Impulse and Continuous Control of Piecewise Deterministic Markov Processes *

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Abstract

In this paper we consider the problem of impulse and continuous control on the jump rate and post jump location parameters of piecewise-deterministic Markov processes (PDP's). In a companion paper we studied the optimal stopping with continuous control problem of PDP's assuming only absolutely continuity along trajectories hypothesis on the final cost function. In this paper we apply these results to obtain optimality equations for the impulse and continuous control problem of PDP's in terms of a set of quasi-variational inequalities as well as on the first jump time operator of the process. No continuity or differential assumptions on the whole state space, neither stability assumptions on the parameters of the problem are required. It is shown that if the post intervention operator satisfies some locally lipschitz continuity along trajectories properties then so will the value function of the impulse and continuous control problem.

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1 Introduction

A general family of non-diffusion stochastic models suitable for formulating many optimization problems in several areas of operations research, namely Piecewise-deterministic Markov processes (PDP's), were introduced in [10],[11]. These processes are determined by three local characteristics; the flow ϕ, the jump rate λ and the transition measure Q, which specifies the post-jump location. Starting from x the motion of the process follows the flow $\phi(t,x)$ until the first jump time T_1 which occurs either spontaneously in a Poisson-like fashion with rate $\lambda(\phi(t,x))$ or when the flow hits the boundary of the state space. In either case the location of the process at the jump time T_1 is selected by the transition measure $Q(.;\phi(T_1,x))$ and the motion restarts from this new point as before. Several results are now available on the theory of PDP's, including invariant measures [16], continuous control [10],[12],[13],[15],[28], optimal stopping [6],[8]-[10], and impulse control [5],[7],[8],[10],[14], [17]-[19], with some specific problems being solved [10]. Note that PDP's are not, in general, Feller processes, and therefore the general theory for such processes cannot be applied directly to PDP's. Related to PDP's are the so-called Markov Decision Drift Processes, which deals with problems involving interventions (similar to impulse control) and continuous control on the rate of jumps and post-jump location measure. Such problems have been analyzed in [21],[22], [27] via time-discretization, and in [29],[30] via Bellman inequalities.

In a companion paper [9] we have considered the optimal stopping with continuous control of PDP's. In this case we allow for continuous control on the local parameters λ and Q, besides the optimal stopping problem. We considered the final cost function to be absolutely continuous along trajectories. It was shown in [9] that the optimal value function also has this property. No boundary conditions neither global continuity conditions were considered there. Moreover it was shown that the value function is the unique fixed point of the operator associated to the first jump operator of the process, as well as the unique solution of a set of variational inequalities, defined in terms of the infinitesimal operator of the process.

In many problems the control action has to be taken by intervention, that is, the decision to act is taken at discrete times, in response to the random evolution of the system, and the process moved to a new point in the state space. This kind of control is known in the international literature as impulse control. Optimal stopping problems are interesting not only by its own, but also because they are the first step to study this more general problem. Bensoussan and Lions [1] studied this problem in the context of diffusion processes, as an application of variational and quasi-variational inequalities, and further developed in [2],[3]. Kushner [23],[24] considered similar problems, approximating the diffusion by controlled Markov chains. In [25] Lepeltier and Marchal considered the impulse control problem of right processes under very general assumptions by formulating it as a sequence of optimal stopping problems [31],[32], an approach due to Robin [26]. For PDP's this class of problem was analyzed in [5],[7],[8],[10],[14], [17]-[19], under different approaches.

In this paper we will apply the results in [9] to analyze the impulse with continuous control problem of PDP's. The main differences between this paper and previous ones are the following. Firstly we allow for continuous control on the local parameters λ and transition measure Q, besides the impulse control problem. To do that, we consider a more general family of stochastic processes, not necessarily time-homogeneous Markovian processes. Secondly we do not impose continuity or differentiability assumptions on the whole state space of the process neither stability conditions, as in previous papers, but instead we require only locally lipschtz continuity along trajectories assumptions for the post intervention operator. We assume that after an intervention the process can only be sent to points along a finite number of trajectories of the flow $\phi(.,.)$. Under these assumptions we show that a set of quasi-variational inequalities is necessary and sufficient condition to characterize the optimal value function of the problem, which can also be characterized as the unique fixed point of the first jump or intervention operator. It is also shown that the value function is locally lipschitz continuous along trajectories.

This work is organized in the following way. In Section 2 we present some notations, definitions, assumptions, and preliminary results, while in Section 3 we introduce the continuous control and the process generated by it, and recall the main results for the optimal stopping with continuous control problem obtained in [9]. In Section 4 we define the impulse and continuous control problem for PDP's. In section 5 we present the main results of the paper. Theorem 5.1 characterizes the optimal value function of the problem as the unique fixed point of the first jump operator, or as the unique solution of a set of quasi-variational inequalities in the domain of the absolutely continuous along trajectories functions.

2 Definitions, Assumptions, and Preliminary Results

2.1 Definitions and Notation

Let E^0 be an open non-empty subset of R^d, ∂E^0 its boundary. We consider $\phi(t,x)$ the flow of a lipschitz continuous vector field $\mathcal{L}$. Define

$$\partial^{\pm} E^0 = \{z \in \partial E^0; z = \phi(\pm t, x) \text{ for some } x \in E^0\}$$

and $\partial_1 E^0 = \partial^- E^0 - \partial^+ E^0$. Set $E = E^0 \cup \partial_1 E^0, \sigma(E)$ the Borel σ-field of E, $\mathcal{P}(E)$ the set of probability measures on E, $\partial E^* = \partial^+ E^0$, and for each $x \in E$, write

$$t^*(x) = inf\{t > 0; \phi(t,x) \in \partial E^0\},$$
$$t^*_-(x) = inf\{t \geq 0; \phi(-t,x) \in \partial E^0\}$$
$$\mathcal{Z}(x) = \begin{cases} [t^*_-(x), t^*(x)) & if\, t^*_-(x) < \infty \\ (-\infty, t^*(x)) & if\, t^*_-(x) = \infty \end{cases}$$

where $inf\{\emptyset\} := \infty$. Therefore for every $t \in \mathcal{Z}(x)$, $\phi(t,x) \in E$. It is also convenient to define $t^*(z) = 0$ for $z \in \partial E^*$, and

$$\mathcal{I}_t(x) = \begin{cases} [t, \infty) & \text{if } t^*(x) = \infty \\ [t, t^*(x)] & \text{if } t^*(x) < \infty \end{cases}.$$

We set $\mathbb{B}(\mathbb{X})$ the set of all bounded Borel measurable functions from a Borel set $\mathbb{X}$ into R, and write

$$\|g\| = \sup_{x \in \mathbb{X}} |g(x)|.$$

We define

$$\mathbb{B}^c(E) = \{f \in \mathbb{B}(E); f(\phi(.,x)) : [0, t^*(x)) \mapsto R \text{ is continuous for each } x \in E, \text{ and } lim_{t \uparrow t^*(x)} f(\phi(t,x)) \text{ exists}\}.$$
$$\mathbb{B}^{ac}(E) = \{f \in \mathbb{B}^c(E); f(\phi(.,x)) : [0, t^*(x)) \mapsto R \text{ is absolutely continuous for each } x \in E\}$$
$$\mathbb{B}^{lp}(E) = \{f \in \mathbb{B}^c(E); f(\phi(.,x)) : [0, t^*(x)) \mapsto R \text{ is locally lipschitz continuous for each } x \in E\}$$

and for $f \in \mathbb{B}^c(E)$, $f(z) := lim_{t \uparrow t^*(x)} f(\phi(t,x))$ for all $z = lim_{t \uparrow t^*(x)} \phi(t,x) \in \partial^* E$. For $f \in \mathbb{B}^{ac}(E)$, we denote by $\mathcal{X}h(.)$ a measurable function from E to R such that for any $x \in E$, $t \in \mathcal{Z}(x)$,

$$h(\phi(t,x)) = h(x) + \int_0^t \mathcal{X}h(\phi(s,x))ds.$$

2.2 Continuous Control Parameters

We consider the following parameters for the continuous control problem.

a) $U(.)$ is a Borel set-valued function such that for each $x \in E \cup \partial^* E$, $U(x) \subset \mathbb{U}$, where $\mathbb{U}$ is a non-empty Borel space, and

a.1) $U(x)$ is compact for each point $x \in E \cup \partial^* E$,

a.2) $U(\phi(.,x)) : [0, t^*(x)) \mapsto \mathbb{U}$ is right continuous for each $x \in E$.

b) $\lambda(.,.) : E \times \mathbb{U} \mapsto R_+$ is a bounded Borel measurable function such that for each $x \in E$

$$\lambda(x,.) : U(x) \mapsto R_+ \text{ is continuous,}$$

c) $Q(.;.,.) : E \cup \partial^* E \times \mathbb{U} \mapsto \mathcal{P}(E)$ are such that for each $g \in \mathbb{B}(E)$, and $x \in E$, $z \in \partial^* E$,

c.1) $\lambda(x,.)Qg(x,.) : U(x) \mapsto R$ is lower semi continuous,

c.2) $Qg(z,.) : U(z) \mapsto R$ is lower semi continuous,

where we use the notation,

$$Qg(x,.) = \int_E g(y) Q(dy; x,.).$$

d) $f(.,.) : E \times \mathbb{U} \mapsto R_+$ is a bounded Borel measurable function such that for each $x \in E$,

$$f(x,.) : U(x) \mapsto R \text{ is lower semi continuous.}$$

e) $r(.,.) : \partial^* E \times \mathbb{U} \mapsto R_+$ is a bounded Borel measurable function and for each $z \in \partial^* E$

$$r(z,.) : U(z) \mapsto R \text{ is lower semi continuous.}$$

Define for each $x \in E$

$$V(x) = \{v(.) : \mathcal{I}(x) \mapsto \mathbb{U} \text{ measurable; } v(s) \in U(\phi(s,x)) \text{ for each } s \in \mathcal{I}(x)\}$$

and for any $v \in V(x)$, set $\lambda^v(\phi(s,x)) = \lambda(\phi(s,x), v(s)), Q^v(.;\phi(s,x)) = Q(.;\phi(s,x), v(s)), f^v(\phi(s,x)) = f(\phi(s,x), v(s))$, $r^v(\phi(t^*(x),x)) = r(\phi(t^*(x),x), v(t^*(x)))$ (for $t^*(x) < \infty$). We also define

$$\mathcal{U} = \{u(.) : E \cup \partial^* E \mapsto \mathbb{U} \text{ a measurable function such that } u(\phi(t,x)) \in U(\phi(t,x)) \\ \text{for each } x \in E \text{ and } t \in \mathcal{I}(x)\}.$$

and for $u \in \mathcal{U}$, we write $\lambda^u(x) = \lambda(x, u(x)), Q^u(.;x) = Q(.;x,u(x)), f^u(x) = f(x,u(x))$ and $r^u(z) = r(z, u(z))$ for $z \in \partial^* E$.

Remark 2.1. *From the assumptions b), c.1), c.2), d), and e) we have that for $g_1, g_2 \in \mathbb{B}(E)$,*

$$f(x,.) - \lambda(x,.) \int_E (g_2(x) - g_1(y)) Q(dy; x,.) \text{ and } r(z,.) + Qg_2(z,.)$$

are lower semi-continuous in $U(x)$. Therefore from the compactness assumption in a.1) and the results in Proposition D5) in [20], page 183, there exists a measurable selector $u^* \in \mathcal{U}$ such that for each $x \in E$, $z \in \partial^* E$

$$\begin{aligned} & f^{u^*}(x) - \lambda^{u^*}(x) \int_E (g_2(x) - g_1(y)) Q^{u^*}(dy; x) \\ & = min_{a \in U(x)} \{ f(x,a) - \lambda(x,a) \int_E (g_2(x) - g_1(y)) Q(dy; x, a) \} \\ r^{u^*}(z) + Q^{u^*} g_2(z) & = min_{a \in U(z)} \{ r(z,a) + Qg_2(z,a) \}. \end{aligned} \tag{1}$$

2.3 Impulse Control Parameters

In this subsection we define the set $K(x)$, the points the controller can send the process to after an intervention at x, and the operator M, namely the post intervention operator, which gives the optimal value of the function after an intervention. We will assume that after an intervention at $x \in E$, the process can restart from a set of points in a finite number $\ell(x)$ of trajectories, crossing the points $y_k(x)$, $k = 1, \ldots, \ell(x)$. Thus $y_k(x)$, $k = 1, \ldots, \ell(x)$, are reference points for the flow trajectories the impulse controller can send the process to after an intervention at x. More formally, we consider the following parameters:

i) a Borel measurable function

$$\ell(.) : E \to \mathbb{I}_+$$

where $\mathbb{I}_+ = \{1, 2, , \ldots\}$.

ii) a set of Borel measurable functions

$$y_k(.) : E \to E$$

for $k = 1, \ldots, \ell(.)$

Along the flow $\phi(t, x)$ we assume that the functions $\ell(.)$ and $y_k(.)$ are constant, so that the trajectories to where the process can be sent to after an intervention at $\phi(t, x)$ are kept constant. More formally, we assume that

Assumption I1: For each $x \in E \cup \partial^* E$,

$$\begin{aligned}
&\text{I1.a) } \ell(\phi(t, x)) = \ell(x), \\
&\text{I1.b) } y_k(\phi(t, x)) = y_k(x), k = 1, \ldots, \ell(x) \\
&\text{I1.c) } y_r(x) \neq \phi(s, y_k(x)) \text{ for } k \neq r,\ s \in \mathcal{Z}(y_k(x))
\end{aligned}$$

for all $t \in \mathcal{Z}(x)$.

Assumptions I1.a), I1.b) means that the number $\ell(\phi(t, x))$ and reference points $y_k(\phi(t, x))$, $k = 1, \ldots, \ell(x)$, are kept constant along the flow $\phi(t, x)$, $t \in \mathcal{Z}(x)$. Assumption I1.c) means that there is only one reference point along each flow trajectory $\phi(s, y_k(x))$ for $s \in \mathcal{Z}(y_k(x))$, $k = 1, \ldots, \ell(x)$.

It is convenient to define $\ell(\phi(t^*(x), x)) = \ell(x)$ and $y_k(\phi(t^*(x), x)) = y_k(x)$. The reference points $\{y_1(x), \ldots, y_{\ell(x)}(x)\}$ define the trajectories to where the process can be sent to after an intervention at x. We shall define now for each trajectory $\phi(s, y_k(x))$ the admissible points s to where the system can be sent to after an intervention at x.

iii) we consider $\mathcal{S}_k(.) : E \cup \partial^* E \to \sigma(R)$ a Borel set valued mapping such that

$$\mathcal{S}_k(x) \subset \mathcal{Z}(y_k(x)),\ k \leq \ell(x).$$

This means that when an intervention is taken at x, the set of points $K(x)$ the process can be sent to is given by

$$K(x) = \{\phi(s, y_k(x)); s \in \mathcal{S}_k(x), k = 1, \ldots, \ell(x)\}.$$

We make the following assumptions regarding the sets $\mathcal{S}_k(x)$.

Assumption I2: $\bar{\mathcal{S}}_k(x)$ (the closure of $\mathcal{S}_k(x)$) are compact sets in R for $k = 1, \ldots, \ell(x)$, and for each interval $[a, b] \subset \bar{\mathcal{Z}}(x)$, we can find an interval $[c_k, d_k] \subset \mathcal{Z}(y_k(x))$ such that for $k = 1, \ldots, \ell(x)$,

$$\bigcup_{t \in [a,b]} \bar{\mathcal{S}}_k(\phi(t, x)) \subset [c_k, d_k].$$

Assumption I3: $\bar{\mathcal{S}}_k(\phi(t,x))$ is locally lipschitz continuous in t, that is, for any interval $[a,b] \subset \bar{\mathcal{Z}}(x)$, there exists T_S (which may depend on the interval $[a,b]$) such that

$$d_H(\bar{\mathcal{S}}_k(\phi(t,x)), \bar{\mathcal{S}}_k(\phi(s,x)) \leq T_S \Big| t-s \Big|$$

for all $(t,s) \in [a,b]$, $k = 1, \ldots, \ell(x)$ (where $d_H(.,.)$ is the Hausdorff distance).

We shall now define the cost per intervention parameter $c(.,.)$ associated to the impulse control problem.

iv) $c(.,.) : E \cup \partial^* E \times E \cup \partial^* E \to R_+$ is a Borel measurable function.

Assumption I4: $c(\phi(t,x), \phi(r,y))$ is locally lipschitz continuous in $(t,r) \in \bar{\mathcal{Z}}(x) \times \mathcal{Z}(y)$, that is, for any rectangle $[a,b] \times [c,d] \in \bar{\mathcal{Z}}(x) \times \mathcal{Z}(y)$, there exists T_C (which may depend on the interval $[a,b] \times [c,d]$) such that

$$\Big| c(\phi(t,x), \phi(r,y)) - c(\phi(s,x), \phi(p,y)) \Big| \leq T_C \left(\Big| t-s \Big| + \Big| r-s \Big| \right)$$

for all $(t,s) \times (r,p) \in [a,b] \times [c,d]$.

We define now the post intervention operator M, which gives the optimal value for a function $\varphi \in \mathbb{B}(E)$ after an intervention at x. Define for $x \in E \cup \partial^* E$

$$M\varphi(x) = \min_{k \in \{1,\ldots,\ell(x)\}} M_k \varphi(x) = \inf_{y \in K(x)} (c(x,y) + \varphi(y)).$$

Define $\mathcal{D}_k = \{x \in E \cup \partial^* E; \ell(x) \leq k\}$, which is, by definition, a Borel set. The following results are straightforward (in what follows, we assume that $\varphi \in \mathbb{B}^{ac}(E)$).

Proposition 2.1. *$M_k\varphi(.)$ is measurable. Moreover there is a measurable selector $s_k(.) : \mathcal{D}_k \to E \cup \partial^* E$ such that*

$$M_k\varphi(x) = c(x, \phi(s_k(x), y_k(x))) + \varphi(\phi(s_k(x), y_k(x))),\ s_k(x) \in \bar{\mathcal{S}}_k(x).$$

Proof. From Assumption I4, for any $x \in \mathcal{D}_k$, $c(x, \phi(s, y_k(x))) + \varphi(\phi(s, y_k(x)))$ is continuous in $s \in \bar{\mathcal{S}}_k(x)$, that is, $c(x,.) + \varphi(.)$ is continuous in $\{\phi(s, y_k(x)); s \in \bar{\mathcal{S}}_k(x)\}$. Therefore from Assumption I2,

$$\begin{aligned} M_k\varphi(x) &= \inf_{s \in \mathcal{S}_k(x)} \Big(c(x, \phi(s, y_k(x))) + \varphi(\phi(s, y_k(x))) \Big) \\ &= \min_{s \in \bar{\mathcal{S}}_k(x)} \Big(c(x, \phi(s, y_k(x))) + \varphi(\phi(s, y_k(x))) \Big) \end{aligned}$$

and from Proposition D.5 in [20], we have the result. □

Proposition 2.2. *$M\varphi(.)$ is measurable. Moreover there is a measurable selector $\kappa(.) : E \cup \partial^* E \to \mathbb{I}_+$, such that $\kappa(x) \in \{1, \ldots, \ell(x)\}$ and*

$$M\varphi(x) = c(x, \phi(s_{\kappa(x)}(x), y_{\kappa(x)}(x))) + \varphi(\phi(s_{\kappa(x)}(x), y_{\kappa(x)}(x))).$$

Proof. This is immediate from Proposition 2.1. □

We define the measurable selector $\mathfrak{y}(x) \in \bar{K}(x) (= \{\phi(s, y_k(x)); s \in \bar{\mathcal{S}}_k(x)\})$ for each $x \in E \cup \partial^* E$ in the following way:

$$\mathfrak{y}(x) = \phi(s_{\kappa(x)}(x), y_{\kappa(x)}(x)).$$

Therefore for each $x \in E \cup \partial^* E$,

$$M\varphi(x) = c(x, \mathfrak{y}(x)) + \varphi(\mathfrak{y}(x)),\ \mathfrak{y}(x) \in \bar{K}(x).$$

Finally we consider the following assumptions:

Assumption I5: $c(x,y) \geq c_0$ for some $c_0 > 0$ and every $x \in E \cup \partial^* E$, $y \in \bar{K}(x)$.

Assumption I6: for every $x \in E \cup \partial^* E$, and $y \in \bar{K}(x)$, $\bar{K}(y) \subset \bar{K}(x)$.

Assumption I7: for every $x \in E \cup \partial^* E$, $y \in \bar{K}(x)$, $z \in \bar{K}(y) \subset \bar{K}(x)$, $c(x,y) + c(y,z) > c(x,z)$.

Assumptions I5 implies that each time an intervention is taken, at least c_0 is paid. Assumptions I6 and I7 imply that it is never worth taking two simultaneous interventions.

3 Optimal Stopping with Continuous Control Problem

3.1 Problem Formulation

Let Ω denote the space of right-continuous functions on R_+ taking values in E such that the left limit exists for all $t > 0$, and follows the flow $\phi(.,.)$ between discontinuity points. For $\omega = \omega(t) \in \Omega$, set $\mathcal{F}_t^0 = \sigma\{\omega(s); 0 \leq s \leq t\}$ and $\mathcal{F}^0 = \bigvee_{t \geq 0} \mathcal{F}_t^0$. For $k = 0, 1, \ldots$, let

$$\varpi_k(\omega) = (Z_0(\omega), T_1(\omega), Z_1(\omega), \ldots, T_k(\omega), Z_k(\omega))$$

where $T_i(\omega)$ represents the i^{th} jump of the function $\omega(t)$ and $Z_i(\omega)$ its post jump location (for a more detailed presentation, see [9]). We shall define the set of admissible continuous control strategies Ψ in the following way. A point $\psi = (\zeta_1, \zeta_2, \ldots) \in \Psi$ if it is predictable (see [10], page 264 or [9] for further details) and $\zeta_k : \Omega \times R_+ \to \mathbb{U}$ is a Borel measurable function (see [9]) satisfying

$$\zeta_k(\varpi_{k-1}(\omega), s) \in U(\phi(s, Z_{k-1}(\omega)))$$

for all $s \in \mathcal{I}(Z_{k-1}(\omega))$. We define the motion of the process $\{X_t^\psi\}$ associated to the strategy ψ, starting from a point $x \in E$, in the following way. Take a random variable T_1^ψ such that

$$P_x(T_1^\psi > t) = \begin{cases} exp\{-\int_0^t \lambda^{\zeta_1}(\phi(s,x))ds\} & \text{for } t < t^*(x) \\ 0 & \text{for } t \geq t^*(x) \end{cases}$$

where $\zeta_1(x, .) \in V(x)$ (see subsection 2.2). If T_1^ψ generated according to the above probability is equal to infinity, then $X_t^\psi = \phi(t,x)$ for all $t \geq 0$. Otherwise select independently an E-valued random variable having distribution $Q^{\zeta_1}(.; \phi(T_1^\psi, x))$. The trajectory of $\{X_t^\psi\}$ starting at x, for $t \leq T_1^\psi$, is given by

$$X_t^\psi = \begin{cases} \phi(t,x) & \text{for } t < T_1^\psi \\ Z_1^\psi & \text{for } t = T_1^\psi \end{cases}.$$

Starting from $X_{T_1^\psi}^\psi = Z_1^\psi$, we now select the next inter-jump time $T_2^\psi - T_1^\psi$ and post-jump location $X_{T_2^\psi}^\psi = Z_2^\psi$ in a similar way, using $\zeta_2(x, T_1^\psi, Z_1^\psi, .)$ (which by definition belongs to $V(Z_1^\psi)$ for T_1^ψ

and Z_1^ψ fixed) instead of $\zeta_1(x,.)$. This gives a time-variant piecewise-deterministic trajectory for the process $\{X_t^\psi\}$ with jump times $T_1^\psi, T_2^\psi, \ldots$, and post-jump location $Z_1^\psi, Z_2^\psi, \ldots$. Note that in general we can not guarantee that the process $\{X_t^\psi\}$ is a Markov process, since ζ_k may depend on the whole history of the process up to k^{th}-jump. For the case in which we have, for all $k = 0, 1, \ldots$

$$\zeta_k(\varpi_{k-1}(\omega), .) = u_k(\phi(., Z_{k-1}(\omega)))$$

for $u_k \in \mathcal{U}$ (see subsection 2.2), $k = 0, 1, \ldots$,we write $\psi = (u_1, u_2, \ldots)$ and it can be shown by arguments similar to the ones in [10], page 62-66, that $\{X_t^\psi\}$ is a strong Markov process (although not necessarily homogeneous). The particular case in which $\psi = (u, u, \ldots)$ for some $u \in \mathcal{U}$ corresponds to the definition of a piecewise-deterministic Markov process (PDP for short), as presented in [10], [11]. For this case we shall just write u instead of ψ for all the definitions of the previous section. As proved in [10], [11], $\{X_t^u\}$ is a homogeneous strong Markov process, characterized by the following parameters: the flow $\phi(.,.)$, the jump rate $\lambda^u(.)$, the transition measure $Q^u(.;.)$.

The procedure above defines a family of measures $\{P_x^\psi; x \in E\}$ on $(\Omega, \mathcal{F}^0)$. The final assumption on ψ to be an admissible control is that for every $x \in E$,

$$E_x^\psi(\sum_{k=1}^{\infty} 1_{\{T_k^\psi \leq t\}}) < \infty \tag{2}$$

for all $t \in R_+$. In particular (2) implies that $T_k^\psi \to \infty$ as $k \to \infty$ almost surely. We shall assume that (2) is satisfied for all $\psi = u \in \mathcal{U}$. For any $\mu \in \mathcal{P}(E)$, define P_μ^ψ on $(\Omega, \mathcal{F}^0)$ as

$$P_\mu^\psi(A) = \int_A P_x^\psi(A)\mu(dx).$$

Let $\mathcal{F}_t^{\psi,\mu}$ be the completion of $\mathcal{F}_t^0$ with respect to all P_μ^ψ-null set of $\mathcal{F}^0$, and

$$\mathcal{F}_t^\psi = \bigcap_{\mu \in \mathcal{P}(E)} \mathcal{F}_t^{\psi,\mu}.$$

By the same arguments as in [10], Theorem 25.3, it follows that $\mathcal{F}_t^\psi$ is right-continuous. Define

$$p^{\psi *}(t) = \sum_{i=1}^{\infty} 1_{\{t \geq T_i^\psi\}} 1_{\{X^\psi_{T_i(\omega)-} \in \partial^* E\}}$$

which is a measure that counts the number of jumps from the boundary. We make the following assumption:

Assumption I8: We assume that there exists $d_1 \geq 0$ such that for some $\psi \in \Psi$,

$$E_x^\psi(\int_0^\infty e^{-\alpha s} r^\psi(X_{s-}^\psi) dp^{\psi *}(s)) \leq d_1, \, \forall \, x \in E. \tag{3}$$

Associated to an admissible control strategy ψ and an $\mathcal{F}_t^\psi$-stopping time τ, we have the following cost, for $\alpha > 0$:

$$\mathcal{J}^{\psi,\tau}(x) = E_x^\psi(\int_0^\tau e^{-\alpha s} f^\psi(X_s^\psi) ds + \int_0^\tau e^{-\alpha s} r^\psi(X_{s-}^\psi) dp^{\psi *}(s) + e^{-\alpha\tau}\varphi(X_\tau^\psi)), \tag{4}$$

where $\varphi \in \mathbb{B}^{ac}(E)$ and

$$f^\psi(X_s^\psi(\omega)) = f(\phi(s - T_{i-1}^\psi(\omega), Z_{i-1}^\psi(\omega)), \zeta_i(\varpi_{i-1}(\omega), s - T_{i-1}^\psi(\omega))) \text{ for } T_{i-1}^\psi(\omega) \leq s < T_i^\psi(\omega)$$
$$r^\psi(X_{s-}^\psi(\omega)) = r(X^\psi_{T_i(\omega)-}(\omega), \zeta_i(\varpi_{i-1}(\omega), T_i^\psi(\omega))) \text{ for } s = T_i^\psi(\omega) \text{ and } X^\psi_{T_i(\omega)-}(\omega) \in \partial^* E.$$

We consider the following optimal stopping with continuous control problem:

$$\rho(x) = \inf_{\psi \in \Psi} \inf_{\tau \in \mathcal{M}^{\psi}} \mathcal{J}^{\psi,\tau}(x) \tag{5}$$

where $\mathcal{M}^{\psi}$ denotes the set of all $\mathcal{F}_t^{\psi}$-stopping times.

3.2 First Jump Operator

We shall now define the first jump operator associated to the optimal stopping with continuous control problem posed in subsection 3.1. For $h_2 \in \mathbb{B}(E)$, $h_1 \in \mathbb{B}^{ac}(E)$, define for each $x \in E$ and $v \in V(x)$,

$$J^v(h_1,h_2)(t,x) = \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds + e^{-\alpha t - \Lambda^v(t,x)} h_1(\phi(t,x))$$

where

$$Q^v h_2(\phi(s,x)) = \int_E h_2(y) Q(dy; \phi(s,x), v(s)),\ \Lambda^v(t,x) = \int_0^t \lambda(\phi(s,x), v(s))ds$$

and

$$K^v(h_2)(x) = \int_0^{t^*(x)} e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_2(\phi(s,x)))ds + e^{-\alpha t^*(x) - \Lambda^v(t^*(x),x)}(r^v(\phi(t^*(x),x)) + Q^v h_2(\phi(t^*(x),x))).$$

Set

$$L(h_1,h_2)(x) = \inf_{v \in V(x)} \Big(\Big\{ \inf_{0 \le t < t^*(x)} J^v(h_1,h_2)(t,x) \Big\} \wedge K^v(h_2)(x) \Big). \tag{6}$$

The operator $L(.,.)(.)$ defined above is the first jump operator.

3.3 Main Results

The following result was proved in [9], and characterizes the value function $\rho(x)$ as the unique fixed point of $L(\varphi, .)$ in $\mathbb{B}^{ac}(E)$ or, equivalently, as the unique solution in $\mathbb{B}^{ac}(E)$ of a set of variational inequalities.

Theorem 3.1. *The following assertions are equivalent:*
1) There exists a unique solution ϑ in the class $\mathbb{B}^{ac}(E)$ to the system

$$w(x) = L(\varphi, w)(x)$$

for all $x \in E$.
2) There exists a unique solution ϱ in the class $\mathbb{B}^{ac}(E)$ to the system,

$$\begin{aligned}
&2.a)\ w(x) \le \varphi(x)\\
&2.b) \mathcal{X}w(x) - \alpha w(x)\\
&+ \min_{a \in U(x)} \{ f(x,a) - \lambda(x,a) \int_E (w(x) - w(y)) Q(dy; x,a) \} \ge 0\\
&2.c) \Big(\varphi(x) - w(x)\Big)\Big(\mathcal{X}w(x) - \alpha w(x)\ +\\
&\min_{a \in U(x)} \{ f(x,a) - \lambda(x,a) \int_E (w(x) - w(y)) Q(dy; x,a) \}\Big) = 0
\end{aligned}$$

for all $x \in E$, and

$$2.d)\ w(z) = \varphi(z) \wedge min_{a \in U(z)} \{r(z,a) + Qw(z,a)\}$$

for $z \in \partial^* E$. Moreover, $\rho(x) = \varrho(x) = \vartheta(x)$ for all $x \in E$ and, given any $\epsilon > 0$, we have that

$$\mathcal{J}^{u^*,\tau_\epsilon}(x) \le \rho(x) + \epsilon$$

where $u^* \in \mathcal{U}$ is as defined in (1) with $\rho(x)$ replacing $g_1(x)$, $g_2(x)$, and $\tau_\epsilon \in \mathcal{M}^{u^*}$ as

$$\tau_\epsilon = inf\{t \ge 0; e^{-\alpha t}\varphi(X_t^{u^*}) \le e^{-\alpha t}\vartheta(X_t^{u^*}) + \epsilon\}.$$

4 Impulse and Continuous Control Problem

4.1 Problem Formulation

The formal definition of the construction of the state space of the impulse and continuous control problem follows the same steps as in Davis [10], and can be found in the Appendix I. Associated to an admissible strategy $\xi = \{(\psi_k, R_k, S_k),\ k = 1, 2, \ldots\} \in \mathbb{S}$ (see Appendix I), we define the following cost $\mathbb{J}^\xi(x)$:

$$\mathbb{J}^\xi(x) = E_x^\xi\Big(\int_0^\infty e^{-\alpha s} f^\xi(Y_s^\xi) ds + \int_0^\infty e^{-\alpha s} r^\xi(Y_{s-}^\xi) 1_{\{Y_{s-}^\xi \in \partial^* E\}} ds + \sum_{k=1}^\infty e^{-\alpha \tau_k} c(Y_{\tau_k}, Y_{\tau_k^+})\Big)$$

where E_x^ξ denotes the expectation with respect to the measure $\mathbb{P}_x^\xi$ on $\mathcal{W}$, $Y_t^\xi(w)$ is the sample path associated to the strategy ξ, τ_k is the k^{th} intervention time (see Appendix I) and

$$f^\xi(Y_t^\xi(w)) = f^{\psi_{i+1}}(\hat{\omega}_{i+1}(t - \tau_i(w))) \text{ for } \tau_i(w) < t \le \tau_{i+1}(\omega),$$
$$r^\xi(Y_{t-}^\xi(w)) = r^{\psi_{i+1}}(\hat{\omega}_{i+1}((t - \tau_i(w))^-)) \text{ for } \tau_i(w) < t \le \tau_{i+1}(\omega).$$

We want to solve the following impulse with continuous control problem:

$$\hat{\rho}(x) = \inf_{\xi \in \mathbb{S}} \mathbb{J}^\xi(x).$$

We will consider also the following auxiliary problem:

$$\hat{\rho}_n(x) = \inf_{\xi \in \mathbb{S}_n} \mathbb{J}^\xi(x),$$

where, as defined in Appendix 1, $\mathbb{S}_n$ denotes the set of admissible strategies with at most n interventions. Since $\mathbb{S}_n \subset \mathbb{S}$, we have that $\hat{\rho}(x) \le \hat{\rho}_n(x)$.

4.2 Property of the Operator M

In this section we shall present a property of the operator M which will be crucial for the development of the main results of our paper.

Proposition 4.1. *M maps $\mathbb{B}^{ac}(E)$ into $\mathbb{B}^{ac}(E)$.*

Proof. Consider $\varphi(.) \in \mathbb{B}^{ac}(E)$. Let us prove that $M_k\varphi(\phi(.,x)) : \mathcal{I}(x) \to R$ is absolutely continuous for each $k = 1, \dots, \ell(x)$. Consider an arbitrary interval $[a,b] \subset \mathcal{I}(x)$ and c_k, d_k such that (see Assumption I2)

$$\underset{t\in[a,b]}{\cup} \bar{\mathcal{S}}_k(\phi(t,x)) \subset [c_k, d_k] \subset \mathcal{Z}(y_k(x)).$$

We have (see proof of Proposition 2.1) that

$$\begin{aligned} &M_k\varphi(\phi(t,x)) - M_k\varphi(\phi(s,x)) = \\ &\underset{r\in\bar{\mathcal{S}}_k(\phi(t,x))}{min}\Big(c(\phi(t,x),\phi(r,y_k(x))) + \varphi(\phi(r,y_k(x)))\Big) - \\ &\underset{p\in\bar{\mathcal{S}}_k(\phi(s,x))}{min}\Big(c(\phi(s,x),\phi(p,y_k(x))) + \varphi(\phi(p,y_k(x)))\Big). \end{aligned} \tag{7}$$

Consider $\varsigma = max\{T_{C,k}; k = 1, \dots, \ell(x)\} \vee T_S$ where $T_{C,k}$ is as in Assumption I4 for $[a,b]\times[c_k,d_k]$, and T_S as in Assumptions I3. Therefore,

$$\Big|c(\phi(t,x),\phi(r,y(x))) - c(\phi(s,x),\phi(p,y(x)))\Big| \le \varsigma\Big(\Big|t-s\Big| + \Big|r-p\Big|\Big) \tag{8}$$

and

$$d_H(\bar{\mathcal{S}}_k(\phi(t,x)), \bar{\mathcal{S}}_k(\phi(s,x))) \le \varsigma\Big|t-s\Big| \tag{9}$$

for all $(t,s)\times(r,p) \in [a,b]\times[c_k,d_k]$. Choose $p \in \bar{\mathcal{S}}_k(\phi(s,x))$ such that

$$c(\phi(s,x),\phi(p,y_k(x))) + \varphi(\phi(p,y_k(x))) = M_k\varphi(\phi(s,x)). \tag{10}$$

From (9) we can find $r \in \bar{\mathcal{S}}_k(\phi(t,x))$ such that

$$\Big|r-p\Big| \le d_H(\bar{\mathcal{S}}_k(\phi(t,x)), \bar{\mathcal{S}}_k(\phi(s,x))) \le \varsigma\Big|t-s\Big|. \tag{11}$$

Therefore, from (7)-(11),

$$\begin{aligned} &M_k\varphi(\phi(t,x)) - M_k\varphi(\phi(s,x)) \le \Big(c(\phi(t,x),\phi(r,y_k(x))) + \\ &\varphi(\phi(r,y_k(x)))\Big) - \Big(c(\phi(s,x),\phi(p,y_k(x))) + \varphi(\phi(p,y_k(x)))\Big)\Big) \le \\ &\Big|c(\phi(t,x),\phi(r,y_k(x))) - c(\phi(s,x),\phi(p,y_k(x)))\Big| + \\ &\Big|\varphi(\phi(r,y_k(x))) - \varphi(\phi(p,y_k(x)))\Big| \le \\ &\varsigma\Big(\Big|t-s\Big| + \Big|r-p\Big|\Big) + \Big|\varphi(\phi(r,y_k(x))) - \varphi(\phi(p,y_k(x)))\Big| \le \\ &\varsigma\Big(\varsigma+1\Big)\Big|t-s\Big| + \Big|\varphi(\phi(r,y_k(x))) - \varphi(\phi(p,y_k(x)))\Big|. \end{aligned} \tag{12}$$

Given $\epsilon > 0$, we have from the fact that $\varphi \in \mathbb{B}^{ac}(E)$ that we can find $\delta' > 0$ such that whenever $\{(r_{i,k}, p_{i,k}); i = 1, \dots, n\}$ are disjoint intervals in $[c_k, d_k]$, satisfying

$$\sum_{i=1}^{n}\Big|r_{i,k} - p_{i,k}\Big| < \delta' \tag{13}$$

we have

$$\sum_{i=1}^{n} \Big| \varphi(\phi(r_{i,k}, y_k(x))) - \varphi(\phi(p_{i,k}, y_k(x))) \Big| < \frac{1}{2}\epsilon. \tag{14}$$

Set

$$\delta = min\{\frac{\delta'}{\varsigma}, \frac{1}{2(\varsigma+1)\varsigma}\epsilon\} \tag{15}$$

and consider an arbitrary sequence $\{(t_i, s_i); i = 1, \ldots, n\}$ of disjoint intervals in $[a, b]$ satisfying

$$\sum_{i=1}^{n} \Big| t_i - s_i \Big| < \delta. \tag{16}$$

Consider $r_{i,k} \in \bar{\mathcal{S}}_k(\phi(t_i, x))$ and $p_{i,k} \in \bar{\mathcal{S}}_k(\phi(s_k, x))$ such that (10) and (11) are satisfied replacing t, r, s, p by $t_i, r_{i,k}, s_i, p_{i,k}$ respectively. Therefore from (11) and (15),

$$\sum_{i=1}^{n} \Big| r_{i,k} - p_{i,k} \Big| \le \varsigma \sum_{i=1}^{n} \Big| t_i - s_i \Big| < \varsigma\delta \le \delta'$$

and therefore from (13) we have that (14) is satisfied. From (12), (14) and (15),

$$\begin{aligned}\sum_{i=1}^{n} \Big(M_k\varphi(\phi(t_i, x)) - M_k\varphi(\phi(s_i, x)) \Big) &< \varsigma\Big(\varsigma+1\Big) \sum_{i=1}^{n} \Big| t_i - s_i \Big| + \frac{1}{2}\epsilon \\ &\le \varsigma\Big(\varsigma+1\Big)\delta + \frac{1}{2}\epsilon \le \epsilon.\end{aligned}$$

Similarly it can be shown that

$$\sum_{i=1}^{n} \Big(M_k\varphi(\phi(s_i, x)) - M_k\varphi(\phi(t_i, x)) \Big) < \epsilon$$

and thus, if (16) is satisfied then

$$\sum_{i=1}^{n} \Big| M_k\varphi(\phi(t_i, x)) - M_k\varphi(\phi(s_i, x)) \Big| \le \epsilon$$

proving that $M_k\varphi(\phi(t, x))$ is absolutely continuous in t. Notice now that

$$M\varphi(\phi(t, x)) = \underset{k \in \{1,\ldots,\ell(x)\}}{min} M_k\varphi(\phi(t, x))$$

so that

$$\begin{aligned}\sum_{i=1}^{n} (M\varphi(\phi(t_i, x)) - M\varphi(\phi(s_i, x))) &\le \sum_{k=1}^{\ell(x)} \sum_{i=1}^{n} (M_k\varphi(\phi(t_i, x)) - M_k\varphi(\phi(s_i, x))) \\ &< \ell(x)\epsilon\end{aligned}$$

and similarly

$$\sum_{i=1}^{n}(M\varphi(\phi(s_i,x)) - M\varphi(\phi(t_i,x))) < \ell(x)\epsilon$$

showing that

$$\sum_{i=1}^{n}\Big|M\varphi(\phi(s_i,x)) - M\varphi(\phi(t_i,x))\Big| < \ell(x)\epsilon.$$

Since $t^*(x) \in \mathcal{I}(x)$ if $t^*(x) < \infty$, we have from the above results that

$$\lim_{t\to t^*(x)} M\varphi(\phi(t,x)) = M\varphi(\phi(t^*(x),x))$$

completing the proof. $\square$

4.3 The Cost of No-Intervention

Set (see (3) for definition of d_1)

$$b = \|f\| + \alpha d_1.$$

Consider the following auxiliary problems: $h_0(x) = \frac{b}{\alpha}$ and

$$h(x) = \inf_{\psi\in\Psi} E_x^{\psi}\Big(\int_0^{\infty} e^{-\alpha s} f^{\psi}(X_s^{\psi})ds + \int_0^{\infty} e^{-\alpha s} r(X_{s-}^{\psi})1_{\{X_{s-}^{\psi}\in\partial^* E\}}ds\Big),$$
$$h_n(x) = \inf_{\psi\in\Psi} E_x^{\psi}\Big(\int_0^{T_n^{\psi}} e^{-\alpha s} f^{\psi}(X_s^{\psi})ds + \int_0^{T_n^{\psi}} e^{-\alpha s} r(X_{s-}^{\psi})1_{\{X_{s-}^{\psi}\in\partial^* E\}}ds + e^{-\alpha T_n^{\psi}}\frac{b}{\alpha}\Big).$$

Therefore h denotes the optimal cost for the continuous control without any intervention. From the fact that $0\leq f^{\psi}(x) \leq \|f\|$, and (3), it follows that

$$E_x^{\psi}\Big(\int_0^{\infty} e^{-\alpha s} f^{\psi}(X_s^{\psi})ds + \int_0^{\infty} e^{-\alpha s} r(X_{s-}^{\psi})1_{\{X_{s-}^{\psi}\in\partial^* E\}}ds\Big) \leq$$
$$E_x^{\psi}\Big(\int_0^{T_{n+1}^{\psi}} e^{-\alpha s} f^{\psi}(X_s^{\psi})ds + \int_0^{T_{n+1}^{\psi}} e^{-\alpha s} r(X_{s-}^{\psi})1_{\{X_{s-}^{\psi}\in\partial^* E\}}ds + e^{-\alpha T_{n+1}^{\psi}}\frac{b}{\alpha}\Big) \leq$$
$$E_x^{\psi}\Big(\int_0^{T_n^{\psi}} e^{-\alpha s} f^{\psi}(X_s^{\psi})ds + \int_0^{T_n^{\psi}} e^{-\alpha s} r(X_{s-}^{\psi})1_{\{X_{s-}^{\psi}\in\partial^* E\}}ds + e^{-\alpha T_n^{\psi}}\frac{b}{\alpha}\Big) \leq \frac{b}{\alpha}$$

and therefore, $0\leq h(x) \leq h_{n+1}(x) \leq h_n(x) \leq \frac{b}{\alpha}$. For any $v \in V(x)$ and $g \in \mathbb{B}(E)$, define,

$$\mathcal{N}^v(g)(t,x) = \inf_{w\in V(\phi(t,x))} K^{q_t(v,w)}(g)(x)$$

where $q_t(v,w) \in V(x)$ is defined as $q_t(v,w) = v(s)$, $0 \leq s < t$, $w(s)$ for $s \geq t$, $s \in \mathcal{I}(x)$. By repeating the same reasoning as in the proof of Proposition 4.5 in [9], we have that $\mathcal{N}^v(g)(.,x) \in \mathbb{B}^{ac}(E)$ whenever $g \in \mathbb{B}(E)$.

Proposition 4.2. *$h \in \mathbb{B}^{ac}(E)$.*

Proof. As in the proof of Proposition 4.1 of [9], for any $v \in V(x)$, $t \in [0, t^*(x))$,

$$h_{n+1}(\phi(t,x)) = \Big\{\mathcal{N}^v(h_n)(t,x) - \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v h_n(\phi(s,x)))ds\Big\} e^{\alpha t + \Lambda^v(t,x)}. \tag{17}$$

Consider $\psi_\epsilon \in \Psi$ such that

$$E_x^{\psi_\epsilon}\Big(\int_0^\infty e^{-\alpha s} f^{\psi_\epsilon}(X_s^{\psi_\epsilon})ds + \int_0^\infty e^{-\alpha s} r(X_{s-}^{\psi_\epsilon}) 1_{\{X_{s-}^{\psi_\epsilon} \in \partial^* E\}} ds\Big) \leq h(x) + \epsilon.$$

Since

$$E_x^{\psi_\epsilon}\Big(\int_0^\infty e^{-\alpha s} f^{\psi_\epsilon}(X_s^{\psi_\epsilon})ds + \int_0^\infty e^{-\alpha s} r(X_{s-}^{\psi_\epsilon}) 1_{\{X_{s-}^{\psi_\epsilon} \in \partial^* E\}} ds\Big) \geq E_x^{\psi_\epsilon}\Big(\int_0^{T_n^{\psi_\epsilon}} e^{-\alpha s} f^{\psi_\epsilon}(X_s^{\psi_\epsilon})ds + \int_0^{T_n^{\psi_\epsilon}} e^{-\alpha s} r(X_{s-}^{\psi_\epsilon}) 1_{\{X_{s-}^{\psi_\epsilon} \in \partial^* E\}} ds\Big)$$

it follows that

$$0 \leq h_n(x) - h(x) \leq e^{-\alpha T_n^{\psi_\epsilon}} \frac{b}{\alpha} + \epsilon$$

and recalling by assumption that $T_n^{\psi_\epsilon} \to \infty$ a.s., we obtain that $h_n(x) \downarrow h(x)$ as $n \to \infty$. From (17) and the bounded convergence Theorem, it is clear that

$$\begin{aligned} h(\phi(t,x)) &= \lim_{n\to\infty} h_{n+1}(\phi(t,x)) \\ &= \Big\{ \inf_n \{\mathcal{N}^v(h_n)(t,x)\} - \lim_{n\to\infty} \Big\{ \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) + \\ &\lambda^v(\phi(s,x))Q^v h_n(\phi(s,x)))ds\Big\} e^{\alpha t + \Lambda^v(t,x)} \Big\} \\ &= \Big\{\mathcal{N}^v(\inf_n h_n)(t,x) - \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) \\ &+ \lambda^v(\phi(s,x))Q^v(\lim_{n\to\infty} h_n(\phi(s,x))))ds\Big\} e^{\alpha t + \Lambda^v(t,x)} \\ &= \Big\{\mathcal{N}^v(h)(t,x)\Big\} - \int_0^t e^{-\alpha s - \Lambda^v(s,x)}(f^v(\phi(s,x)) \\ &+ \lambda^v(\phi(s,x))Q^v h(\phi(s,x)))ds\Big\} e^{\alpha t + \Lambda^v(t,x)}. \end{aligned} \tag{18}$$

and since each term on the right hand side of the last equality belongs to $\mathbb{B}^{ac}(E)$ (recall that $h \in \mathbb{B}(E)$), we have that h is absolutely continuous along trajectories. From (18)

$$h(x) = \inf_{v \in V(x)} K^v(h)(x).$$

and therefore, as $t \uparrow t^*(x)$, we get that

$$h(\phi(t,x)) \to \min_{a \in U(z)} \{r(z,a) + Qh(z,a)\}$$

where $z = \phi(t^*(x), x)$. Thus, $h \in \mathbb{B}^{ac}(E)$. □

By repeating the same arguments as in the proof of Theorem 4.1 in [9], we get that

$$\mathcal{X}h(x) - \alpha h(x) + min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (h(x) - h(y))Q(dy,x,a)\} = 0,\ x \in E,$$
$$h(z) = \min_{a\in U(z)}\{r(z,a) + Qh(z,a)\},\ z \in \partial^* E. \tag{19}$$

In particular we have from (19) that (recall that $\|h\| \leq \frac{b}{\alpha}$)

$$\|\mathcal{X}h\| \leq \gamma,\ \gamma := \|f\| + \frac{b}{\alpha}(\alpha + 2\|\lambda\|). \tag{20}$$

5 Main Results

5.1 The Sequence of Optimal Stopping with Continuous Control Problem

Define the following operator $\mathcal{G}$ on $\mathbb{B}(E)$:

$$\mathcal{G}\varphi(x) = \underset{\psi\in\Psi}{inf}\ \underset{\tau\in\mathcal{M}^\psi}{inf}\ E_x^\psi\Big(\int_0^\tau e^{-\alpha s} f^\psi(X_s^\psi)ds + \int_0^\tau e^{-\alpha s} r^\psi(X_{s-}^\psi)dp^{\psi *}(s) + e^{-\alpha\tau}M\varphi(X_\tau^\psi)\Big).$$

By taking $\tau = \infty$ it follows that $\mathcal{G}\varphi(x) \leq h(x)$ for any $\psi(x) \in \mathbb{B}(E)$. It is easy to see that M and $\mathcal{G}$ are monotone operators, that is, if $\varphi_1 \leq \varphi_2$ then $M\varphi_1 \leq M\varphi_2$ and $\mathcal{G}\varphi_1 \leq \mathcal{G}\varphi_2$.

Proposition 5.1. *$\mathcal{G}$ maps $\mathbb{B}^{ac}(E)$ into $\mathbb{B}^{ac}(E)$.*

Proof. The result follows from Proposition 4.1 (M maps $\mathbb{B}^{ac}(E)$ into $\mathbb{B}^{ac}(E)$) and Theorem 3.1 in Section 3. □

Set now the following sequence of functions $\{\hat{\vartheta}_n\}$ in $\mathbb{B}^{ac}(E)$:

$$\hat{\vartheta}_{n+1} = \mathcal{G}\hat{\vartheta}_n,\ n = 0, 1, \ldots,\ \hat{\vartheta}_0 = h.$$

Note that since $h \in \mathbb{B}^{ac}(E)$, it follows from Proposition 5.1 that indeed $\hat{\vartheta}_n \in \mathbb{B}^{ac}(E)$ for all $n = 0, 1, \ldots$. Note also that $0 \leq \hat{\vartheta}_1 = \mathcal{G}\hat{\vartheta}_0 = \mathcal{G}h \leq h$, and from the monotone property of M and $\mathcal{G}$, $0 \leq \hat{\vartheta}_{n+1} \leq \hat{\vartheta}_n \leq h$ for all $n = 0, 1, \ldots$. From the results for the optimal stopping with continuous control problem of PDP's (Theorem 3.1 in Section 3), we have that $\{\hat{\vartheta}_n\}$ satisfies, for $n \geq 1$, the following variational inequalities for $x \in E$,:

$$\begin{aligned}
&\hat{\vartheta}_{n+1}(x) \leq M\hat{\vartheta}_n(x)\\
&\mathcal{X}\hat{\vartheta}_{n+1}(x) - \alpha\hat{\vartheta}_{n+1}(x)\\
&+ min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (\hat{\vartheta}_{n+1} - \hat{\vartheta}_{n+1}(y))Q(dy;x,a)\} \geq 0\\
&\Big(M\hat{\vartheta}_n(x) - \hat{\vartheta}_{n+1}(x)\Big)\Big(\mathcal{X}\hat{\vartheta}_{n+1}(x) - \alpha\hat{\vartheta}_{n+1}(x) +\\
&min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (\hat{\vartheta}_{n+1}(x) - \hat{\vartheta}_{n+1}(y))Q(dy;x,a)\}\Big) = 0\\
&\hat{\vartheta}_{n+1}(z) = M\hat{\vartheta}_n(z) \wedge min_{a\in U(z)}\{r(z,a) + Q\hat{\vartheta}_{n+1}(z,a)\},\ z \in \partial^* E.
\end{aligned} \tag{21}$$

and for $n = 0$, $\hat{\vartheta}_0(x) = h(x)$ satisfies (19), with (see (20)) $\|\mathcal{X}h\| \leq \gamma$. The next result follows from arguments similar to those presented in [10], and the proof is presented in Appendix II.

Proposition 5.2. *For $n = 1, 2, \ldots$, $\hat{\vartheta}_n(x) = \hat{\rho}_n(x)$.*

Proof. See Appendix II. $\square$

Proposition 5.3. *Consider $y \in E$ such that $\hat{\vartheta}_{n+1}(y) < M\hat{\vartheta}_n(y)$. Then,*

$$\left|\mathcal{X}\hat{\vartheta}_{n+1}(y)\right| \leq \gamma.$$

Proof. If $\hat{\vartheta}_{n+1}(y) < M\hat{\vartheta}_n(y)$ then from (21) we must have that

$$\begin{aligned}
&\mathcal{X}\hat{\vartheta}_{n+1}(y) - \alpha\hat{\vartheta}_{n+1}(y) + \\
&min_{a \in U(y)}\{f(y,a) - \lambda(y,a)\int_E (\hat{\vartheta}_{n+1}(y) - \hat{\vartheta}_{n+1}(z))Q(dz; y, a)\} = 0.
\end{aligned}$$

Therefore,

$$\left|\mathcal{X}\hat{\vartheta}_{n+1}(y)\right| \leq \|f\| + \left\|\hat{\vartheta}_{n+1}\right\|(2\|\lambda\| + \alpha)$$

and since $0 \leq \hat{\vartheta}_{n+1} \leq h$, it follows that $\left\|\hat{\vartheta}_{n+1}\right\| \leq \frac{b}{\alpha}$ and we obtain the desired result. $\square$

Proposition 5.4. *Consider $y \in E$ such that $\hat{\vartheta}_{n+1}(y) = M\hat{\vartheta}_n(y)$. Then for $\mathfrak{y}(y) \in \bar{K}(y)$ (see subsection 2.3) so that*

$$M\hat{\vartheta}_n(y) = c(y, \mathfrak{y}(y)) + \hat{\vartheta}_n(\mathfrak{y}(y))$$

we can find $\epsilon_n(y) > 0$ such that

$$\left|\hat{\vartheta}_n(\phi(s, \mathfrak{y}(y))) - \hat{\vartheta}_n(\mathfrak{y}(y))\right| \leq \gamma\left|s\right| \tag{22}$$

for all $s \in \mathcal{Z}(\mathfrak{y}(y))$ such that $\left|s\right| \leq \epsilon_n(y)$.

Proof. Suppose first that $n > 0$. Let us show that $\hat{\vartheta}_n(\mathfrak{y}(y)) < M\hat{\vartheta}_{n-1}(\mathfrak{y}(y))$. Suppose by contradiction that

$$\hat{\vartheta}_n(\mathfrak{y}(y)) = M\hat{\vartheta}_{n-1}(\mathfrak{y}(y)).$$

Consider $z \subset \bar{K}(\mathfrak{y}(y))(\subset \bar{K}(y)$ from Assumption I6)) such that

$$M\hat{\vartheta}_{n-1}(\mathfrak{y}(y)) = c(\mathfrak{y}(y), z) + \hat{\vartheta}_{n-1}(z).$$

Then from Assumption I7 and the fact that $\hat{\vartheta}_{n-1}(z) \geq \hat{\vartheta}_n(z)$,

$$\begin{aligned}
M\hat{\vartheta}_n(y) &= c(y, \mathfrak{y}(y)) + \hat{\vartheta}_n(\mathfrak{y}(y)) = c(y, \mathfrak{y}(y)) + M\hat{\vartheta}_{n-1}(\mathfrak{y}(y)) \\
&= c(y, \mathfrak{y}(y)) + c(\mathfrak{y}(y), z) + \hat{\vartheta}_{n-1}(z) \\
&\geq c(y, \mathfrak{y}(y)) + c(\mathfrak{y}(y), z) + \hat{\vartheta}_n(z) \\
&> c(y, z) + \hat{\vartheta}_n(z) \geq M\hat{\vartheta}_n(y)
\end{aligned}$$

which is a contradiction. Therefore we must have $\hat{\vartheta}_n(\mathfrak{y}(y)) < M\hat{\vartheta}_{n-1}(\mathfrak{y}(y))$, and from continuity in s of $\hat{\vartheta}_n(\phi(s, \mathfrak{y}(y)))$, $M\hat{\vartheta}_{n-1}(\phi(s, \mathfrak{y}(y))))$, we have that for some $\epsilon_n(y) > 0$,

$$\hat{\vartheta}_n(\phi(s, \mathfrak{y}(y))) < M\hat{\vartheta}_{n-1}(\phi(s, \mathfrak{y}(y)))) \text{ for all } s \in \mathcal{Z}(\mathfrak{y}(y)) \text{ such that } \left|s\right| \leq \epsilon_n(y).$$

From Proposition 5.3, $\left|\mathcal{X}\hat{\vartheta}_n(\phi(s,\mathfrak{y}(y)))\right| \leq \gamma$ for all $s \in \mathcal{Z}(\mathfrak{y}(y))$ such that $\left|s\right| \leq \epsilon_n(y)$, and therefore, since $\hat{\vartheta}_n \in B^{ac}(E)$

$$\left|\hat{\vartheta}_n(\phi(s,\mathfrak{y}(y))) - \hat{\vartheta}_n(\mathfrak{y}(y))\right| = \Big|\int_0^s \mathcal{X}\hat{\vartheta}_n(\phi(\ell,\mathfrak{y}(y)))d\ell\Big| \leq \int_0^{\left|s\right|} \left|\mathcal{X}\hat{\vartheta}_n(\phi(\ell,\mathfrak{y}(y)))\right|d\ell \leq \left|s\right|\gamma$$

for all $s \in \mathcal{Z}(\mathfrak{y}(y))$ such that $\left|s\right| \leq \epsilon_n(y)$. For $n = 0$ we have that $\|\mathcal{X}h\| \leq \gamma$ and the same reasoning as above applies, proving the Proposition. □

Proposition 5.5. *Consider a fixed $x \in E$ and an arbitrary interval $[a,b] \subset [0,t^*(x))$. Consider ς as in the proof of Proposition 4.1 and define $\mathit{\Gamma} = \varsigma\Big(\varsigma + 1 + \gamma\Big) + \gamma > 0$. Then if $t \in [a,b]$ is such that $\frac{d\hat{\vartheta}_{n+1}(\phi(t,x))}{dt}$ exists, we have*

$$\Big|\frac{d\hat{\vartheta}_{n+1}(\phi(t,x))}{dt}\Big| \leq \mathit{\Gamma}.$$

Proof. Set $y = \phi(t,x)$. Suppose first that $\hat{\vartheta}_{n+1}(y) < M\hat{\vartheta}_n(y)$. Then from Proposition 5.3,

$$\left|\mathcal{X}\hat{\vartheta}_{n+1}(y)\right| \leq \gamma < \mathit{\Gamma}.$$

Suppose now that $\hat{\vartheta}_{n+1}(y) = M\hat{\vartheta}_n(y)$. From (21) we get that

$$\mathcal{X}\hat{\vartheta}_{n+1}(y) \geq -\gamma > -\mathit{\Gamma}.$$

From Proposition 5.4 we can find $\epsilon_n(y) > 0$ such that (22) is satisfied for all $s \in \mathcal{Z}(\mathfrak{y}(y))$ with $\left|s\right| \leq \epsilon_n(y)$. From (11), (12), (22), and $0 < \ell \leq \frac{\epsilon_n(y)}{\varsigma}$, we have

$$M\hat{\vartheta}_n(\phi(\ell,y)) - M\hat{\vartheta}_n(y) \leq \varsigma\Big(\varsigma + 1 + \gamma\Big)\ell$$

that is,

$$\hat{\vartheta}_{n+1}(\phi(\ell,y)) - \hat{\vartheta}_n(y) \leq M\hat{\vartheta}_n(\phi(\ell,y)) - M\hat{\vartheta}_n(y) \leq \mathit{\Gamma}\ell$$

and taking the limit as $\ell \downarrow 0$ we get that

$$\mathcal{X}\hat{\vartheta}_{n+1}(y) \leq \mathit{\Gamma}.$$

Therefore, $-\mathit{\Gamma} \leq \mathcal{X}\hat{\vartheta}_{n+1}(y) \leq \mathit{\Gamma}$, and the result follows. □

Proposition 5.6. *Consider the same conditions and notations as in Proposition 5.5. Then for any $t, t+\ell$ in $[a,b]$, we have that*

$$\left|\hat{\vartheta}_n(\phi(t+\ell,x)) - \hat{\vartheta}_n(\phi(t,x))\right| \leq \mathit{\Gamma}\left|\ell\right|. \tag{23}$$

Proof. From Proposition 5.1, $\{\hat{\vartheta}_n\}$ in $\mathbb{B}^{ac}(E)$, and therefore,

$$\hat{\vartheta}_n(\phi(t+\ell,x)) - \hat{\vartheta}_n(\phi(t,x)) = \int_t^{t+\ell} \mathcal{X}\hat{\vartheta}_n(\phi(s,x))ds. \tag{24}$$

From Proposition 5.5, $\left|\mathcal{X}\hat{\vartheta}_n(\phi(t,x))\right| \leq \mathit{\Gamma}$, and therefore from (24),

$$\left|\hat{\vartheta}_n(\phi(t+\ell,x)) - \hat{\vartheta}_n(\phi(t,x))\right| = \Big|\int_t^{t+\ell} \mathcal{X}\hat{\vartheta}_n(\phi(s,x))ds\Big| \leq \int_t^{t+\ell} \left|\mathcal{X}\hat{\vartheta}_n(\phi(s,x))\right|ds \leq \mathit{\Gamma}\left|\ell\right|$$

proving (23). □

5.2 Characterization Results

As pointed out above, the sequence of functions $\{\hat{\vartheta}_n\}$ is positive and decreasing in n, and thus for some $\hat{\vartheta}$ in $\mathbb{B}(E)$,

$$\lim_{n\to\infty} \hat{\vartheta}_n(x) = \inf_n \hat{\vartheta}_n(x) = \hat{\vartheta}(x),\ x \in E.$$

Since $\hat{\vartheta}_n(z)$ for $z \in \partial^* E$ is well defined (see subsection 2.1), the above definition can be extended to $\hat{\vartheta}(z)$ for $z \in \partial^* E$ as well. The next result is proved in the Appendix II.

Proposition 5.7. $\hat{\vartheta}(x) = \hat{\rho}(x)$.

Proof. See Appendix II ☐

Following the same arguments as in Theorem (54.19) of [10], we have the next result (the proof will be omitted).

Proposition 5.8. *$\hat{\vartheta}$ is the unique positive fixed point in $\mathbb{B}(E)$ of the operator $\mathcal{G}$, that is, $\hat{\vartheta}$ is the unique function in $\mathbb{B}(E)$ satisfying $\mathcal{G}w(x) = w(x)$, $w(x) \geq 0$, $x \in E$. Moreover for some $0 < \nu < 1$, $b_1 \geq 0$, we have that for all $x \in E \cup \partial^* E$*

$$0 \leq \mathcal{G}^n h(x) - \hat{\rho}(x) \leq \nu^n b_1.$$

Proof. Same arguments as in Theorem (54.12) of [10], and Proposition 5 in [7]. ☐

Proposition 5.9. $\hat{\vartheta} \in \mathbb{B}^{lp}(E)$.

Proof. Consider a fixed $x \in E$ and an arbitrary interval $[a, b] \subset [0, t^*(x))$. As seen in Proposition 5.6, for any $t, t + \ell$ in $[a, b]$, we have that

$$\left|\hat{\vartheta}_n(\phi(t+\ell, x)) - \hat{\vartheta}_n(\phi(t, x))\right| \leq \Gamma \left|\ell\right|$$

where Γ depends only on the choice of the interval $[a, b]$. Taking the limit as n goes to infinity, we have that

$$\left|\hat{\vartheta}(\phi(t+\ell, x)) - \hat{\vartheta}(\phi(t, x))\right| \leq \Gamma \left|\ell\right|.$$

Continuity of $\hat{\vartheta}(\phi(., x))$ at the boundary follows from uniform convergence of $\hat{\vartheta}_n$ to $\hat{\vartheta}$, and continuity of $\hat{\vartheta}_n(\phi(., x))$ at the boundary. ☐

We have now the main result of the paper, which characterizes the value function of the impulse and continuous control problem of a PDP as the unique positive fixed point operator of the first jump operator, or equivalently, as the solution of a set of quasi variational inequalities.

Theorem 5.1. *The following assertions are equivalent:*

1) There exists a unique solution $\hat{\vartheta}$ in the class of positive functions in $\mathbb{B}^{ac}(E)$ to the system

$$w(x) = L(Mw, w)(x)$$

for all $x \in E$.

2) There exists a unique solution $\hat{\varrho}$ in the class of positive functions in $\mathbb{B}^{ac}(E)$ to the system,

$$
\begin{aligned}
&2.a)\ w(x) \leq Mw(x) \\
&2.b)\ \mathcal{X}w(x) - \alpha w(x) \\
&+ min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (w(x) - w(y))Q(dy;x,a)\} \geq 0 \\
&2.c)\ \Big(Mw(x) - w(x)\Big)\Big(\mathcal{X}w(x) - \alpha w(x) + \\
&min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E (w(x) - w(y))Q(dy;x,a)\}\Big) = 0
\end{aligned}
$$

for all $x \in E$, and

$$2.d)\ w(z) = Mw(z) \wedge min_{a\in U(z)}\{r(z,a) + Qw(z,a)\}$$

for $z \in \partial^* E$. Moreover, $\hat{\rho}(x) = \hat{\varrho}(x) = \hat{\vartheta}(x)$ for all $x \in E$ and $\hat{\rho} \in \mathbb{B}^{lp}(E)$.

Proof. From Propositions 5.7, 5.8 and 5.9, we have that $0 \leq \hat{\rho}(x) = \hat{\vartheta}(x) = \mathcal{G}\hat{\vartheta}(x)$, and $\hat{\vartheta} \in \mathbb{B}^{lp}(E) \subset \mathbb{B}^{ac}(E)$. From Proposition 4.1, $M\hat{\vartheta} \in \mathbb{B}^{ac}(E)$, and from 3.1, $\hat{\vartheta}(x)$ satisfies conditions 1) and 2) of the Theorem. Suppose now that $w \in \mathbb{B}^{ac}(E)$ satisfies $w(x) \geq 0$ for all $x \in E$, and 1) or 2) of the Theorem. From Proposition 4.1, $Mw \in \mathbb{B}^{ac}(E)$, and from Theorem 3.1, $w(x) = \mathcal{G}w(x)$ for all $x \in E$. From Proposition 5.8, $w = \hat{\vartheta}$. □

The construction of ϵ-optimal strategies is as in [7]. Consider $\mu^* \in \mathcal{U}$ such that (1) is satisfied with $\hat{\rho}$ replacing g_1, g_2. For $\delta > 0$, set $\varphi_\delta(.) : E \cup \partial^* E \to E$ a Borel measurable selector such that $\varphi_\delta(x) \in K(x)$ and $c(x, \varphi_\delta(x)) + \hat{\rho}(\varphi_\delta(x)) \leq M\hat{\rho}(x) + \delta$ (see Proposition 7.34 in [4] for the existence of such selector), and $\tau_\delta \in \mathcal{M}^{\mu^*}$ defined as in Theorem 3.1 with $\hat{\rho}$ and $M\hat{\rho}$ replacing ρ and φ respectively. Consider n and $\delta > 0$ such that for $\epsilon > 0$, $n\delta + \frac{b_1}{c_0 n}(\frac{b}{\alpha} + b_1) + \frac{b_1}{c_0}\delta < \epsilon$ (b_1 as in Proposition 5.8 and b as in subsection 4.3). Define the strategy $\xi_\epsilon = \{(\mu^*, R_k, S_k); k = 1, 2, \ldots\}$ such that $R_k(w_k) = \varphi_\delta(\hat{x}_{\eta(\hat{\omega}_n)}(\hat{\omega}_n))$, $S_k(w_{k-1}, \omega_k) = \tau_\delta(\omega_k)$, $k = 1, 2, \ldots, n$, $S_{n+1}(w_n, \omega_{n+1}) = \infty$ (see Appendix I). According to Proposition 7 in [7]

$$\mathbb{J}^{\xi_\epsilon}(x) \leq \hat{\rho}(x) + \epsilon.$$

Appendix I: Admissible Strategies

This Appendix follows [10], page 228-230. We set Ω the space of right continuous with left limit functions on $\mathbb{R}_+$, taking values on E, and following the flow $\phi(.,.)$ between discontinuity points. Let Δ be a fictitious state, and define $\hat{\Omega}$ as the set of functions $\hat{\omega} : [0,\infty) \to E \cup \{\Delta\}$ such that one of the possibilities below holds:

1) $\hat{\omega} \in \Omega$

2) for some $a \in \mathbb{R}_+$, $\hat{\omega}$ is right continuous with left-hand limit on $[0, a)$ and has left-hand limit at $t = a$, and $\hat{\omega}(t) = \Delta$ for $t > a$.

3) $\hat{\omega}(t) = \Delta$ for all $t \in \mathbb{R}_+$.

Consider $\eta : \hat{\Omega} \to \mathbb{R}_+ \cup \{\infty\}$ defined by:

$$\eta(\hat{\omega}) = \begin{cases} \infty, \text{ for } \hat{\omega} \text{ satisfying } 1, \\ a, \text{ for } \hat{\omega} \text{ satisfying } 2, \\ 0, \text{ for } \hat{\omega} \text{ satisfying } 3, \end{cases}$$

and set $\hat{x}_t$ the coordinate function $\hat{x}_t(\hat{\omega}) = \hat{\omega}(t)$, $\hat{\mathfrak{F}}_t^0 = \sigma\{\hat{x}_s; s \le t\}$, $\hat{\mathfrak{F}}$ the universal completion of $\hat{\mathfrak{F}}_\infty^0$, and $\hat{\mathfrak{F}}_t$ the $\hat{\mathfrak{F}}$-completion of $\hat{\mathfrak{F}}_t^0$. Let

$$\mathcal{W}_k = \prod_{i=1}^{k} \hat{\Omega}_i, \ \mathfrak{G}_k^0 = \sigma\Big\{\prod_{i=1}^{k} \hat{\mathfrak{F}}_i\Big\},$$
$$\mathcal{W} = \prod_{i=1}^{\infty} \hat{\Omega}_i, \ \mathfrak{G}^0 = \sigma\Big\{\prod_{i=1}^{\infty} \hat{\mathfrak{F}}_i\Big\},$$

where $\hat{\Omega}_i$ is a copy of $\hat{\Omega}$, $i = 1, 2, \ldots$, and $\hat{\mathfrak{F}}_i$ a copy of $\hat{\mathfrak{F}}$. We set $\mathfrak{G}_k$ the universal completion of $\mathfrak{G}_k^0$ and $\mathfrak{G}$ the universal completion of $\mathfrak{G}^0$. An element $w \in \mathcal{W}$ is then a sequence $w = (\hat{\omega}_1, \hat{\omega}_2, \ldots)$ where $\hat{\omega}_i \in \hat{\Omega}_i$. We also set

$$\text{i) } w_k(w) = (\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_k) \in \mathcal{W}_k,$$
$$\text{ii) } \tau_k(w) = \sum_{i=1}^{k} \eta_i(\hat{\omega}_i) \ (\text{ where } a + \infty := \infty,\ a \in \mathbb{R}_+ \ )$$
$$\text{iii) } \tau_\infty(w) = \lim_{k\to\infty} \tau_k(w).$$

Let Δ_∞ be another fictitious state. We define the $E \cup \{\Delta_\infty\}$ valued function Y on $\mathbb{R}_+ \times \mathcal{W}$ by:

$$Y_t(w) = \begin{cases} \hat{\omega}_1(0), & \text{if } t = 0, \\ \hat{\omega}_{i+1}(t - \tau_i(w)), & \text{if } \tau_i(w) < t \le \tau_{i+1}(\omega), \\ \Delta_\infty, & \text{if } t \ge \tau_\infty(\omega). \end{cases}$$

Note that the function $t \to Y_t(w)$ is right continuous with left-hand limit except when $t = \tau_i(w)$, at which points it is generally neither left nor right continuous, although both right and left limits exit.

An intervention with continuous control strategy is a sequence $\xi = \{(\psi_k, R_k, S_k),\ k = 1, 2, \ldots\}$, where for each $k = 1, 2, \ldots$, $\psi_k \in \Psi$, $\{\mathcal{F}_t^{\psi_k}\}$ is the filtration defined previously on Ω, and

i) $R_n : \mathcal{W}_n \to E \cup \{\Delta\}$ is a universally measurable function satisfying

$$R_n(w_n) \begin{cases} = \Delta, & \text{if } \tau_n(w_n) = \infty \\ \in K(\hat{x}_{\eta(\hat{\omega}_n)}(\hat{\omega}_n)), & \text{otherwise} \end{cases}.$$

ii) $S_1 : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is an $\mathcal{F}_t^{\psi_1}$–stopping time such that $S_1(\omega) > 0$ for all $\omega \in \Omega$. For $k = 2, 3, \ldots$, $S_k : \mathcal{W}_{n-1} \times \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a universally measurable function such that $S_k(w_{k-1}, .)$ is an $\mathcal{F}_t^{\psi_k}$–stopping time for all $w_{k-1} \in \mathcal{W}_{k-1}$. If $\tau_{k-1}(w_{k-1}) = \infty$ then $S_k(w_{k-1}, \omega) = 0$ for all $w \in \Omega$, otherwise $S_k(w_{k-1}, \omega) > 0$ for all $\omega \in \Omega$.

With an strategy $\xi = \{(\psi_k, R_k, S_k),\ k = 1, 2, \ldots\}$ we can construct sample paths of the controlled process in the following way: From S_1, we define $\chi_1 : \Omega \to \hat{\Omega}_1$ by:

$$\chi_1(\omega(t)) = \begin{cases} \omega(t), & \text{if } t \le S_1(\omega) \\ \Delta, & \text{if } t > S_1(\omega) \end{cases}.$$

For $k \ge 2$, given $w_{k-1} \in \mathcal{W}_{k-1}, \xi$ defines $\chi_k : \Omega \to \hat{\Omega}_k$ from the following rule:

1) If $R_{k-1}(w_{k-1}) = \Delta$, then $\chi_k(\omega(t)) = \Delta$, for all $t \in \mathbb{R}_+$.

2) If $R_{k-1}(w_{k-1}) \ne \Delta$, then:

2.a) If $S_k(w_{k-1},\omega)=\infty$, then $\chi_k(\omega)=\omega$; otherwise

2.b)

$$\chi_k(\omega(t)) = \begin{cases} \omega(t), & \text{if } t \le S_k(w_{k-1},\omega) \\ \Delta, & \text{if } t > S_k(w_{k-1},\omega) \end{cases}.$$

Recall that, given a strategy $\xi = \{(\psi_k, R_k, S_k),\ k=1,2,\dots\}$, $P_x^{\psi_k}$ is a measure on Ω for each $x\in E$. We can define in this way a measure μ_x^1 on $\hat{\Omega}_1$ by

$$\mu_x^1(F) = (P_x^{\psi_1}\chi_1^{-1})(F_1) = P_x^{\psi_1}\Big\{\omega\in\Omega : \chi_1(\omega)\in F_1\Big\},\ F_1\in\hat{\mathfrak{F}}_1$$

and for $k=2,3,\dots$ and each $w_{k-1}\in\mathcal{W}_{k-1}$ such that $R_{k-1}(w_{k-1})\neq\Delta$, a conditional measure on $\hat{\Omega}_k$ by

$$\mu_x^k(F_k; w_{k-1}) = (P^{\psi_k}_{R_{k-1}(w_{k-1})}\chi_k^{-1})(F_k),\ \ F_k\in\hat{\mathfrak{F}}_k.$$

There exists a unique probability measure $\mathbb{P}_x^\xi$ in $\mathcal{W}$ which is consistent with this family of conditional measures, that is, such that if $F\in\mathfrak{G}_k$ with $F=F_1\times F_2\times\ldots\times F_k, F_i\in\hat{\mathfrak{F}}_i$, then

$$\mathbb{P}_x^\xi(F) = \int_{F_1}\cdots\int_{F_{k-2}}\int_{F_{k-1}} \mu_x^k(F_k;(\hat{\omega}_1,\dots,\hat{\omega}_{k-1}))\mu_x^k(d\hat{\omega}_{k-1};(\hat{\omega}_1,\dots,\hat{\omega}_{k-2}))\dots\mu_x^1(d\hat{\omega}_1).$$

Definition: A sequence $\xi = \{(\psi_k, R_k, S_k),\ k=1,2,\dots\}$ is an admissible strategy if it satisfies the above conditions and if $\mathbb{P}_x^\xi\ \{w : \tau_\infty(w)=\infty\}=1$, for all $x\in E$. We denote by $\mathbb{S}$ the set of all admissible strategies and by $\mathbb{S}_n$ the set of all admissible strategies such that $\tau_{n+1}=\infty$ almost surely, that is, not more than n interventions are taken.

From the above construction we have that S_k determines the moment of the k^{th} intervention, and R_k the starting point for the $(k+1)^{th}$ "piece" of the process, that is, the initial point of the process after the k^{th} intervention. Indeed, if Y_t is the sample path defined above, we have that $Y_0(w)=x, Y_t(w)=\hat{x}(\hat{\omega}_1)$, for $t\le\tau_1(\hat{\omega}_1)$, and $\tau_1(\hat{\omega}_1)=S_1(\hat{\omega}_1)$. If $\tau_1(\hat{\omega}_1)=\infty$ then $R_1(\hat{\omega}_1)=\Delta$, and therefore, $\chi_k(\omega)=\Delta$ for all $k\ge 2$. If $\tau_1(\hat{\omega}_1)<\infty$, then $R_1(\hat{\omega}_1)\in K(\hat{x}_{\tau_1(\hat{\omega}_1)}(\hat{\omega}_1))$, and $P_x^\xi(\hat{\omega}_2(0)=R_1(\hat{\omega}_1))=1$. Notice now that $Y_{\tau_1+t}(w)=\hat{\omega}_2(t)$, for $t\in(0,\tau_2]$, and thus

$$Y_{\tau_1^+}(w) = \lim_{t\downarrow 0} Y_{\tau_1+t}(w) = R_1(\hat{\omega}_1).$$

The construction goes on like this.

Appendix II: Proofs of Propositions 5.2 and 5.7

Proof of Proposition 5.2: For $n=0$ we have that $\mathbb{S}_0$ is formed by the set of all admissible strategies such that $\tau_1=\infty$ almost surely, that is, $\hat{\vartheta}_0(x)=h(x)=\hat{\rho}_0(x)$. Suppose now that $\hat{\vartheta}_n(x)=\hat{\rho}_n(x)$, and for arbitrary $\epsilon>0$, consider

$$\xi = \{(\psi_k, R_k, S_k),\ k=1,2,\dots,\}\in\mathbb{S}_{n+1}\ (\text{thus } \tau_{n+2}=\infty \text{ almost surely})$$

such that

$$E^\xi_x(\int_0^\infty e^{-\alpha s} f^\xi(Y^\xi_s)ds + \int_0^\infty e^{-\alpha s} r^\xi(Y^\xi_{s-})1_{\{Y^\xi_{s-}\in\partial^* E\}}ds + \sum_{k=1}^\infty e^{-\alpha\tau} c(Y_{\tau_k}, Y_{\tau_k^+})) \leq \hat{\rho}_{n+1}(x) + \epsilon.$$

Given $\omega \in \Omega$, set $\xi' = \{(\psi'_k, R'_k, S'_k),\ k = 1, 2, \dots, \} \in \mathbb{S}_n$ by

$$\psi'_k = \psi_{k+1}, R'_k(.) = R_{k+1}(\chi_1(\omega), .), S'_k(.) = S_{k+1}(\chi_1(\omega), .).$$

We have from the construction of $\{Y^\xi_s\}$ that

$$\begin{aligned}
&E^\xi_x(\int_0^\infty e^{-\alpha s} f^\xi(Y^\xi_s)ds + \int_0^\infty e^{-\alpha s} r^\xi(Y^\xi_{s-})1_{\{Y^\xi_{s-}\in\partial^* E\}}ds + \sum_{k=1}^\infty e^{-\alpha\tau} c(Y^\xi_{\tau_k}, Y^\xi_{\tau_k^+})) = \\
&E^{\psi_1}_x(\int_0^{S_1^{\psi_1}} e^{-\alpha s} f^{\psi_1}(X^{\psi_1}_s)ds + \int_0^{S_1^{\psi_1}} e^{-\alpha s} r^{\psi_1}(X^{\psi_1}_{s-})1_{\{X^{\psi_1}_{s-}\in\partial^* E\}}ds + \\
&e^{-\alpha S_1^{\psi_1}}(c(X_{S_1^{\psi_1}}, R_{S_1^{\psi_1}}) + E(\mathbb{J}^{\xi'}(R_{S_1^{\psi_1}})\big|\mathcal{F}^{\psi_1}_{S_1^{\psi_1}}))) \geq \\
&E^{\psi_1}_x(\int_0^{S_1^{\psi_1}} e^{-\alpha s} f^{\psi_1}(X^{\psi_1}_s)ds + \int_0^{S_1^{\psi_1}} e^{-\alpha s} r^{\psi_1}(X^{\psi_1}_{s-})1_{\{X^{\psi_1}_{s-}\in\partial^* E\}}ds + \\
&e^{-\alpha S_1^{\psi_1}}(c(X_{S_1^{\psi_1}}, R_{S_1^{\psi_1}}) + \hat{\rho}_n(R_{S_1^{\psi_1}}))) \geq \\
&E^{\psi_1}_x(\int_0^{S_1^{\psi_1}} e^{-\alpha s} f^{\psi_1}(X^{\psi_1}_s)ds + \int_0^{S_1^{\psi_1}} e^{-\alpha s} r^{\psi_1}(X^{\psi_1}_{s-})1_{\{X^{\psi_1}_{s-}\in\partial^* E\}}ds + \\
&e^{-\alpha S_1^{\psi_1}} M\hat{\vartheta}_n(X_{S_1^{\psi_1}})) \geq \mathcal{G}\hat{\vartheta}_n(x) = \hat{\vartheta}_{n+1}(x)
\end{aligned}$$

showing that $\hat{\vartheta}_{n+1}(x) \leq \hat{\rho}_{n+1}(x)$ (since ϵ is arbitrary). On the other hand, consider $\xi' = \{(\psi'_k, R'_k, S'_k),\ k = 1, 2, \dots\} \in \mathbb{S}_n$ such that it is ϵ-optimal for the problem with value function $\hat{\rho}_n$, and consider (ψ_1, R_1, S_1) such that $\psi_1 = u$,

$$\begin{aligned}
&f^u(x) - \lambda^u(x)\int_E(\hat{\vartheta}_{n+1}(x) - \hat{\vartheta}_{n+1}(y))Q^u(dy; x) = \\
&min_{a\in U(x)}\{f(x,a) - \lambda(x,a)\int_E(\hat{\vartheta}_{n+1}(x) - \hat{\vartheta}_{n+1}(y))Q(dy; x, a)\} \\
&r^u(z) + Q^u\hat{\vartheta}_{n+1}(z) = min_{a\in U(z)}\{r(z,a) + Q\vartheta(z,a)\} \\
&S_1 = inf\{t \geq 0; e^{-\alpha t}M\hat{\vartheta}_n(X^u_t) \leq e^{-\alpha t}\hat{\vartheta}_{n+1}(X^u_t) + \epsilon\}, \\
&M\hat{\vartheta}_n(x) = c(x, R_1) + \hat{\vartheta}_n(R_1).
\end{aligned}$$

Set $\xi = \{(\psi_k, R_k, S_k),\ k = 1, 2, \dots, \} \in \mathbb{S}_{n+1}$ with for any $\omega \in \Omega$, $\psi_{k+1}(\chi_1(\omega), .) = \psi'_k(.)$,

$R_{k+1}(\chi_1(\omega),.) = R'_k(.), S_{k+1}(\chi_1(\omega),.) = S'_k(.)$. Then

$$\hat{\vartheta}_{n+1}(x) + 2\epsilon \geq E_x^{\psi_1}\Big(\int_0^{S_1^{\psi_1}} e^{-\alpha s} f^{\psi_1}(X_s^{\psi_1})ds + \int_0^{S_1^{\psi_1}} e^{-\alpha s} r^{\psi_1}(X_{s-}^{\psi_1})1_{\{X_{s-}^{\psi_1}\in\partial^* E\}}ds +$$
$$e^{-\alpha S_1^{\psi_1}}(c(X_{S_1^{\psi_1}}, R_{S_1^{\psi_1}}) + (\hat{\rho}_n(R_{S_1^{\psi_1}}) + \epsilon))\Big) \geq$$
$$E_x^{\psi_1}\Big(\int_0^{S_1^{\psi_1}} e^{-\alpha s} f^{\psi_1}(X_s^{\psi_1})ds + \int_0^{S_1^{\psi_1}} e^{-\alpha s} r^{\psi_1}(X_{s-}^{\psi_1})1_{\{X_{s-}^{\psi_1}\in\partial^* E\}}ds +$$
$$e^{-\alpha S_1^{\psi_1}}(c(X_{S_1^{\psi_1}}, R_{S_1^{\psi_1}}) + E(\mathbb{J}^{\xi'}(R_{S_1^{\psi_1}})\big|\mathcal{F}^{\psi_1}_{S_1^{\psi_1}}))\Big) =$$
$$E_x^{\xi}\Big(\int_0^{\infty} e^{-\alpha s} f^{\xi}(Y_s^{\xi})ds + \int_0^{\infty} e^{-\alpha s} r^{\xi}(Y_{s-}^{\xi})1_{\{Y_{s-}^{\xi}\in\partial^* E\}}ds + \sum_{k=1}^{\infty} e^{-\alpha\tau}c(Y_{\tau_k}, Y_{\tau_k^+})\Big) \geq \hat{\rho}_{n+1}(x)$$

showing that $\hat{\rho}_{n+1}(x) \leq \hat{\vartheta}_{n+1}(x)$ (recall again that ϵ is arbitrary), completing the proof of the Proposition. $\square$

Proof of Proposition 5.7: Since $\hat{\rho}(x) \leq \hat{\rho}_n(x) = \hat{\vartheta}_n(x)$, it follows that $\hat{\rho}(x) \leq \hat{\vartheta}(x)$. On the other hand consider an ϵ-optimal solution for the problem with value function $\hat{\rho}$ given by $\xi = \{(\psi_k, R_k, S_k),\ k = 1, 2, \dots,\} \in \mathbb{S}$, so that

$$\begin{aligned}\mathbb{J}^{\xi}(x) = E_x^{\xi}\Big(&\int_0^{\infty} e^{-\alpha s} f^{\xi}(Y_s^{\xi})ds + \int_0^{\infty} e^{-\alpha s} r^{\xi}(Y_{s-}^{\xi})1_{\{Y_{s-}^{\xi}\in\partial^* E\}}ds + \sum_{k=1}^{\infty} e^{-\alpha\tau}c(Y^{\xi}_{\tau_k}, Y^{\xi}_{\tau_k^+})\Big)\\ &\leq \hat{\rho}(x) + \epsilon.\end{aligned}$$

Clearly we must have from the bounded convergence theorem that

$$\lim_{n\to\infty} E_x^{\xi}\Big(\sum_{k=n}^{\infty} e^{-\alpha\tau}c(Y^{\xi}_{\tau_k}, Y^{\xi}_{\tau_k^+})\Big) = 0.$$

Consider the strategy $\xi_n = \{(\psi_k, R_k, S_k),\ k = 1, 2, \dots, n, \psi_{n+1}, S_{n+1} = \infty\} \in \mathbb{S}_n$ and set

$$\gamma^{\psi}(x) = E_x^{\psi}\Big(\int_0^{\infty} e^{-\alpha s} f^{\psi}(X_s^{\psi})ds + \int_0^{\infty} e^{-\alpha s} r(X_{s-}^{\psi})1_{\{X_{s-}^{\psi}\in\partial^* E\}}ds\Big).$$

Then we have that

$$\begin{aligned}\mathbb{J}^{\xi}(x) - \mathbb{J}^{\xi_n}(x) =& E_x^{\xi}\Big(\int_{\tau_n}^{\infty} e^{-\alpha s} f^{\xi}(Y_s^{\xi})ds + \int_{\tau_n}^{\infty} e^{-\alpha s} r^{\xi}(Y_{s-}^{\xi})1_{\{Y_{s-}^{\xi}\in\partial^* E\}}ds\Big)\\ &- E_x^{\xi}(e^{-\alpha\tau_n}\gamma^{\psi_{n+1}}(Y^{\xi}_{\tau_n})) + E_x^{\xi}\Big(\sum_{k=n}^{\infty} e^{-\alpha\tau}c(Y^{\xi}_{\tau_k}, Y^{\xi}_{\tau_k^+})\Big)\end{aligned}$$

and thus,

$$\Big|\mathbb{J}^{\xi}(x) - \mathbb{J}^{\xi_n}(x)\Big| \leq 2aE_x^{\xi}(e^{-\alpha\tau_n}) + E_x^{\xi}\Big(\sum_{k=n}^{\infty} e^{-\alpha\tau}c(Y^{\xi}_{\tau_k}, Y^{\xi}_{\tau_k^+})\Big) \to 0 \text{ as } n \to \infty$$

since ξ is admissible, and therefore by definition we must have $\tau_n \to \infty$ as $n \to \infty$ almost surely. Since $\hat{\rho}_n(x) \leq \mathbb{J}^{\xi_n}(x)$, it follows that

$$\hat{\vartheta}(x) = \lim_{n\to\infty}\hat{\vartheta}_n(x) = \lim_{n\to\infty}\hat{\rho}_n(x) \leq \mathbb{J}^{\xi}(x) \leq \hat{\rho}(x) + \epsilon$$

and since ϵ is arbitrary, we get $\hat{\vartheta}(x) \leq \hat{\rho}(x)$, completing the proof of the proposition. $\square$

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