Continuous-Time State-Feedback H_2-Control of Markovian Jump Linear Systems Via Convex Analysis ⋆

O.L.V. Costa [1]

University of São Paulo, Department of Electronic Engineering, 05508-900 - São Paulo - SP - Brazil

J.B.R. do Val and J.C. Geromel

LAC-DT/Faculty of Electrical Engineering, UNICAMP, CP 6101, 13081-970 - Campinas - SP - Brazil

Abstract

Continuous-time H_2-control problem for the class of linear systems with Markovian jumps (MJLS) using convex analysis is considered in this paper. The definition of the H_2-norm for continuous-time MJLS is presented and related to the appropriate observability and controllability Gramians. A convex programming formulation for the H_2-control problem of MJLS is developed. That enables us to tackle the optimization problem of MJLS under the assumption that the transition rate matrix $\Pi = [\pi_{ij}]$ for the Markov chain may not be exactly known, but belongs to an appropriate convex set. An equivalence between the convex formulation when Π is exactly known and the usual dynamic programming approach of quadratic optimal control of MJLS is established. It is shown that there exists a solution for the convex programming problem if and only if there exists the mean square stabilizing solution for a set of coupled algebraic Riccati equations. These results are compared with other related works in the current literature.

Key words: convex programming, linear systems, Markov parameters, jump process, quadratic control, parameter uncertainty.

⋆ This work was supported in part by CNPq (Brazilian National Research Council) and FAPESP (Research Council of the State of São Paulo).
A short version of the paper was presented in 1996 at the 13^{th} IFAC World Congress.
[1] Corresponding author. Postal Address: Escola Politécnica da Universidade de São Paulo, Departamento de Engenharia Eletrônica, 05508-900 São Paulo, SP, Brazil, fax: 55 11 8185718, telephone: 55 11 8185771, e-mail: oswaldo@lac.usp.br

1 Introduction

Continuous-time LQ control problems for Markovian jump linear systems (MJLS) have been studied in the current literature by several authors, usually under a dynamic programming formulation that leads to a set of coupled algebraic Riccati equations (CARE) for the solution of the problem, cf. [1,2,7–10,12]. More recently a convex approach ([3]) has been adopted to study MJLS, leading to numerical algorithms via linear matrix inequalities (LMI) techniques, see [5,11,12].

In this paper we present a convex programming approach for the H_2-control problem of a continuous-time MJLS, tracing a parallel with the discrete-time results presented in [5]. Initially the definition of the H_2-norm for the continuous-time MJLS is presented, together with the connection with the controllability and observability Gramians. These definition and connections can be seen as generalization of those known in the linear deterministic case. Next, the H_2-control problem is defined, and written as a convex optimization problem. The power of this formulation is twofold. Firstly, it allows to address not only the classical situation in which all system parameters are known but also the important situation in which the transition rate of the Markov chain is not precisely known. Here we assume, like in [11], that the transition matrix of the Markov chain, Π, belongs to a convex set with the vertices themselves formed by some transition rate matrices and, for this case, the H_2-guaranteed cost control is derived via convex analysis.

Secondly we can show that for the case in which Π is exactly known, the existence of a solution to the convex programming problem is completely equivalent to the existence of the mean square stabilizing solution to the CARE, and one solution can be recovered from the other.

It is important to stress the differences between the results presented in this paper and the ones in [11,12]. In [11], LMI conditions were derived ensuring robust mean square stability for continuous-time MJLS. Robustness there was referred to uncertainties in the transition probability matrix describing the random jumps of the systems, assumed to belong to a convex set. Applications to the optimization of the decay rate and stability margins of MJLS were also presented. Note however that the quadratic or H_2-optimization was not considered there, nor any connection with the CARE was made. In this paper we use the robustness ideas presented in [11], which are based on the coupled Lyapunov equations, and apply it to the H_2-guaranteed cost control of the MJLS. Furthermore, for the case with no uncertainties on the transition matrix, we establish a link between the present convex formulation and the existence of the mean square stabilizing solution of the CARE. This result is different from the one in [12], which establishes a connection between the maximal solution of the CARE and the solution of a LMI optimization problem. In that paper, it was shown that under the hypothesis that the system is mean square stabilizable, there exists the maximal solution to the CARE and this solution can be obtained by solving a LMI optimization problem.

Thus the LMI optimization problems presented here and in [12] are distinct problems. The coupled Lyapunov equations approach, used in this paper, guarantees mean square stability of the solution, at a price of involving more variables in the optimization problem than in [12]. In that paper the maximal solution of the CARE is sought, which involves a LMI optimization problem with less variables, but cannot guarantee mean square stability of the solution. Another advantage of the Lyapunov approach is that it can handle the case of uncertainties on the transition matrix Π, providing a robust mean square stability solution for the H_2-guaranteed cost control problem.

The paper is organized in the following way. The main notation to be employed throughout is presented in section 2, as well as some basic results concerning stability in the mean square sense. The definition of the H_2-norm for continuous-time MJLS and its connection with the observability and controllability Gramians are presented in section 3. The main results of this paper are presented in sections 4 and 5. They consist of deriving a convex programming formulation to the H_2-optimal problems when Π is exactly known and when Π belongs to a convex set. In section 5 we establish the link between the convex and the CARE formulations. In section 6 we present numerical examples to illustrate the developed results and, in section 7, we conclude the paper with some final comments.

2 Notation and Stability Results

Set $\mathcal{H}^{m,n}$ the linear space made up of all N-sequence of matrices $V = (V_1, \dots, V_N)$, where each V_i is a n by m matrix. For $V, H \in \mathcal{H}^{m,n}$ we define the following inner product:

$$< V; H >= \sum_{i=1}^{N} tr\Big(V_i^* H_i)\Big).$$

For simplicity we set $\mathcal{H}^{n,n} = \mathcal{H}^n$ and $\mathcal{H}^{n+} = \{V = (V_1, \dots, V_N) \in \mathcal{H}^n;\ V_i \geq 0\}$. For $H = (H_1, \dots, H_N)$ and $V = (V_1, \dots, V_N)$ in $\mathcal{H}^{n+}$ the notation $H \leq L$ ($H < L$) indicates that $H_i \leq L_i$ ($H_i < L_i$) for each $i = 1, \dots, N$. We denote by $\mathbb{B}(\mathcal{H}^n)$ the Banach space of all bounded linear operators of $\mathcal{H}^n$ into $\mathcal{H}^n$, and write $\mathrm{Re}(\lambda(\mathcal{L}))< 0$ if all the eigenvalues of $\mathcal{L} \in \mathbb{B}(\mathcal{H}^n)$ have real part less than zero.

We define ℓ_2^r as the Hilbert space formed by the stochastic processes $z = \{z(t); t \geq 0\}$ such that, for each $t \geq 0$, $z(t)$ is a second order r-dimensional random variable, and

$$\|z\|_2^2 := \int_0^\infty E(\|x(t)\|^2)\, dt < \infty.$$

Consider the following system on a probability space $(\Omega, \mathcal{F}, P)$,

$$\begin{cases} \dot{x}(t) = \widetilde{A}_{\theta(t)}\, x(t) \\ x(0) = x_0, \quad \theta(0) = \theta \end{cases} \tag{1}$$

where $\{\theta(t); t \geq 0\}$ is a continuous-time Markov chain with finite state space $\{1, \ldots, N\}$ and transition rate matrix $\Pi = [\pi_{ij}]$, $\widetilde{A} = (\widetilde{A}_1, \ldots, \widetilde{A}_N) \in \mathcal{H}^n$, and x_0 a second order random variable. The transition rate matrix Π will be either exactly known or belong to a convex set $\mathbb{D}$, where (see also [11])

$$\mathbb{D} = \{\Pi; \Pi = \sum_{\nu=1}^{\ell} \rho^\nu \Pi^\nu;\ \rho^\nu \geq 0,\ \sum_{\nu=1}^{\ell} \rho^\nu = 1\}$$

and $\Pi^\nu = [\pi_{ij}^\nu]$, $\nu = 1, \ldots, \ell$, is a set of known transition rate matrices.

Define $Q(t) = (Q_1(t), \ldots, Q_N(t))$, where

$$Q_j(t) := E(x(t)x(t)' 1_{\{\theta(t)=j\}}) \geq 0 \tag{2}$$

and $1_{\{.\}}$ stands for the Dirac measure.

For $S = (S_1, \ldots, S_N) \in \mathcal{H}^n$ we define the operator $\mathcal{T} \in \mathbb{B}(\mathcal{H}^n)$ as: $\mathcal{T}(S) = (\mathcal{T}_1(S), \ldots, \mathcal{T}_N(S))$ where

$$\mathcal{T}_j(S) = \widetilde{A}_j S_j + S_j \widetilde{A}_j' + \sum_{i=1}^{N} \pi_{ij} S_i \tag{3}$$

With the inner product as defined above we have that $\mathcal{L} := \mathcal{T}^*$ is given by:

$$\mathcal{L}_i(S) = \widetilde{A}_i' S_i + S_i \widetilde{A}_i + \sum_{j=1}^{N} \pi_{ij} S_j.$$

In particular, we have that $\mathcal{T}$ and $\mathcal{L}$ possess the same eigenvalues. The following result provides a connection between (2) and (3) (cf. [9,10]):

Proposition 1 *For every $t \geq 0, \dot{Q}(t) = \mathcal{T}(Q(t))$.*

The definition below represents a generalization of the definition of stable systems from the deterministic to the Markovian jump case (see [10]):

Definition 1 *We say that model (1) is mean square stable (MSS) if*

$$E(\|x(t)\|^2) \to 0 \text{ as } t \to \infty$$

for any initial condition x_0 and initial distribution for θ_0.

Note that MSS implies that $\|x(t)\| \to 0$ as $t \to \infty$ with prob. 1 (cf. [9]). We have the following result, which can be derived from [8], and [9]:

Proposition 2 *The assertions below are equivalent:*

a) Model (1) is MSS
b) $Re(\lambda(\mathcal{T})) < 0$.
c) $Re(\lambda((\mathcal{L})) < 0$.
d) There exists $b > 0$ and $a > 0$, such that for each $t \geq 0, 1, \ldots,$

$$E(\|x(t)\|^2) \leq a\ exp(-bt).$$

e) (coupled Lyapunov equations) given any $S = (S_1, \ldots, S_N) > 0$ in $\mathcal{H}^{n+}$ there exists $P = (P_1, \ldots, P_N) > 0$ in $\mathcal{H}^{n+}$ satisfying $\mathcal{L}(P) + S = 0$.
f) (adjoint coupled Lyapunov equations) given any $S = (S_1, \ldots, S_N) > 0$ in $\mathcal{H}^{n+}$ there exists $P = (P_1, \ldots, P_N) > 0$ in $\mathcal{H}^{n+}$ satisfying $\mathcal{T}(P) + S = 0$.

Moreover, if $Re(\lambda(\mathcal{L}))< 0$ then, for any $S \in \mathcal{H}^n$ we have that,

i) there exists a unique $P \in \mathcal{H}^n$ such that $\mathcal{L}(P) + S = 0$.
ii) if $S \geq T$ ($S > T$) and $\mathcal{L}(P) + S = 0$, $\mathcal{L}(L) + T = 0$ then $P \geq L$ ($P > L$).

These results also hold replacing $\mathcal{L}$ by $\mathcal{T}$.

The following definition of mean square stabilizability, also considered in [9,11], represents a generalization of those known from the deterministic control theory ($N = 1$). Consider $A = (A_1, \ldots, A_N) \in \mathcal{H}^n$ and $B = (B_1, \ldots, B_N) \in \mathcal{H}^{m,n}$.

Definition 2 *We say that (A,B) is mean square stabilizable if there exists $K = (K_1, \ldots, K_N) \in \mathcal{H}^{n,m}$ such that model (1) is MSS with $\widetilde{A}_i = A_i - B_i K_i$. In this case we say that K stabilizes (A,B) in the mean square sense. We set $\mathbb{K} = \{K \in \mathcal{H}^{n,m}$; K stabilizes (A,B) in the mean square sense$\}$.*

If Π is not exactly known, we make the following definitions:

Definition 3 *We say that (A,B) is robustly mean square stabilizable if there exists $K = (K_1, \ldots, K_N) \in \mathcal{H}^{n,m}$ such that model (1) is MSS with $\widetilde{A}_i = A_i - B_i K_i$ for every $\Pi \in \mathbb{D}$. In this case we say that K stabilizes robustly (A,B) in* *the mean square sense. We set $\mathbb{K}_r = \{K \in \mathcal{H}^{n,m}$; K stabilizes robustly (A,B) in the mean square sense$\}$.*

Definition 4 *We say that (A,B) is quadratically mean square stabilizable if there exists $K = (K_1, \ldots, K_N) \in \mathcal{H}^{n,m}$ and $P = (P_1, \ldots, P_N) > 0$, $P \in \mathcal{H}^{n+}$, such that*

$$(A_j - B_jK_j)P_j + P_j(A_j - B_jK_j)' + \sum_{i=1}^{N} \pi_{ij}P_i < 0, \;\; j = 1, \ldots, N$$

for every $\Pi = [\pi_{ij}] \in \mathbb{D}$. In this case we say that K stabilizes quadratically (A,B) in the mean square sense. We set $\mathbb{K}_q = \{K \in \mathcal{H}^{n,m}; K$ stabilizes quadratically (A,B) in the mean square sense$\}$.

From Proposition 2 it is clear that $\mathbb{K}_q \subset \mathbb{K}_r$.

3 Controllability, Observability Gramians and the H_2-Norm

Consider the following system $\mathcal{G}$

$$\mathcal{G} = \begin{cases} \dot{x}(t) = \widetilde{A}_{\theta(t)}x(t) + J_{\theta(t)}w(t) \\ x(0) = 0, \quad \theta(0) = \theta_0 \\ z(t) = \widetilde{C}_{\theta(t)}x(t) \end{cases} \tag{4}$$

where $\widetilde{A} = (\widetilde{A}_1, \ldots, \widetilde{A}_N) \in \mathcal{H}^n$, $\widetilde{C} = (\widetilde{C}_1, \ldots, \widetilde{C}_N) \in \mathcal{H}^{n,p}$ and $J = (J_1, \ldots, J_N) \in \mathcal{H}^{r,n}$ with $J_iJ_i' > 0$ for each $i = 1, \ldots, N$. The initial distribution for θ_0 is given by $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i > 0$, $i = 1, \ldots, N$, $\sum_{i=1}^{N} \alpha_i = 1$.

If Re$(\lambda(\mathcal{T}))< 0$ (that is, model (1) is MSS) and $w = \{w(t); t \geq 0\}$ is an impulse input then, from Proposition 2.d) we have that $z = \{z(t); t \geq 0\} \in \ell_2^p$. The next definition is a generalization of the H_2-norm from continuous-time deterministic systems to the stochastic Markovian jump case.

Definition 5 *We define the H_2-norm of system $\mathcal{G}$ as*

$$\|\mathcal{G}\|_2^2 = \sum_{s=1}^{r} \sum_{j=1}^{N} \alpha_j \, \|z_{s,j}\|_2^2$$

where $z_{s,j}$ represents the output sequence $\{z(t);\, t \geq 0\}$ given by (4) when *a) the input is given by* $w = \{w(t);\, t \geq 0\}$, $w(t) = e_s\delta(t)$, $\delta(t)$ *the unitary impulse, and* e_s *the r-dimensional unitary vector formed by 1 at the* s^{th} *position and zero elsewhere, and*

b) $\theta(0) = j$.

For the deterministic case ($N = 1$) the above definition reduces to the usual H_2-norm. At the end of this section we also present a connection between the above definition and the expected value of the stationary variance of the output, when the input is the unit variance white noise.

We will show next that the H_2-norm as defined above can be calculated as the solution of the continuous-time coupled Gramian of observability and controllability, a result that mirrors its deterministic counterpart. For this, define $\mathcal{M} = (\widetilde{C}_1'\widetilde{C}_1, \ldots, \widetilde{C}_N'\widetilde{C}_N) \in \mathcal{H}^{n+}$, $\alpha\mathcal{N} = (\alpha_1 J_1 J_1', \ldots, \alpha_N J_N J_N') \in \mathcal{H}^{n+}$, $S = (S_1, \ldots, S_N) \in \mathcal{H}^{n+}$, $P = (P_1, \ldots, P_N) \in \mathcal{H}^{n+}$ the unique solution of the equations (see Proposition 2)

$$\begin{aligned} \mathcal{L}(S) + \mathcal{M} &= 0 \quad \text{(observability Gramian)} \\ \mathcal{T}(P) + \alpha\mathcal{N} &= 0 \quad \text{(controllability Gramian).} \end{aligned}$$

From Proposition 2 and $\alpha\mathcal{N} > 0$ it follows that $P > 0$.

Theorem 1 $\|\mathcal{G}\|_2^2 = \sum_{j=1}^N \alpha_j\, tr(J_j' S_j J_j) = \sum_{j=1}^N tr(\widetilde{C}_j P_j \widetilde{C}_j')$.

PROOF. Let us show the first equality. For $i = 1, \ldots, N$,

$$\mathcal{L}_i(S) + \widetilde{C}_i\widetilde{C}_i' = \widetilde{A}_i' S_i + S_i \widetilde{A}_i + \sum_{j=1}^N \pi_{ij} S_j + \widetilde{C}_i\widetilde{C}_i' = 0.$$

Consider $z = \{z(t); t \geq 0\}$ an impulse response of (4). Then, from (2) and Proposition 1, and recalling that $\mathcal{L}^* = \mathcal{T}$, we get

$$\begin{aligned} E(z(t)'z(t)) &= E(x(t)'\widetilde{C}_{\theta(t)}'\widetilde{C}_{\theta(t)}x(t)) = -E\Big(x(t)'\mathcal{L}_{\theta(t)}(S)x(t)\Big) \\ &= -\sum_{i=1}^N E\Big(x(t)'\mathcal{L}_i(S)x(t)1_{\{\theta(t)=i\}}\Big) = -\sum_{i=1}^N tr\Big(E\Big(x(t)x(t)'1_{\{\theta(t)=i\}}\Big)\mathcal{L}_i(S)\Big) \\ &= -\sum_{i=1}^N tr\Big(Q_i(t)'1_{\{\theta(t)=i\}}\Big)\mathcal{L}_i(S)\Big) = -<Q(t);\mathcal{L}(S)> \\ &= -<\mathcal{T}(Q(t)); S> = -<\dot{Q}(t); S>. \end{aligned}$$

Taking the integral over t from 0 to ∞, and recalling that $Q(t) \to 0$ as $t \to \infty$

(since the system is MSS) and $\theta(0)= j$, $x(0) = J_j e_s$, we get that

$$\|z_{s,j}\|_2^2 = \int_0^\infty E(\|z(t)\|^2)dt = -\int_0^\infty < \dot{Q}(t); S > \, dt = - < Q(t); S > \Big|_0^\infty$$
$$=< Q(0); S >= \sum_{i=1}^N tr(Q_i(0)S_i) = e_s' J_j' S_j J_j e_s$$

Therefore,

$$\|\mathcal{G}\|_2^2 = \sum_{s=1}^n \sum_{j=1}^N \alpha_j \|z_{s,j}\|_2^2 = \sum_{s=1}^n \sum_{j=1}^N \alpha_j e_s' J_j' S_j J_j e_s = \sum_{j=1}^N \alpha_j tr(J_j' S_j J_j)$$

proving the first equality. For the second equality, we have that

$$\|\mathcal{G}\|_2^2 = \sum_{k=1}^N \alpha_k tr(J_k' S_k J_k) = \sum_{k=1}^N tr(\alpha_k J_k' S_k J_k + P_k(\widetilde{C}_k' \widetilde{C}_k + \mathcal{L}_k(S))$$
$$= \sum_{k=1}^N tr(\alpha_k J_k J_k' S_k + P_k \widetilde{C}_k' \widetilde{C}_k) + < P; \mathcal{L}(S) >$$
$$=< P; \mathcal{M} > + < \alpha\mathcal{N}; S > + < \mathcal{T}(P); S >$$
$$=< P; \mathcal{M} > + < (\mathcal{T}(P) + \alpha\mathcal{N}); P >=< P; \mathcal{M} >= \sum_{k=1}^N tr(\widetilde{C}_k P_k \widetilde{C}_k')$$

completing the proof of the Theorem. □

If the Markov process $\{\theta(t); t \geq 0\}$ is irreducible recurrent then $P(\theta(t) = i) \rightarrow \beta_i$ as $t \rightarrow \infty$ exponentially fast for some $\beta_i > 0$, $i = 1, \ldots, N$, $\sum_{i=0}^N \beta_i = 1$. In this case, if $\alpha_i = \beta_i$ above, we get that, like in the case with no jumps, the definition of the H_2-norm will coincide with the expected value of the stationary variance of the output, when the input is the unit variance white noise. Indeed, consider the stochastic differential equation

$$dx(t) = \widetilde{A}_{\theta(t)} x(t) + J_{\theta(t)} dv(t)$$

where $\{v(t); t \geq 0\}$ is a Wiener process with incremental covariance equal to the identity. The following proposition can be easily proved, and we shall omit the proof.

Proposition 3 *For every* $t \geq 0, \dot{Q}_j(t) = \mathcal{T}_j(Q(t)) + P(\theta(t) = j) J_j J_j'$, $j = 1, \ldots, N$.

Therefore, $Q(t) \rightarrow P$ *as* $t \rightarrow \infty$, *and*

$$\lim_{t\rightarrow\infty} E(\|z(t)\|^2) = \lim_{t\rightarrow\infty} \sum_{i=1}^N tr(\widetilde{C}_i Q_i(t) \widetilde{C}_i') = \sum_{i=1}^N tr(\widetilde{C}_i P_i \widetilde{C}_i') = \|\mathcal{G}\|_2^2 .$$

However, as it will be seen in the next sections, for the H_2-optimal control problem with Π exactly known, the optimal controller will not depend on $J = (J_1, \ldots, J_N)$. In particular we have that the choice of $\alpha = (\alpha_1, \ldots, \alpha_N)$ in the definition of the H_2-norm will not affect the optimal solution of the problem (see Remark 1 below).

4 H_2-Control via Convex Analysis

4.1 Preliminaries

Consider the jump controlled system $\mathcal{G}$

$$\mathcal{G} = \begin{cases} \dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + J_{\theta(t)}w(t) \\ x(0) = 0, \quad \theta(0) = \theta_0 \\ z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \end{cases} \tag{5}$$

where $A = (A_1, \ldots, A_N) \in \mathcal{H}^n$, $B = (B_1, \ldots, B_N) \in \mathcal{H}^{m,n}$, $C = (C_1, \ldots, C_N) \in \mathcal{H}^{n,p}$, $D = (D_1, \ldots, D_N) \in \mathcal{H}^{m,p}$, with $D_i' D_i > 0$ and $C_i' D_i = 0$ for each $i = 1, \ldots, N$.

For $K = (K_1, \ldots, K_N) \in \mathbb{K}$ set $\mathcal{G}_K$ as system (5) with $u(t) = -K_{\theta(t)}x(t)$. The H_2-optimal control problem for Π exactly known is defined as

$$\min\{\|\mathcal{G}_K\|_2 ; K \in \mathbb{K}\}, \tag{6}$$

whereas the H_2-guaranteed cost control for continuous-time Markovian jump linear systems with uncertain transition probability matrix Π is defined as: find $K \in \mathbb{K}_r$ and $\beta < \infty$, such that

$$\left\|\mathcal{G}_K^{\Pi}\right\|_2^2 \leq \beta \text{ for every } \Pi \in \mathbb{D}, \tag{7}$$

where $\left\|\mathcal{G}_K^{\Pi}\right\|_2$ represents the H_2-norm of system (5) with feedback control law defined by $K \in \mathbb{K}_r$ and transition probability matrix given by $\Pi \in \mathbb{D}$ (see section 2 for definition of $\mathbb{D}$).

4.2 Π *exactly known*

Define $R = (R_1, \dots, R_N) \in \mathcal{H}^{(n+m)+}$ and $W = (W_1, \dots, W_N) \in \mathcal{H}^{(n+m)+}$ as

$$R_i = \begin{bmatrix} C_i'C_i & 0 \\ 0 & D_i'D_i \end{bmatrix}, \quad W_i = \begin{bmatrix} W_{i1} & W_{i2} \\ W_{i2}' & W_{i3} \end{bmatrix},$$

and set

$$\Theta_j(W) = A_j W_{j1} + W_{j1}A_j' - B_j W_{j2}' - W_{j2}B_j' + \sum_{i=1}^{N} \pi_{ij} W_{i1} + \alpha_j J_j J_j', \quad j = 1, \dots, N$$
$$\mathcal{C} = \{W \geq 0; \quad W_{j1} > 0, \Theta_j(W) \leq 0, \quad j = 1, \dots, N\} \tag{8}$$

Clearly $\mathcal{C}$ is a convex set. The following result was proved in [11].

Proposition 4 *$\mathbb{K} \neq \emptyset$ iff $\mathcal{C} \neq \emptyset$. Moreover, $\mathbb{K} = \{(W_{12}'W_{11}^{-1}, \dots, W_{N2}'W_{N1}^{-1});$ $W = (W_1, \dots, W_N) \in \mathcal{C}\}$.*

Define the following convex optimization problem:

$$\mu := \min\{\sum_{i=1}^{N} tr(R_i W_i); W = (W_1, \dots, W_N) \in \mathcal{C}\} \tag{9}$$

We have the following theorem, which parallels the results presented in [5]:

Theorem 2 *The following assertions are equivalent:*

a) There exists $\widehat{W} = (\widehat{W}_1, \dots, \widehat{W}_N)$ such that

$$\widehat{W} = (\widehat{W}_1, \dots, \widehat{W}_N) = arg\ min\{\sum_{i=1}^{N} tr(R_i W_i); \ W = (W_1, \dots, W_N) \in \mathcal{C}\}. \tag{10}$$

b) There exists $\widehat{K} = (\widehat{K}_1, \dots, \widehat{K}_N) \in \mathbb{K}$ such that

$$\left\| \mathcal{G}_{\widehat{K}} \right\|_2^2 = \min\{\|\mathcal{G}_K\|_2^2 ; K \in \mathbb{K}\}. \tag{11}$$

Moreover if a) holds, then (11) holds with

$$\widehat{K}_i = \widehat{W}_{i2}'\widehat{W}_{i1}^{-1}, \quad i = 1, \dots, N, \tag{12}$$

and

$$\mu = \sum_{i=1}^{N} tr(R_i \widehat{W}_i) = \min\{\|\mathcal{G}_K\|_2^2 ; \ K \in \mathbb{K}\}.$$

On the other hand if b) holds for $\widehat{K} = (\widehat{K}_1, \dots, \widehat{K}_N) \in \mathbb{K}$, then (10) is satisfied with

$$\widehat{W}_i = \begin{bmatrix} \widehat{P}_i & \widehat{P}_i \widehat{K}_i' \\ \widehat{K}_i \widehat{P}_i & \widehat{K}_i \widehat{P}_i \widehat{K}_i' \end{bmatrix} \tag{13}$$

and $\widehat{W} = (\widehat{W}_1, \dots, \widehat{W}_N)$, where $\widehat{P} = (\widehat{P}_1, \dots, \widehat{P}_N) \in \mathcal{H}^{n+}$ satisfies

$$(A_j - B_j \widehat{K}_j)\widehat{P}_j + \widehat{P}_j (A_j - B_j \widehat{K}_j)' + \sum_{i=1}^N \pi_{ij} \widehat{P}_i + \alpha_j J_j J_j' = 0.$$

PROOF. For any $W \in \boldsymbol{\mathcal{C}}, tr(R_i W_i) = tr(C_i W_{i1} C_i') + tr(D_i W_{i3} D_i')$ and, from (8), $\mathcal{T}(Q) + \alpha \mathcal{N} \leq 0$, where $\widetilde{A}_i = A_i - B_i K_i$, $K_i = W_{i2}' W_{i1}^{-1}$, $Q = (Q_1, \dots, Q_N), Q_i = W_{i1} > 0$. Therefore, from Proposition 2, $W_{i1} \geq P_i$ for $i = 1, \dots, N$ where $P = (P_1, \dots, P_N) \in \mathcal{H}^{n+}$ satisfies $\mathcal{T}(P) + \alpha \mathcal{N} = 0$. From Theorem 1,

$$\begin{aligned} \|\mathcal{G}_K\|_2^2 &= \sum_{j=1}^N tr((C_j - D_j K_j) P_j (C_j - D_j K_j)') \\ &\leq \sum_{j=1}^N tr((C_j - D_j K_j) W_{j1} (C_j - D_j K_j)') \\ &\leq \sum_{j=1}^N tr(C_j W_{j1} C_j' + D_j W_{j3} D_j') = \sum_{j=1}^N tr(R_j W_j) \end{aligned}$$

showing that $\mu \geq \min\{\|\mathcal{G}_K\|_2^2; K \in \mathbb{K}\}$. On the other hand, for any $K = (K_1, \dots, K_N) \in \mathbb{K}$ we have from Theorem 1 that,

$$\|\mathcal{G}_K\|_2^2 = \sum_{j=1}^N tr((C_j - D_j K_j) P_j (C_j - D_j K_j)')$$

where $P = (P_1, \dots, P_N) > 0$ satisfies $\mathcal{T}(P) + \alpha \mathcal{N} = 0$, with $\widetilde{A}_i = A_i - B_i K_i$. Defining W as $W_{i1} = P_i$, $W_{i2} = P_i K_i'$, $W_{i3} = W_{i2}' W_{i1}^{-1} W_{i2}$ we get that $W \in \boldsymbol{\mathcal{C}}$, and

$$\sum_{j=1}^N tr(R_j W_j) = \sum_{j=1}^N \Big(tr(C_j W_{j1} C_j') + tr(D_j W_{j3} D_j') \Big) = \\ \sum_{j=1}^N tr((C_j - D_j K_j) P_{j1} (C_j - D_j K_j)') = \|\mathcal{G}_K\|_2^2$$

that is, $\mu \leq \min\{\|\mathcal{G}_K\|_2^2 \; ; \; K \in \mathbb{K}\}$. Thus $\mu = \min\{\|\mathcal{G}_K\|_2^2 \; ; \; K \in \mathbb{K}\}$ and one solution can be recovered from the other according to the equations (12) and (13) above. $\square$

4.3 $\mathbf{\Pi}$ *not exactly known*

We consider now the case that Π is not exactly known but belongs to $\mathbb{D}$ (see section 2). Set for $\nu = 1, \ldots, \ell$,

$$\mathcal{C}^{\nu} = \{W \geq 0; W_{j1} > 0, \Theta_j^{\nu}(W) \leq 0, j = 1, \ldots, N\}$$

where

$$\Theta_j^{\nu}(W) = A_j W_{j1} + W_{j1} A_j' - B_j W_{j2}' - W_{j2} B_j' + \sum_{i=1}^{N} \pi_{ij}^{\nu} W_{i1} + \alpha_j J_j J_j',$$

$j = 1, \ldots, N$, with $\Pi^{\nu} = [\pi_{ij}^{\nu}]$. It is clear that $\mathcal{C}^{\nu}$ is a convex set, and thus

$$\mathcal{C}_q := \cap_{\nu=1}^{\ell} \mathcal{C}^{\nu}$$

is also a convex set. The following result has been proved in [11].

Proposition 5 *$\mathbb{K}_q \neq \emptyset$ iff $\mathcal{C}_q \neq \emptyset$. Moreover, $\mathbb{K}_q = \{(W_{12}' W_{11}^{-1}, \ldots, W_{N2}' W_{N1}^{-1});$ $W = (W_1, \ldots, W_N) \in \mathcal{C}_q\}$.*

We have the following corollary concerning problem (7).

Corollary 1 *Let $\widehat{W} = (\widehat{W}_1, \ldots, \widehat{W}_N) = arg\ min\{\sum_{i=1}^{N} tr(R_i W_i); W = (W_1, \ldots, W_N) \in \mathcal{C}_q\}$. Then $\widehat{K}_i = \widehat{W}_{i2}' \widehat{W}_{i1}^{-1}$, $i = 1, \ldots, N$, solves the H_2-guaranteed cost control problem with $\beta = \sum_{i=1}^{N} tr(R_i \widehat{W}_i)$.*

PROOF. For any $W \in \mathcal{C}_q$ and every $\Pi = [\pi_{ij}] \in \mathbb{D}$ (see Theorem 2 and Proposition 5), $tr(R_i W_i) = tr(C_i W_{i1} C_i') + tr(D_i W_{i3} D_i')$, and

$$\sum_{i=1}^{N} \pi_{ij} W_{i1} + (A_j - B_j W_{j2}' W_{j1}^{-1}) W_{j1} + W_{j1} (A_j - B_j W_{j2}' W_{j1}^{-1})' + \alpha_j J_j J_j' \leq 0.$$

Therefore, from Proposition 2, $W_{i1} \geq P_i$ for $i = 1, \ldots, N$, where $P = (P_1, \ldots, P_N) \in \mathcal{H}^{n+}$ satisfies

$$\sum_{i=1}^{N} \pi_{ij} P_i + (A_j - B_j W_{j2}' W_{j1}^{-1}) P_j + P_j (A_j - B_j W_{j2}' W_{j1}^{-1})' + \alpha_j J_j J_j' = 0$$

for arbitrary $\Pi \in \mathbb{D}$. From Theorem 2, $\|\mathcal{G}_K^{\Pi}\|_2^2 \leq \sum_{j=1}^{N} tr(R_j W_j)$ and since Π is arbitrary in $\mathbb{D}$, $\max_{\Pi \in \mathbb{D}} \|\mathcal{G}_K^{\Pi}\|_2^2 \leq \sum_{j=1}^{N} tr(R_j W_j)$, showing that all $W \in \mathcal{C}_q$ generates an upper bound to $\left\|\mathcal{G}_K^{\Pi}\right\|_2^2$. Clearly the smallest one is given by the optimal solution of the problem posed above, so that $\left\|\mathcal{G}_K^{\Pi}\right\|_2^2 \leq \beta$ for all $\Pi \in \mathbb{D}$. $\square$

5 The Convex Approach and the CARE

This section establishes a link between the convex approach, based on the solution of problem (9) that corresponds to the H_2-control problem of a MJLS with Π exactly known, and the CARE

$$A_i'X_i + X_iA_i - X_iB_i(D_i'D_i)^{-1}B_i'X_i + \sum_{j=1}^{N} \pi_{ij}X_j + C_i'C_i = 0, \quad i = 1, \dots, N. \tag{14}$$

This kind of equation arises when one applies the dynamic programming formulation to the quadratic optimization problem of a MJLS

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t)$$

with minimization cost given by

$$J(u) = E(\int_0^{\infty} \left\| C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \right\|^2 dt),$$

cf. [8]. Previous works in the area of LQ-optimization of MJLS have been mainly concerned with the theoretical study of (14), and the associated development of numerical procedures for the solutions of (14), see for instance, [1,6,8,12].

We define the non-linear operator $\mathcal{R}(\cdot) : \mathcal{H}^n \to \mathcal{H}^n$ in the following way: for any $X = (X_1, \dots, X_N) \in \mathcal{H}^n$, $\mathcal{R}(X) = (\mathcal{R}_1(X), \dots, \mathcal{R}_N(X))$ is defined as

$$\mathcal{R}_i(X) = A_i'X_i + X_iA_i - X_iB_i(D_i'D_i)^{-1}B_i'X_i + \sum_{j=1}^{N} \pi_{ij}X_j + C_i'C_i.$$

The following definitions are related to the solutions of equation (14) (see also [12]).

Definition 6 *We say that $V = (V_1, \dots, V_N) \in \mathcal{H}^n$ is a maximal solution of (14) if $\mathcal{R}(V) = 0$ and, for any $S = (S_1, \dots, S_N) \in \mathcal{H}^n$ such that $\mathcal{R}(S) \geq 0$, we have that $V \geq S$.*

Definition 7 *We say that $V = (V_1, \dots, V_N) \in \mathcal{H}^n$ is the mean square stabilizing solution of (14) if $\mathcal{R}(V) = 0$ and for $K = (K_1, \dots, K_N) \in \mathcal{H}^{n,m}$ defined as $K_i = (D_i'D_i)^{-1}B_i'V_i, i = 1, \dots, N$, we have that $K \in \mathbb{K}$.*

The next result is an immediate consequence of Theorem 3.1 in [12].

Proposition 6 *Suppose that (A, B) is mean square stabilizable. Then there exists a maximal solution $V = (V_1, \dots, V_N)$ to (14). Moreover if V is the mean square stabilizing solution of (14) then it is the unique mean square stabilizing solution of (14). Furthermore, defining the following sequence of CARE,*

$$A_i'X_i + X_iA_i - X_iB_i(D_i'D_i)^{-1}B_i'X_i + \sum_{j=1}^{N} \pi_{ij}X_j + C_i'C_i + \frac{I}{k} = 0, \tag{15}$$

$i = 1, \dots, N$, we have that there exists the mean square stabilizing solution $V^k = (V_1^k, \dots, V_N^k)$ to (15) and $V_i^k \geq V_i^{k+1}$, $V^k \to V$ as $k \to \infty$.

We can now show the main result of this section, which relates the CARE and the convex problem (9).

Theorem 3 *The following assertions are equivalent:*

a) There exists the mean square stabilizing solution $\widehat{V} = (\widehat{V}_1, \dots, \widehat{V}_N) \in \mathcal{H}^{n+}$ to (14).
b) There exists $\widehat{W} = (\widehat{W}_1, \dots, \widehat{W}_N)$ such that

$$\widehat{W} = (\widehat{W}_1, \dots, \widehat{W}_N) = arg\ min\{\sum_{i=1}^{N} tr(R_iW_i);\ W = (W_1, \dots, W_N) \in \boldsymbol{\mathcal{C}}\}.$$

Moreover if $\widehat{V} = (\widehat{V}_1, \dots, \widehat{V}_N) \in \mathcal{H}^{n+}$ is the mean square stabilizing solution to (14), then setting $\widehat{K}_i = (D_i'D_i)^{-1}B_i'\widehat{V}_i$, $i = 1, \dots, N$, $\widehat{K} = (\widehat{K}_1, \dots, \widehat{K}_N)$, $\widehat{P} = (\widehat{P}_1, \dots, \widehat{P}_N) > 0$ satisfying

$$(A_j - B_j\widehat{K}_j)\widehat{P}_j + \widehat{P}_j(A_j - B_j\widehat{K}_j)' + \sum_{i=1}^{N} \pi_{ij}\widehat{P}_i + \alpha_jJ_jJ_j' = 0, \quad j = 1, \dots, N,$$

and $\widehat{W} = (\widehat{W}_1, \dots, \widehat{W}_N)$ as

$$\widehat{W}_i = \begin{bmatrix} \widehat{P}_i & \widehat{P}_i\widehat{K}_i' \\ \widehat{K}_i\widehat{P}_i & \widehat{K}_i\widehat{P}_i\widehat{K}_i' \end{bmatrix}, \quad i = 1, \dots, N,$$

we have that $\widehat{W}$ satisfies b). On the other hand if $\widehat{W}$ satisfies b), then setting $\widehat{K}_i = \widehat{W}_{i2}'\widehat{W}_{i1}^{-1}$ and $\widehat{V} = (\widehat{V}_1, \dots, \widehat{V}_N) \in \mathcal{H}^{n+}$ satisfying for $i = 1, \dots, N$,

$$(A_i - B_i\widehat{K}_i)'\widehat{V}_i + \widehat{V}_i(A_i - B_i\widehat{K}_i) + \sum_{j=1}^{N} \pi_{ij}\widehat{V}_j + (C_i - D_i\widehat{K}_i)'(C_i - D_i\widehat{K}_i) = 0, \tag{16}$$

we have that $\widehat{V}$ is the mean square stabilizing solution of (14).

PROOF. We shall prove that a) is equivalent to b). The second part of the theorem follows as a by product of this proof. Let us prove first that b) implies a). Indeed, let $\widehat{W} = (\widehat{W}_1, \ldots, \widehat{W}_N) = \arg\min\{\sum_{i=1}^{N} tr(R_i W_i); W = (W_1, \ldots, W_N) \in \mathcal{C}\}$, and write $\widehat{K}_i = \widehat{W}_{i2}' \widehat{W}_{i1}^{-1}$, $i = 1, \ldots, N$, $\widehat{K} = (\widehat{K}_1, \ldots, \widehat{K}_N)$. As seen in Proposition 4, $\widehat{K} \in \mathbb{K}$ and from Proposition 2, there exists a unique solution $\widehat{V} = (\widehat{V}_1, \ldots, \widehat{V}_N) \in \mathcal{H}^{n+}$ satisfying equation (16). From the fact that (A,B) is mean square stabilizable (indeed, $\widehat{K} \in \mathbb{K} \neq \emptyset$) we know, from Proposition 6, that there exists a maximal solution $V = (V_1, \ldots, V_N)$ to (14). Moreover from equation (14) we have that

$$(A_i - B_i\widehat{K}_i)'V_i + V_i(A_i - B_i\widehat{K}_i) + \sum_{j=1}^{N} \pi_{ij} V_j + (C_i - D_i\widehat{K}_i)'(C_i - D_i\widehat{K}_i)$$
$$- (K_i - \widehat{K}_i)'(D_i'D_i)(K_i - \widehat{K}_i) = 0 \tag{17}$$

where $K_i = (D_i'D_i)^{-1}B_i'V_i, i = 1, \ldots, N$, and $K = (K_1, \ldots, K_N)$. From $\widehat{K} \in \mathbb{K}$, (16), (17) and Proposition 2, we get that $\widehat{V} \geq V$. As stated in Proposition 6, $V = \lim_{k\to\infty} V^k$ where V^k is the mean square stabilizing solution of (15) with $K_i^k = (D_i'D_i)^{-1}B_i'V_i^k$, $K^k = (K_1^k, \ldots, K_N^k) \in \mathbb{K}$. For each $k = 1, 2, \ldots$, let $\widehat{V}^k = (\widehat{V}_1^k, \ldots, \widehat{V}_N^k) \in \mathcal{H}^{n+}$ be the unique solution satisfying (see Proposition 2)

$$(A_i - B_iK_i^k)'\widehat{V}_i^k + \widehat{V}_i^k(A_i - B_iK_i^k) + \sum_{j=1}^{N} \pi_{ij}\widehat{V}_j^k + (C_i - D_iK_i^k)'(C_i - D_iK_i^k) = 0. \tag{18}$$

From $K^k \in \mathbb{K}$, (18), (15) and Proposition 2 it follows that $V^k \geq \widehat{V}^k$. By the optimality of $\widehat{W}$, Theorem 1, Theorem 2, and the fact that $K^k \in \mathbb{K}$ for each $k = 1, 2, \ldots$, we get that

$$\left\|\mathcal{G}_{\widehat{K}}\right\|_2^2 = \sum_{i=1}^{N} \alpha_i tr(J_i'\widehat{V}_iJ_i) \leq \|\mathcal{G}_{K^k}\|_2^2 = \sum_{i=1}^{N} \alpha_i tr(J_i'\widehat{V}_i^kJ_i) \leq \sum_{i=1}^{N} \alpha_i tr(J_i'V_i^kJ_i)$$

and taking the limit as $k \to \infty$, and recalling that $V^k \to V$ as $k \to \infty$, we obtain that

$$\sum_{i=1}^{N} \alpha_i tr(J_i'\widehat{V}_iJ_i) \leq \sum_{i=1}^{N} \alpha_i tr(J_i'V_iJ_i).$$

Thus, $\widehat{V} \geq V$ and $\sum_{i=1}^{N} \alpha_i tr(J_i'(\widehat{V}_i - V_i)J_i) \leq 0$, which yields $tr(J_i'(\widehat{V}_i - V)J_i) = 0$ for each $i = 1, \ldots, N$. Since $J_iJ_i' > 0$ by assumption, it follows that $tr(\widehat{V}_i - V_i) = 0$ for $i = 1, \ldots, N$, with $\widehat{V} \geq V$. It can only hold if $\widehat{V} = V$. Subtracting equations (16) and (17) yields for $i = 1, \ldots, N$,

$$(K_i - \widehat{K}_i)'(D_i'D_i)(K_i - \widehat{K}_i) = 0$$

and since $D_i'D_i > 0$ by assumption, it follows that $K_i = \widehat{K}_i$, showing that b) implies a). Let us now prove that a) implies b). Suppose $\widehat{V} = (\widehat{V}_1, \dots, \widehat{V}_N) \in \mathcal{H}^{n+}$ is the mean square stabilizing solution of (14). Then $\widehat{K} = (\widehat{K}_1, \dots, \widehat{K}_N) \in \mathbb{K}$, where for $i = 1, \dots, N$, $\widehat{K}_i = (D_i'D_i)^{-1}B_i'\widehat{V}_i$. For any $G = (G_1, \dots, G_N) \in \mathbb{K}$, define $S = (S_1, \dots, S_N) \in \mathcal{H}^{n+}$ as the unique solution satisfying for $i = 1, \dots, N$,

$$(A_i - B_iG_i)'S_i + S_i(A_i - B_iG_i) + \sum_{j=1}^{N} \pi_{ij}S_j + (C_i - D_iG_i)'(C_i - D_iG_i) = 0. \tag{19}$$

Equation (14) with $\widehat{V}$ can be written as

$$\begin{aligned}(A_i - B_iG_i)'\widehat{V}_i + \widehat{V}_i(A_i - B_iG_i) + \sum_{j=1}^{N} \pi_{ij}\widehat{V}_j + (C_i - D_iG_i)'(C_i - D_iG_i)\\ - (\widehat{K}_i - G_i)'(D_i'D_i)(\widehat{K}_i - G_i) = 0, \;\; i = 1, \dots, N.\end{aligned} \tag{20}$$

and from (19), (20) and Proposition 2, $S \geq \widehat{V}$. Moreover, since

$$(A_i - B_i\widehat{K}_i)'\widehat{V}_i + \widehat{V}_i(A_i - B_i\widehat{K}_i) + \sum_{j=1}^{N} \pi_{ij}\widehat{V}_j + (C_i - D_i\widehat{K}_i)'(C_i - D_i\widehat{K}_i) = 0,$$

$i = 1, \dots, N$, we have, from Theorem 1, that

$$\left\|\mathcal{G}_{\widehat{K}}\right\|_2^2 = \sum_{i=1}^{N} \alpha_i tr(J_i'\widehat{V}_iJ_i) \leq \sum_{i=1}^{N} \alpha_i tr(J_i'S_iJ_i) = \|\mathcal{G}_G\|_2^2,$$

and since G is arbitrary in $\mathbb{K}$, it follows that

$$\left\|\mathcal{G}_{\widehat{K}}\right\|_2^2 = \min\{\|\mathcal{G}_G\|_2^2; G \in \mathbb{K}\}.$$

The remainder of the proof is immediate from Theorem 2. $\square$

Remark 1 *Since the solution of the CARE (14) does not depend on J_jJ_j', we can see that the optimal controller does not depend on $\alpha\mathcal{N}$. In particular, the choice of α in definition 5 will not affect the optimal controller for the H_2-optimal control problem.*

6 Example

In this section we consider an example using the algorithm presented in the previous section. For the sake of comparisons we also implemented other solutions via algorithms to be found in the literature, whenever applicable. Provided that the Markov chain parameters are perfectly known, we can apply the algorithm I in [1], the algorithm in section III of [6], or the LMI optimization problem presented in [12] for handling the correspondent CARE. These methods stand as the most updated numerical solutions for the CARE to our knowledge. The convex solutions were implemented using the LMItool Matlab interface [4] designed for the LMI solver SP [13], and the methods in [1] and [6] were implemented by specific Matlab routines.

The example is a simple economic system based on Samuelson's multiplier-accelerator model (see [2]). Three modes of operation for an economic system are considered: 1 = "Norm", 2 = "Boom", 3 = "Slump". The parameters for each of these modes of operation are:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.545 & 0.626 \\ 0 & -1.570 & 1.465 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.106 & 0.087 \\ 0 & -3.810 & 3.861 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1.80 & -0.3925 & 4.52 \\ 3.14 & 0.100 & -0.28 \\ -19.06 & -0.148 & 1.56 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -0.283 \\ 0.333 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.087 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0.064 \\ 0.195 \\ -0.080 \end{bmatrix},$$

$$C_1 = C_2 = C_3 = \begin{bmatrix} I_3 \\ 0_{1\times 3} \end{bmatrix}, \quad D_1 = D_2 = D_3 = \begin{bmatrix} 0_{3\times 1} \\ 1 \end{bmatrix}.$$

and $J_1 = J_2 = J_3 = \mathrm{I}_3$, where I_r represents the r by r identity matrix and $0_{\mathrm{r}\times\mathrm{s}}$ the null matrix of correspondent dimension. We adopt $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, and regarding the transition matrix Π, two situations were considered: i) Π is exactly known, and in this case we also solved the problem using the methods in [1,6,12]; ii) Π is not exactly known, but belongs to a set $\mathbb{D}$ as defined in section 2.

i) Π exactly known, and equal to

$$\Pi = \begin{bmatrix} -0.53 & 0.32 & 0.21 \\ 0.50 & -0.88 & 0.38 \\ 0.40 & 0.13 & -0.53 \end{bmatrix}.$$

In this case, the optimal solutions obtained are given in table 1. The results for the method in [6] are omitted in table 1 because it was not possible to

Table 1
Results for Example 1. (μ = H_2-norm; n =number of iterations)

Convex Method	Method in [1]	Convex Method in [12]
μ=2338	μ=2338	μ=2338
n = 26	n = 48	n = 29
K_1=[-1.951 -17.49 25.49]	K_1= [-1.951 -17.49 25.49]	K_1=[-1.951 -17.49 25.49]
K_2=[-1.106 -73.09 79.01]	K_2=[-1.101 -73.10 79.00]	K_2=[-1.101 -73.10 79.00]
K_3=[-93.41 10.41 -12.60]	K_3=[-93.41 10.41 -12.60]	K_3=[-93.41 10.41 -12.60]

attain convergence within the required precision, using the standard Matlab routine for the solution of the Lyapunov equation. The method could attain a precision of 10^{-3}, but iterations afterwards would led to a drifting from the correct solution and divergence. The methods otherwise agree, as table 1 shows; the control solution stabilizes the system in the MSS with $\max\{Re(\lambda(\mathcal{L}))\} = -0.2503$.

In order to exemplify the differences between the method proposed here and the convex method in [12], let us introduce the following modifications in the matrix Π : $\pi_{i2} = \pi_{2j} = 0$, for all i and j, so that $\pi_{11} = -\pi_{13} = -0.21$ and $\pi_{33} = -\pi_{31} = -0.404$. Thus, there are not jumps to and from the Markov state 2, and notice that the pair (A_2,B_2) is not controllable, with the eigenvalue in the origin lying on the noncontrollable part of the pair (A_2,B_2). No stabilizing solution can be expected in this situation and, in fact, the method we proposed yields a reply from the LMItool package stating that the problem is unfeasible. On the other hand, the LMI problem devised by the methodology in [12] provides the solution: K_1 =[0.1262 -0.9989 1.7054], K_2 =[0 -0.162 0.1199] and K_3 =[-3.650 0.4780 -1.484], but one can check for this solution that $\max\{Re(\lambda(\mathcal{L}))\} = 7.53$, showing that it does not stabilizes the system.

ii) Π is not exactly known, and the set $\mathbb{D}$ is defined by the following extreme values:

$$\Pi^1 = \begin{bmatrix} -0.53 & 0.32 & 0.21 \\ 0.50 & -0.88 & 0.38 \\ 0.40 & 0.13 & 0.53 \end{bmatrix}, \ \Pi^2 = \begin{bmatrix} -0.7 & 0.38 & 0.32 \\ 0.6 & -1 & 0.4 \\ 0.5 & 0.2 & -0.7 \end{bmatrix}.$$

In this case, the H_2-guaranteed cost and associated control are given in table 2. Notice that Π as defined in part i) above belongs to $\mathbb{D}$ ($\Pi = \Pi^1$) and, as expected, $\beta > \mu$.

Table 2
Results for Example 1 – Π not exactly known, Convex Method.

H_2-guaranteed cost (β) = 2861		
K_1=[-2.218 -19.16 28.32]	K_2=[-1.267 -70.88 78.48]	K_3=[-100.8 12.14 -11.81]

7 Conclusions

In this paper we have considered the H_2-control of Markovian jump linear systems (MJLS). A convex programming formulation to the problem was proposed, which leads to efficient numerical algorithms. The power of this technique is that the important case in which the transition rate matrix Π may not be exactly known, can be easily appended to the problem as convex restrictions, leading to an upper bound to the H_2-cost control problem. This problem cannot be handled by using the traditional dynamic programming formulation, which leads to a the solution of a set of coupled algebraic equations (CARE). For the situation in which Π is exactly known, we have shown that the existence of the mean square stabilizing solution to CARE is equivalent to the existence of a solution to the convex programming problem, establishing a link between both formulations. As mentioned in section 1, this link is different from the one established in [12], which deals with a convex problem and the maximal solution of the CARE. In addition, the method provided reliable MSS solution in all the examples that we have implemented.

ACKNOWLEDGMENT: The authors acknowledge a helpful comment of an anonymous reviewer.

References

[1] Abou-Khandil, H., Freiling, G. and Jank, G. (1994). Solution and asymptotic behavior of coupled Riccati equations in jump linear systems. IEEE Trans. on Automat. Control, 39, 1631-1636.

[2] Blair, W.P. Jr and Sworder, D.D. (1975). Continuous-time regulation of a class of econometric models, IEEE Trans. Systems Man Cyber., 5, 341-346.

[3] Boyd, S., Ghaoui, L. El, Feron, E. and Balakrishnan, V. (1994). Linear Matrix Inequalities and Control Theory, SIAM, Philadelphia.

[4] Ghaoui, E. El, Nikoukhah, R. and Delebecque, F. (1995). LMITOOL: a front-end for LMI optimization—User's guide, available via anonymous ftp from ftp.ensa.fr /pub/elghaoui/lmitool.

[5] Costa, O.L.V., do Val, J.B.R, and Geromel, J.C. (1997) A convex programming approach to H$_2$-control of discrete-time Markovian jump linear systems, International Journal of Control, 66, 557-579

[6] Gajic, Z. and Borno, I. (1995). Lyapunov iterations for optimal control of jump linear systems at steady state, IEEE Trans. on Automat. Control, 40, 1971-1975.

[7] Feng, X., Loparo, K.A., Ji, Y. and Chizeck, H.J. (1992). Stochastic stability properties of jump linear systems, IEEE Trans. Automat. Control, 37, 38-53.

[8] Ji, Y. and Chizeck, H.J. (1990). Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control, IEEE Trans. Automat. Control, 35, 777-788.

[9] Mariton, M. (1988). Almost sure and moments stability of jump linear systems, Systems and Control Letters, 11, 393-397.

[10] Mariton, M. (1990). Jump linear systems in automatic control, Marcel Dekker, New York.

[11] Rami, M.A. and Ghaoui, E. El. (1996a). Robust state-feedback stabilization of jump linear systems via LMIs, Int. Journal of Robust and Nonlinear Control, 6, 1015-1022.

[12] Rami, M.A. and Ghaoui, E. El (1996b). LMI optimization for nonstandard Riccati equations arising in stochastic control, IEEE Trans. Automat. Control, 41, 1666-1671.

[13] Vandenberghe, L. and Boyd, S. (1994). Software for semidefinite programming—User's guide, available via anonymous ftp from isl.stanford.edu /pub/boyd/semidef_prog.