3.2 *If $A \subset B$ then $B^\circ \subset A^\circ$, (see equation (5)).*

Remark 3.3 *For any $A \in \mathcal{B}(X)$, $A^\circ \subset A^*$, since from $\mathbf{T'_i}$, $K_a(x,.) >> T(x,.)$.*

We have the following results:

Proposition 3.4 *For $A \in \mathcal{B}(X)$, $A^\circ \subset \left[\overline{A}\right]^\circ$ and if $\lambda(A) > 0$ then $\lambda(\overline{A}) = 1$.*

Proof: If $y \in A^\circ$ then $P^{k+l}(y,A) = \int_X P^k(z,A)\, P^l(y,dz) = 0$ for $(k,l) \in \mathbb{N}_*^2$. Therefore, defining the set $A_k^\circ \doteq \{z \in X \,:\, P^k(z,A) = 0\}$, we have $P^l(y, A_k^\circ) = 1$ and since $A^\circ = \cap_{k=1}^\infty A_k^\circ$ we obtain the first part of the result. Now, from Proposition 10.4.6 in [4] and using *ii)* of Theorem 3.1, we have that for all B in $\mathcal{B}(X)$

$$\lambda(B) = \frac{1}{c} \int_D U_D(y,B)\lambda(dy) \tag{12}$$

where c is a normalizing positive constant and the set D is an arbitrary set such that $\lambda(D) > 0$. Let us suppose that for a set A satisfying $\lambda(A) > 0$ we have $\lambda(A^\circ) > 0$. Then,

$$\int_{A^\circ} U_{A^\circ}(y,A)\lambda(dy) > 0. \tag{13}$$

However, $U_{A^\circ}(y,A) = 0$ for all $y \in A^\circ$ leading to a contradiction. Consequently, if $\lambda(A) > 0$, then $\lambda(A^\circ) = 0$ and the second part of the result follows. □

Lemma 3.5 *Suppose that the set $A \in \mathcal{B}(X)$ is such that $\lambda(A) > 0$. Then $\lambda\left(\widetilde{\overline{A}}\right) > 0$.*

Proof: From the previous lemma, we obtain that $\lambda(\overline{A}) = 1$. Using the fact that λ is an invariant measure, we have

$$\int_X K_a(y, A^\circ)\lambda(dy) \quad = \quad 0$$

which gives $\lambda([A^\circ]^\circ) = 1$.
From remark 3.3, $[A^\circ]^\circ \subset [A^\circ]^*$, and we must have $\lambda([A^\circ]^*) = 1$, that is,

$$\int_X T(y, A^\circ)\lambda(dy) = 0. \tag{14}$$

However, by hypothesis,

$$\int_X T(y, X)\lambda(dy) > 0. \tag{15}$$

Combining equations (14) and (15), we obtain that

$$\int_X T(y, \overline{A})\lambda(dy) > 0.$$

and the result follows. □

For notational convenience, let us introduce the sets

$$\Gamma \doteq \{n \in \mathbb{N} \,:\, \lambda(B_n) > 0\}, \tag{16}$$
$$I(X) \doteq \{y \in X \,:\, P_\lambda^k(y, X) = 0 \text{ for all } k \in \mathbb{N}_*\}. \tag{17}$$

It is easy to check that $I(X) \in \mathcal{B}(X)$.

Lemma 3.6 $I(X) \subset \bigcup_{n\in\Gamma} [B_n]^\circ$.

Proof: If $y \in I(X)$ then for $k \in \mathbb{N}_*$, $1 = P^k(y, X) = P^k_\perp(y, L^c_y)$ and so $P^k(y, L^c_y) = 1$. Consequently,

$$y \in I(X) \quad \Longrightarrow \quad y \in [L_y]^\circ . \tag{18}$$

Now, from proposition 3.4 and since $\lambda(L_y) = 1$, we obtain that

$$\lambda(\overline{L_y}) = 1, \tag{19}$$

$$[L_y]^\circ \subset \left[\overline{L_y}\right]^\circ \subset \left[\overline{\overline{L_y}}\right]^\circ . \tag{20}$$

Therefore, from remarks 3.2 and 3.3

$$[L_y]^\circ \quad \subset \quad \left[\widetilde{\overline{L_y}}\right]^\circ . \tag{21}$$

Recalling that L_y is an absorbing set, we get from condition T_{iii}, that

$$\begin{aligned} \widetilde{\overline{L_y}} &= \{z \in X \ : \ T(z, \overline{L_y}) > 0\} \\ &= \bigcup_{i=1}^{\infty} B_{n_i(y)} \end{aligned} \tag{22}$$

for some subsequence $\{n_i(y)\}_{i=1}^{\infty}$. From lemma 3.5 and using the fact that $\lambda(L_y) = 1$, it follows that $\lambda\left(\widetilde{\overline{L_y}}\right) > 0$. This means that there exists at least one integer (depending on y), denoted by $k(y)$, such that $\lambda(B_{k(y)}) > 0$. Consequently, using equation (22) and remark 3.2, we have

$$\left[\widetilde{\overline{L_y}}\right]^\circ \subset \left[B_{k(y)}\right]^\circ \subset \bigcup_{n \in \Gamma} [B_n]^\circ . \tag{23}$$

Now, combining equations (18), (21) and (23), we obtain the result. □

Proposition 3.7 $\lambda(I(X)) = 0$.

Proof: If Γ is empty then $\bigcup_{n\in\Gamma} [B_n]^\circ = \emptyset$. If Γ is not empty then using proposition 3.4, we have

$$\lambda(\bigcup_{n\in\Gamma} [B_n]^\circ) = 0,$$

since $\lambda(B_n) > 0$ for $n \in \Gamma$.
In any case, $\lambda(\bigcup_{n\in\Gamma} [B_n]^\circ) = 0$ and the result follows from lemma 3.6. □

Lemma 3.8 *Suppose that the set $E \in \mathcal{B}(X)$ is such that $\lambda(E) = 1$. Then $E^\circ \subset I(X)$.*

Proof: For $x \in E^\circ$, we have that $P^k_\lambda(x, E^c) = 0$ and $P^k(x, E^c) = P^k_\perp(x, E^c) = 1$ for all $k \in \mathbb{N}_*$. Thus, $1 = P^k(x, X) = P^k_\lambda(x, X) + P^k_\perp(x, X) = P^k_\lambda(x, X) + 1$ and therefore, $P^k_\lambda(x, X) = 0$ for all $k \in \mathbb{N}_*$, showing the result. □

We present the following theorem which will be very important and useful to obtain the main result of our paper (see theorem 2.6). We define now the sets $Z_0 \doteq S \cap I(X)^c$ and $Z_{j+1} \doteq \{y \in Z_j : P(y, Z_j) = 1\}$ for $j \in \mathbb{N}$. Using proposition 3.7, we obtain that $\lambda(Z_0) = 1$ and by induction, it can be easily shown that $\lambda(Z_j) = 1$ for $j \in \mathbb{N}$ and so $\lambda(\bigcap_{j=0}^{\infty} Z_j) = 1$. Moreover, by construction, $\bigcap_{j=0}^{\infty} Z_j$ is an absorbing set. Consequently, a new Markov chain $\{Y'_k\}_{k\in\mathbb{N}}$ can be defined on $Z \doteq \bigcap_{j=0}^{\infty} Z_j$ with transition kernel P on $Z \times \mathcal{B}(Z)$.

Theorem 3.9 *Assume that the kernel P is weak Feller and satisfies condition T'. The Markov chain $\{Y'_k\}_{k\in\mathbb{N}}$ with state space Z is irreducible and positive recurrent.*

Proof: Let us show that $\{Y'_k\}$ is λ-irreducible. Indeed, suppose by contradiction that there exists a set $A \in \mathcal{B}(Z)$ and $y \in Z$ such that $\lambda(A) > 0$ and $P^k(y, A) = 0$ for $k \geq 1$. Then $y \in A^\circ$. However, using proposition 3.4, we have $A^\circ \subset \left[\overline{A}\right]^\circ$ and $\lambda(\overline{A}) = 1$ since $\lambda(A) > 0$. Applying lemma 3.8, we get that $A^\circ \subset I(X)$. Consequently, $y \in I(X)$ but by definition of Z, $y \in I(X)^c$. This leads to a contradiction. The chain is positive since it admits the invariant probability measure $\lambda(.)$. Now, the result follows from Proposition 10.1.1 in [4]. $\square$

4 Proof of the main result

In this section we present the proof of our main result.

Proof of Theorem 2.6: *(i) $\Rightarrow$ (ii).* Since an invariant probability measure exists, we can apply the results obtained in the previous section. Therefore the Markov chain $\{Y'_k\}$ taking values on Z and with transition probability measure P is irreducible and positive recurrent. Consequently there exists a decomposition $Z = J \cup N$ where the set J is full and absorbing and the chain $\{Y'_k\}$ restricted to J is regular (see Theorem 11.1.4 in [4]). Therefore, $J = \bigcup_{n=1}^{\infty} J_n$ where J_n is a regular set.
Let p be such that $J_p \in \mathcal{B}(Z)^+$; then there exists an integer q such that $D \doteq J_p \cap \bigcup_{n=1}^{q} X_n \in \mathcal{B}(Z)^+$ (since $X = \cup_{n=1}^{\infty} X_n$ for an increasing sequence of compact sets X_n). Therefore, there exists $M \in \mathbb{R}_+$ such that

$$\begin{aligned} \sup_{x\in D} E_x[\tau_D] &\leq \sup_{x\in J_p} E_x[\tau_D] \\ &< M \end{aligned} \tag{24}$$

since J_p is a regular set. Now, let us introduce the function

$$V_D(x) \doteq \begin{cases} E_x[\sigma_D] + 1 & \text{for } x \in Z \\ \infty & \text{for } x \in Z^c \end{cases}$$

Then we have for $x \in Z^c$:

$$PV_D(x) \leq V_D(x) - 1 \tag{25}$$

since $V_D(x) = \infty$. Now using the identities in [5, 4] we obtain that

$$PV_D(x) = V_D(x) - 1, \quad \text{if } x \in Z - D \tag{26}$$

$$PV_D(x) = V_D(x) - 1 + E_x[\tau_D], \quad \text{if } x \in D. \tag{27}$$

By definition $D \subset \bigcup_{n=1}^{q} X_n$, so the set C defined as the closure of D is a compact set. Combining equations (24)-(27), it follows that

$$PV_D(x) \quad \leq \quad V_D(x) - 1 + MI_C(x).$$

Note that for $x_0 \in D$, $E_{x_0}[\tau_D]$ is finite. Moreover, we have $\sigma_D \leq \tau_D$, P_{x_0} a.e. and so $V_D(x_0)$ is finite which proves the result.
(ii) $\Rightarrow$ *(i)*. Immediate from Theorem 1.2.
(iii) $\Rightarrow$ *(i)*. Obvious.
(i) $\Rightarrow$ *(iii)*. The results of section 3 can be applied and using proposition 3.7 and definition 2.5, we obtain the result. □

5 Examples of Feller chains satisfying Condition T'

At first glance **Condition T'** could appear to be restrictive. We shall present in this section three examples of Feller chains which satisfy this condition. The first two examples show that standard classes of Markov chain (such as irreducible chains or T-chains) possess the properties $\mathbf{T}'_{\mathbf{i}}$-$\mathbf{T}'_{\mathbf{iii}}$. Moreover, we present a corollary of theorem 2.6 for Feller Markov T-chains. In the last example, it is shown that the class of chains satisfying condition T' is wider than the classes of irreducible chains and T-chains.

Example 1
T-chains satisfy Condition T'. Indeed, it is easy to check that for a T-chain, the continuous component (labeled T) of K_a (see [4] page 127) satisfies the conditions $\mathbf{T}'_{\mathbf{i}}$ and $\mathbf{T}'_{\mathbf{ii}}$.
Let us choose for $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{B}(X)$ a countable basis for the open sets. Then for all sets $A \in \mathcal{B}(X)$, $T(.,A)$ is lower continuous. Consequently, the continuous component T satisfies $\mathbf{T}'_{\mathbf{iii}}$.

From example 1 and theorem 2.6, the following corollary is immediate.

Corollary 5.1 *Conditions* (*i*), (*ii*) *and* (*iii*) *of thereom 2.6 are equivalent for Feller Markov T-chains.*

Example 2
ψ-irreducible chains satisfy Condition T'. According to definition 2.1, we have $\psi << K_a(x,.)$ for all $x \in X$. Defining $T(y,A) \doteq \psi(A)$ for all $y \in X$ and $A \in \mathcal{B}(X)$, conditions $\mathbf{T}'_{\mathbf{i}}$ and $\mathbf{T}'_{\mathbf{ii}}$ are trivially satisfied. Moreover, for all $A \in \mathcal{B}(X)$, $\{z \in X, T(z,A) > 0\} = X$ or $\emptyset$ which gives the last condition for $\{B_n\}$ defined by the finite family: $\{X, \emptyset\}$.

Example 3
The class of Markov chains satisfying condition T' **is larger** than the union of T chains and ψ-irreducible chains. Indeed consider the following example: $X = [0,\infty)$ and for $y \in (0,\infty)$ and $A \in \mathcal{B}(X)$, $P(y,A) = \dfrac{2}{y}\,\mu_{leb}(A \cap (0,\dfrac{y}{2}))$ and $P(0,0) = 1$. If E is an absorbing set then

$$\mu_{leb}(E \cap (0,\frac{y}{2})) = \frac{y}{2}. \tag{28}$$

Let us show that $E^\circ = \{0\}$ if E is an absorbing set such that $0 \notin E$. Suppose that there exists $z \in E^\circ$ such that $z > 0$. Take an arbitrary $y \in E$. If $y \leq z$, we would have using equation (28)

$$0 = P(z,E) \geq P(z, E \cap (0,\frac{y}{2}) = \frac{y}{z}$$

leading to a contradiction. On the other hand, if $z < y$ then

$$1 = P(y, E) = \frac{2}{y}\, \mu_{leb}(A \cap (\frac{z}{2}, \frac{y}{2})) \leq 1 - \frac{z}{y}$$

which is a contradiction with the fact that $z > 0$, proving the claim.
Therefore, we have the three following possibilities for E°, E being an absorbing set:
a) $E^{\circ} = \{0\}$ if $E \subset (0, \infty)$,
b) $E^{\circ} = (0, \infty)$ if $E = \{0\}$,
c) $E^{\circ} = \emptyset$ if $E \neq \{0\}$ and $\{0\} \subset E$.
Choose the substochastic kernel T equal to P. Then, we have $\{z \in X \, : \, T(z, \overline{E}) > 0\} = (0, \infty)$, $\{0\}$ or $[0, \infty)$. Consequently, conditions $\mathbf{T'_i}$, $\mathbf{T'_{ii}}$ and $\mathbf{T'_{iii}}$ are satisfied for $\{B_n\}$ defined by the finite family: $\{X, X - \{0\}, \{0\}\}$.
It is easy to check that this chain **is not irreducible**. Let us show that it **is not a T-chain**. To be a T-chain, there should exist a sampling distribution a such that K_a possesses a continuous component T with $T(x, X) > 0$ for all $x \in X$. Consequently, we would have $0 = \underline{\lim}_{y \to 0} T(y, 0) \geq T(0, 0)$ and then $T(0, 0) = 0$. Moreover, $K_a(0, (0, \infty)) = 0$ which gives $T(0, (0, \infty)) = 0$. Finally, it follows that $T(0, X) = 0$ in contradiction with the hypothesis that $T(x, X) > 0$ for all $x \in X$. Therefore, it is not a T-chain. However, it can be shown that **this chain is weak Feller**.

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