SIAM J. CONTROL OPTIM.
Vol. 37, No. 5, pp. 1483-1502

STABILITY OF PIECEWISE-DETERMINISTIC MARKOV PROCESSES *

FRANÇOIS DUFOUR † AND OSWALDO L.V. COSTA ‡

Abstract. In this paper, we study a form of stability for a general family of non-diffusion Markov processes, known in the literature as piecewise-deterministic Markov process (PDMP). By stability here we mean the existence of an invariant probability measure for the PDMP. It is shown that the existence of such invariant probability measure is equivalent to the existence of a σ-finite invariant measure for a Markov kernel G linked to the resolvent operator U of the PDMP, satisfying a boundedness condition or, equivalently, a Radon-Nikodym derivative. Here we generalize existing results of the literature [2, 4] since we do not require any additional assumptions to establish this equivalence. Moreover, we give sufficient conditions to ensure the existence of such σ-finite measure satisfying the boundedness condition. They are mainly based on a modified Foster-Lyapunov criterion for the case in which the Markov chain generated by G is either recurrent or weak Feller. To emphasize the relevance of our results, we study three examples and in particular, we are able to generalize the results obtained by Costa and Davis on the capacity expansion model.

Key words. Piecewise-deterministic Markov process, invariant measure, Markov chain

AMS subject classifications. 60J25, 60J10

PII. S0363012997330890

1. Introduction. Piecewise-deterministic Markov processes (PDMPs) were introduced by Davis [3] as a general family of non-diffusion stochastic models suitable for formulating many optimization problems in several areas of applications. The motion of a PDMP $\{x_t\}$ in a state space E depends on three parameters, namely, the flow Φ, the jump rate λ and the transition probability measure Q. Starting from a point x in E, the motion of the process follows the flow $\Phi(x,t)$ until the first jump time T_1. This jump time occurs either when the flow hits the boundary of the state space E or in a Poisson like fashion with rate $\lambda(\Phi(x,t))$. The location Z_1 of the process at the jump time T_1 is selected by the measure $Q(\Phi(x,T_1),.)$ and the motion restarts from this new point Z_1 as before. This gives a piecewise-deterministic trajectory for the PDMP $\{x_t\}$ with jump times $\{T_1,T_2,\ldots\}$ and post-jump location $\{Z_1,Z_2,\ldots\}$. By a suitable choice of the state space E and the parameters Φ, λ and Q it is possible to virtually model all non-diffusion processes found in the international literature. The reader can find in [4] a complete and updated description of the theory of PDMPs as well as several important applications.

A problem of great importance in the theory of stochastic processes that has been in evidence for over the last decades is to obtain conditions for the following problem:

*Received by the editors December 1, 1997; acccepted for publication (in revised form) September 29, 1998; published electronically July 27, 1999. This research was completed during a USP/COFECUB Franco-Brazilian cooperation mission (number UC 40/97).

†Laboratoire des Signaux et Systèmes, C.N.R.S. – E.S.E. Plateau de Moulon, 91192 Gif-sur-Yvette Cedex, France (dufour@lss.supelec.fr).

‡Departamento de Engenharia Eletrônica, Escola Politécnica da Universidade de São Paulo, CEP: 05508 900-Sao Paulo, Brazil (oswaldo@lac.usp.br). This author received financial support from FAPESP (Research Council of the State of São Paulo) and CNPq (Brazilian National Research Council).

$\mathbf{P}_1$. Existence of an invariant probability measure for a stochastic process.

The existence of such a measure corresponds to an important form of stochastic stability and it is the first step towards an ergodic analysis of the process. For a small sample of the huge theory nowadays available on this subject, see [9, 8, 11, 12, 15, 13]. In general it is not a simple matter to determine whether a given process in a general state space has an invariant probability measure and if so, whether it is unique. For discrete-time stochastic processes there is available in [14] a rather complete treatment on this subject.

The main goal of [2] (see also [4]) was to derive conditions for the existence of an invariant probability measure for a PDMP $\{x_t\}$. Associated to the PDMP $\{x_t\}$ there is a Markov chain given by the post-jump location $\{Z_1, Z_2, \ldots\}$ with kernel $\overline{G}$ (see equation (3.8) for its definition). It is natural to ask then if it is possible to link the invariant probability measures associated to the Markov chain $\{Z_1, Z_2, \ldots\}$ and the invariant probability measures associated to the PDMP $\{x_t\}$. Since the conditions for existence and uniqueness of an invariant probability measure are more easily stated in the discrete-time setting, this connection provides a way of analyzing problem $\mathbf{P}_1$ for a PDMP $\{x_t\}$ without relying on the general continuous-time theory. It was shown in [2] that, under some technical assumptions, there is a one to one mapping between the invariant probability measures for the PDMP $\{x_t\}$ and the invariant probability measures of the Markov chain $\{Z_1, Z_2, \ldots\}$. However this result is not entirely satisfactory since the technical assumptions in [2] require that, roughly speaking, the jump rate λ must be strictly positive and bounded. Important applications presented in [4] show that if these conditions are violated then it may exist an invariant probability measure for the PDMP $\{x_t\}$ but not for the associated Markov chain $\{Z_1, Z_2, \ldots\}$, and vice-versa.

In the first part of this paper the problem $\mathbf{P}_1$ for a PDMP $\{x_t\}$ is revisited, under a new approach, that removes the assumptions made in [2]. Instead of using the kernel $\overline{G}$ associated to the Markov chain $\{Z_1, Z_2, \ldots\}$, we introduce a new kernel G (for its definition, see equation (3.4)), related to the resolvent kernel U of the PDMP $\{x_t\}$, and show that problem $\mathbf{P}_1$ for a PDMP $\{x_t\}$ is equivalent to the following one:

$\mathbf{P}_2$. Existence of an invariant σ-finite measure for the Markov chain associated to the kernel G satisfying a boundedness condition.

It is important to stress that this equivalence requires no extra conditions than those usually imposed on the definition of the PDMP $\{x_t\}$, being therefore a significant improvement with respect to those results presented in [2]. Notice that $\mathbf{P}_2$ is related to an invariant σ-finite measure and therefore it is not necessarily finite. As it will be seen in section 3, the boundedness condition in $\mathbf{P}_2$ can also be characterized as a condition in terms of the Randon-Nikodym derivative of the invariant σ-finite measure for G and a certain probability measure. As in [2], a one to one mapping between the invariant probability measures for the PDMP $\{x_t\}$ and the invariant σ-finite measures for the Markov chain associated to the kernel G is established. The examples in [4] are revisited under this new framework and, of course, the problems found in [4] are eliminated with our approach.

It is well known that $\mathbf{P}_1$ is equivalent to the problem of the existence of an invariant probability measure for the resolvent kernel associated to the transition function of the stochastic process (see [1]). It is important to point out that such a direct approach is vain to our problem due to the fact that both the transition function of a general PDMP $\{x_t\}$ as well as its associated resolvent kernel cannot be explicitly calculated. Here our strategy is to use the special structure of the PDMP to provide a tractable answer to the problem $\mathbf{P}_1$.

In the second part of this paper we give weak sufficient conditions for ensuring that $\mathbf{P}_2$ admits a solution. These conditions are based on a modified Foster-Lyapunov criterion for the case in which the kernel G is either recurrent or weak Feller and, we believe, significantly improve previous results presented in [2, 4] for problem $\mathbf{P}_1$ for a PDMP $\{x_t\}$. Indeed, an interesting property of the PDMP's is that under weak conditions (see proposition 4.9) they generate weak Feller kernels G and U although the PDMP's themselves are generally not weak Feller Markov processes. Our results are related to those in Meyn and Tweedie [15, 13], Lin [10], and Foguel [7]. Moreover, it is shown that an invariant probability measure always exists for the PDMP $\{x_t\}$ when G is weak Feller and the union of the state space of $\{x_t\}$ and the active boundary (for its definition see section 2) is a compact set. To illustrate the usefulness of our new conditions, the capacity expansion problem (see [4]) is revisited in a more general set up. A sufficient condition for the existence of an invariant measure is obtained, generalizing previous results in the literature.

The paper is organized in the following way. In section 2 we describe the model and introduce the notation. The equivalence between $\mathbf{P}_1$ for PDMP and $\mathbf{P}_2$ is established in section 3. In section 4 we derive sufficient condition for checking $\mathbf{P}_2$. Illustrative examples are given in section 5. We conclude the paper in section 6 with some final remarks.

2. Model of piecewise deterministic Markov processes. Let E^0 be an open subset of $\mathbb{R}^n$ and ∂E^0 its boundary. As originally defined by Davis [3, 4], a PDMP is determined by its local characteristics $(\mathfrak{X}, \lambda, Q)$ where
— $\mathfrak{X}$ is a locally Lipschitz continuous vector field in E^0 with flow $\Phi(x,t)$.
Now define

$$\Gamma^+ \doteq \{x \in \partial E^0 : x = \Phi(y,t), y \in E^0, t > 0\}$$

and

$$\Gamma^- \doteq \{x \in \partial E^0 : x = \Phi(y,-t), y \in E^0, t > 0\}.$$

$\Gamma^+ \subset \partial E^0$ represents the boundary points at which the flow exits from E^0. $\Gamma^- \subset \partial E^0$ is characterized by the fact that the flow starting from a point in Γ^- will not leave E^0 immediatly. Therefore it is natural to define the state space for the PDMP by

$$E \doteq E^0 \cup \Gamma^- - \Gamma^- \cap \Gamma^+$$

$\mathcal{E}$ will denote the Borel σ-field of E.
Let us denote by $t_*(x) \doteq \inf\{t > 0 : \Phi(x,t) \in \partial E^0\}$ for all x in E. If this set is empty then $t_*(x) = \infty$.
Assume that the explosion time of the flow Φ is equal to infinity when $t_*(x) = \infty$.
— $\lambda : E \to \mathbb{R}^+$ is the jump rate. It is assumed that this function is measurable and

$(\forall x \in E)$ $(\exists \varepsilon > 0)$ such that $\int_0^\varepsilon \lambda(\Phi(x,s))ds$ exists.
— $Q : E \cup \Gamma^+ \times \mathcal{E} \to [0,1]$ is the transition measure satisfying the following property: $(\forall x \in E \cup \Gamma^+)$ $Q(x, E - \{x\}) = 1$.

It is shown in [3, 4] that there exist a probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, P_x)$ on which the motion of the PDMP $\{x_t\}$ is defined as follows. Starting from x, the first jump time T_1 of the process is given by

$$P_x(T_1 > t) \doteq I_{\{t<t_*(x)\}} \exp\left\{-\int_0^t \lambda(\Phi(x,s))ds\right\}.$$

Then

$$(\forall t \in [0, T_1)) \qquad x_t = \Phi(x,t) \qquad \text{and} \qquad x_{T_1} = Z_1$$

where Z_1 is the random variable having as distribution: $Q(\Phi(x,T_1),.)$. Now the process follows the path $\Phi(x_{T_1}, t - T_1)$ up to another jump time T_2 where it jumps to the state $x_{T_2} = Z_2$. The inter-jump time $T_2 - T_1$ and the post-jump random variable Z_2 are defined similarly.

It will be assumed that $(\forall (t,x) \in \mathbb{R}_+ \times E)$, $\quad E_x[\sum_k I_{\{t \geq T_k\}}] < \infty$.

For a more general definition of PDMP the reader is referred to Davis [4, section 29]. However our results are still valid in this setting.

Notation.
Let us denote by μ_{Leb} the Lebesgue measure on $\mathbb{R}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.
Let $B(E)$ be the space of real-valued bounded Borel-measurable functions on E.
$C_b(E)$ ($C_c(E)$ respectively) denotes the space of bounded real-valued continuous functions on E (with compact support respectively).
For all $A \in \mathcal{E}$, $A^c \doteq E - A$ and $\overline{A}$ is the closure of A.
Let ρ be a metric on E; then for all $x \in E$ and $A \in \mathcal{E}$, $d(x,A) \doteq \inf_{y \in A} \rho(x,y)$.
Let X be a metric space and $\mathcal{B}(X)$ its Borel σ-field.
$\mathcal{M}(X)$ ($\mathcal{M}_b(X)$ respectively) denotes the set of all σ-finite (finite respectively) measures defined on $(X, \mathcal{B}(X))$.
Finally, we recall the following classical notations: let K_1 and K_2 be two kernels mapping $(E, \mathcal{E})$ into $\mathbb{R}_+$. Then

$$(\forall \mu \in \mathcal{M}(E)),\ (\forall A \in \mathcal{E}), \quad \mu K_1(A) \doteq \int_E K_1(x,A)\mu(dx)$$

$$(\forall f \in B(E)),\ (\forall x \in E), \quad K_1 f(x) \doteq \int_E f(y) K_1(x,dy)$$

$$(\forall x \in E),\ (\forall A \in \mathcal{E}), \quad K_1 K_2(x,A) \doteq \int_E K_2(y,A) K_1(x,dy)$$

provided that the right members exist (see [17, Definition 1.2 and 1.5, Chapter 1]).

3. Invariant probability measure. The main goal of this section is to show that $\mathbf{P}_1$ for a PDMP is equivalent to $\mathbf{P}_2$ (see theorem 3.5). This new result represents a significant improvement with respect to those of Costa [2] and Davis [4]. Indeed, unlike [2, 4] this equivalence is established without any additional conditions. The key idea is to use a connection between the resolvent kernel U (see equation (3.6)) and the kernel G (see equation (3.4)) which comes from lemma 3.2.

Moreover, in order to provide a complete comparison with the results obtained by Costa and Davis (see for example (34.36) in [4]), we derive transformations which link the invariant probability measures for $\{x_t\}$ and the stochastic kernel G (see corollary 3.6).

For notational convenience, let us define for $x \in E$ and $t \in [0, t_*(x)[$

$$\Lambda(x,t) \doteq \int_0^t \lambda(\Phi(x,s))ds \tag{3.1}$$

and the following kernels mapping $(E, \mathcal{E})$ into $[0,1]$

$$L(x,A) \doteq \int_0^{t_*(x)} \exp -\{s + \Lambda(x,s)\} I_A(\Phi(x,s))ds, \tag{3.2}$$

$$\begin{aligned} K(x,A) \doteq & \int_0^{t_*(x)} \lambda(\Phi(x,s)) \exp -\{s + \Lambda(x,s)\} Q(\Phi(x,s),A)ds \\ & + \exp -\{t_*(x) + \Lambda(x,t_*(x))\} Q(\Phi(x,t_*(x)),A), \end{aligned} \tag{3.3}$$

$$G(x,A) \doteq K(x,A) + L(x,A) \tag{3.4}$$

$$L_n(x,A) \doteq \sum_{i=0}^{n-1} K^i L(x,A) \tag{3.5}$$

$$U(x,A) \doteq E_x \int_0^\infty I_A(x_t) \exp\{-t\} dt \tag{3.6}$$

$$S(x,A) \doteq L(x,E) I_A(x) \tag{3.7}$$

$$\begin{aligned} \overline{G}(x,A) \doteq & \int_0^{t_*(x)} \lambda(\Phi(x,s)) \exp\{-\Lambda(x,s)\} Q(\Phi(x,s),A)ds \\ & + \exp\{-\Lambda(x,t_*(x))\} Q(\Phi(x,t_*(x)),A). \end{aligned} \tag{3.8}$$

In the terminology used by Nummelin [16], the kernels G, U, $\overline{G}$ are stochastic kernels because for all $x \in E$, $G(x,E) = U(x,E) = \overline{G}(x,E) = 1$. $\overline{G}$ is the stochastic kernel for the Markov chain $\{Z_1, Z_2, \ldots\}$. U is the resolvent kernel associated to $\{x_t\}$.

REMARK 3.1. *Note that for all $x \in E$, $0 < L(x,E) < 1$ and that G is a stochastic kernel. Indeed, for all $x \in E$,*

$$L(x,E) + K(x,E) = 1. \tag{3.9}$$

Therefore, we can define the kernel $R : E \times \mathcal{E} \longrightarrow \mathbb{R}_+$ such that

$$R(x,A) \doteq \frac{I_A(x)}{L(x,E)}. \tag{3.10}$$

The following result establishes a connection between the kernels K, L and the resolvent U.

LEMMA 3.2.

$$(\forall x \in E), \quad (\forall A \in \mathcal{E}), \qquad KU(x,A) + L(x,A) = U(x,A). \tag{3.11}$$

Proof. From section 32.2 and Proposition 32.34 in Davis [4] and the definition of L_n (see equation (3.5)), it is easy to obtain:

$$(\forall x \in E), \quad (\forall A \in \mathcal{E}), \qquad \lim_{n\to\infty} \uparrow L_n(x,A) = U(x,A). \tag{3.12}$$

and

$$(\forall x \in E), \quad (\forall A \in \mathcal{E}), \qquad KL_n(x,A) + L(x,A) = L_{n+1}(x,A).$$

Now using the monotone convergence theorem, the result follows. □

REMARK 3.3. *Let us introduce the kernel* $J : E \times \mathcal{E} \longrightarrow \mathbb{R}_+$ *such that*

$$J(x,A) \doteq \sum_{j=0}^{\infty} \int_A L(y,E)K^j(x,dy)\cdot \tag{3.13}$$

Using equations (3.12) and the definition of J *we have that for all* $x \in E$ $J(x,E) = U(x,E) = 1$. *Therefore* J *is a stochastic kernel.*
Before presenting the first main result of this section, let us derive the following technical lemma:

LEMMA 3.4. *Let* α *be a* σ*-finite measure defined on* $(E,\mathcal{E})$. *Then assertions (i) and (ii) are equivalent:*
(i) α *is such that* $\alpha = c\nu R$ *where* ν *is a probability measure and* c *a constant in* $(0,\infty)$.

(ii) α *is such that* $\alpha S(E) < \infty$.

Proof. *(i)* $\Rightarrow$ *(ii).* Obvious.
(ii) $\Rightarrow$ *(i).* Define the probability measure $\nu \doteq \dfrac{\alpha S}{\alpha S(E)}$. Then clearly $\alpha = c\nu R$ with $c = \alpha S(E)$. □

The next theorem establishes the connection between $\mathbf{P}_1$ and $\mathbf{P}_2$.

THEOREM 3.5. *The following assertions are equivalent.*
(i) There exists an invariant probability measure μ *for* $\{x_t\}$.
(ii) There exists a probability measure ν *such that the positive* σ*-finite measure* $\pi = \nu R$ *is invariant for the stochastic kernel* G.
(iii) There exists a positive σ*-finite measure* π *invariant for the stochastic kernel* G *such that* $\pi S(E) < \infty$.

Proof. *(ii)* $\Rightarrow$ *(i).* Assume that there exists a probability measure ν such that $\pi \doteq \nu R$ is invariant for G and define the measure $\mu \doteq \pi L$. Note that for all $A \in \mathcal{E}$, $\mu(A) = \displaystyle\int_E \frac{L(y,A)}{L(y,E)}\nu(dy)$. Therefore μ is a probability measure. Since π is a positive σ-finite measure, there exists a partition $\{E_i\}$ of E such that $\pi(E_i) < \infty$. Recalling that $L = G - K$ (see equation (3.4)) and $\pi = \pi G$ we have that for all $A \in \mathcal{E}$,

$$\begin{aligned} \mu U(A) &= \sum_i \int_{E_i} U(x,A)\pi L(dx) \\ &= \sum_i \int_{E_i} U(x,A)\pi(I-K)(dx). \end{aligned}$$

Now using lemma 3.2, it follows that

$$\begin{aligned}\mu U(A) &= \sum_i \int_{E_i} \{U(x,A) - KU(x,A)\}\pi(dx)\\ &= \sum_i \int_{E_i} L(x,A)\pi(dx)\\ &= \mu(A).\end{aligned}$$

Therefore from Lemma 1 in [1], μ is an invariant measure for $\{x_t\}$.
(i) $\Rightarrow$ (ii). Let μ be an invariant probability measure for the process $\{x_t\}$. Then $\mu U = \mu$ (see [1]). Let us define the measure ν by $\nu \doteq \mu J$ with J defined in equation (3.13). The measure so defined is a probability since $\mu J(E) = \mu U(E) = \mu(E) = 1$ (see remark 3.3). Let us introduce $\pi \doteq \nu R$ (see equation (3.10)). Then, from (3.10) and (3.13) we obtain:

$$\pi = \sum_{j=0}^{\infty} \mu K^j. \tag{3.14}$$

From (3.14) and (3.12) and lemma 3.2, it is easy to obtain that $\pi L = \mu$ and $\pi K + \mu = \pi$.
Consequently $\pi G = \pi(K+L) = \pi K + \mu = \pi$.
(ii) $\Leftrightarrow$ (iii). Using lemma 3.4 the equivalence is straightforward.
This completes the proof. $\square$

As mentioned in the introduction, the condition on the σ-finite invariant measure π for the Markov chain associated to G can be written in terms of the boundedness of $\pi S(E)$ (*(iii)* in the theorem) or in terms of the Radon-Nikodym derivative $L(x,E)^{-1}$ of π with respect to a probability measure ν (*(ii)* in the theorem).
The next result gives explicitely the links between ν and μ using an approach that differs from Costa [2] and Davis [4]. In particular, we avoid to introduce the so-called *boundary measure* associated to the invariant probability measure of $\{x_t\}$ (see for example [4, p.116-118]). Our approach uses directly the kernels J, L and R. Moreover, it is shown that these mappings are one to one, so providing a complete comparison with results known in the literature (see for example Davis, (34.36) in [4]).

COROLLARY 3.6. *(i) If μ is an invariant probability measure for $\{x_t\}$ then the σ-finite measure μJR is invariant for G and $\mu JRL = \mu$.*
(ii) If ν is a probability measure such that νR is invariant for G then the probability measure νRL is invariant for $\{x_t\}$ and $\nu RLJ = \nu$.

Proof. *(i)* and the first part of *(ii)* are easy consequences of theorem 3.5.
In order to show that if ν is a probability measure such that νR is invariant for G then $\nu RLJ = \nu$, let us first note that

$$(\forall x \in E), \quad (\forall A \in \mathcal{E}), \qquad J(x,A) - KJ(x,A) = I_A(x)L(x,E). \tag{3.15}$$

Since νR is a positive σ-finite measure, there exists a partition $\{E_i\}$ of E such that $\nu R(E_i) < \infty$. Moreover, using the fact that $\nu RK + \nu RL = \nu R$ and equation (3.15) we have

$$\nu RLJ(A) = \sum_i \int_{E_i} J(x,A)\nu RL(dx)$$

$$= \sum_i \int_{E_i} I_A(x) L(x,E)\nu(dx)$$
$$= \nu(A).$$

It completes the proof. □

The following result shows that if λ is bounded then $\mathbf{P}_1$ is equivalent to the existence of a finite invariant measure for G. In what follows, we set (see [4, sections 24 and 26]):

$$N_t \doteq \sum_{i=1}^{\infty} I_{\{t \geq T_i\}}$$
$$p^*(t) \doteq \sum_{i=1}^{\infty} I_{\{t \geq T_i\}} I_{\{x_{T_i-} \in \Gamma^+\}}$$

Note that $\{N_t\}$ and $\{p^*(t)\}$ are counting processes, N_t being the number of jumps of the process $\{x_t\}$ in the time interval $[0,t]$ and $p^*(t)$ counts the number of jumps from the boundary up to time t.

PROPOSITION 3.7. *Let μ be an invariant probability measure for the process $\{x_t\}$ with $E_\mu[N_t] < \infty$ for all $t \in \mathbb{R}$ and define $\pi \doteq \mu JR$. Then the following assertions are equivalent:*
(i) $\pi(E) < \infty$.
(ii) $\int_E \lambda(x)\mu(dx) < \infty$.

Proof. By repeating the same arguments as in Proposition 34.13 and Theorem 34.15 of Davis [4, p.116-117] and using the fact that $E_\mu[N_t] < \infty$ for all $t \in \mathbb{R}$, we obtain that

$$E_\mu[\int_0^t \exp\{-s\} dp^*(s)] = (1 - \exp\{-t\})\sigma(\Gamma^+)$$

where σ is a finite measure defined on $(\Gamma^+, \mathcal{B}(\Gamma^+))$ by Theorem 34.15 of Davis (see [4, p.117]).
From the monotone convergence theorem, it follows that

$$E_\mu[\int_0^\infty \exp\{-s\} dp^*(s)] = \sigma(\Gamma^+). \tag{3.16}$$

Consequently, we can define:

$$\Psi_n \doteq E_\mu[\int_0^\infty \exp\{-s\}\lambda_n(x_s)ds + \int_0^\infty \exp\{-s\} dp^*(s)] \tag{3.17}$$

where

$$\lambda_n(x) \doteq \begin{cases} \lambda(x) & \text{if } \lambda(x) < n, \\ n & \text{otherwise.} \end{cases}$$

Using Fubini's Theorem and the fact that μ is an invariant propobability measure for $\{x_t\}$, we get that

$$E_\mu[\int_0^t \exp\{-s\}\lambda_n(x_s)ds] = (1 - \exp\{-t\}) \int_E \lambda_n(x)\mu(dx)$$

and again from the monotone convergence Theorem,

$$E_\mu[\int_0^\infty \exp\{-s\}\lambda_n(x_s)ds] = \int_E \lambda_n(x)\mu(dx). \tag{3.18}$$

Therefore, combining equations (3.16), (3.17) and (3.18), we have:

$$\Psi_n = \int_E \lambda_n(x)\mu(dx) + \sigma(\Gamma^+).$$

Now using the monotone convergence Theorem, we obtain:

$$\lim_{n\to\infty} \Psi_n = \int_E \lambda(x)\mu(dx) + \sigma(\Gamma^+). \tag{3.19}$$

On the other hand, it is easy to show from section 32.2 and Proposition 32.34 of Davis [4] that Ψ_n can be written as

$$\Psi_n = \mu \sum_{i=0}^{\infty} K^i \phi_n \tag{3.20}$$

where

$$\phi_n = \int_0^{t_*(x)} \lambda_n(\Phi(x,s)) \exp -\{s + \Lambda(x,s)\}ds + \exp -\{t_*(x) + \Lambda(x, t_*(x))\} \quad .$$

Recalling the definition of the stochastic kernel K (see equation (3.3)) and from the monotone convergence Theorem, it follows that

$$\lim_{n\to\infty} \uparrow \phi_n = K(x, E)$$

Therefore, using equation (3.20), we have:

$$\lim_{n\to\infty} \Psi_n = \mu \sum_{i=1}^{\infty} K^i(E).$$

Now, with equation (3.19) and the previous one, it follows that

$$\mu \sum_{i=0}^{\infty} K^i(E) = 1 + \mu \sum_{i=1}^{\infty} K^i(E) = 1 + \int_E \lambda(x)\mu(dx) + \sigma(\Gamma^+). \tag{3.21}$$

However, recalling the definitions of the stochastic kernels J and R (see equations (3.13) and (3.10)), we have

$$\mu JR(E) = \mu \sum_{i=0}^{\infty} K^i(E),$$

Thus, combining the previous equation with equation (3.21), we obtain the result. □

REMARK 3.8. *It is clear that if $\lambda(.)$ is bounded then $\pi(E) < \infty$. Therefore, if $\lambda(.)$ is bounded, there exists an invariant probability measure for $\{x_t\}$ if and only if there exists an invariant probability measure for the stochastic kernel G. Assuming moreover that G is weak Feller, necessary and sufficient conditions can be given to ensure that $\{x_t\}$ has an invariant measure (see for example [10, 7, 9, 8]). However these conditions are difficult to check in practice since they need the limit of the sum of iterations of G.*

4. Sufficient conditions. The aim of this part is to give simple sufficient conditions to check if $\mathbf{P}_2$ (and therefore $\mathbf{P}_1$) has a solution. Such test criterion is based on Foster-Lyapunov inequalities. It has been proven in Meyn and Tweedie [14] that this test function implies the existence of invariant probability measure for a class of Markov chain. In our case, the problem is different, since, we have to find a σ-finite measure π for G that satisfy the condition $\pi S(E) < \infty$. Following the line of Meyn and Tweedie [14], Lin [10] and Foguel [7], we are able to show that a modified drift criterion ensure such a property for the case in which the Markov chain generated by G is either recurrent or weak Feller. Moreover, it is shown that an invariant probability measure always exists for the PDMP $\{x_t\}$ when G is weak Feller and the union of the state space of $\{x_t\}$ and the active boundary (for its definition see section 2) is a compact set.
First we recall some classical definitions related to Markov chain. For a complete exposition on the subject the reader is refered to the book of S. Meyn and R. Tweedie [14].

Notation.
We shall denote by $\{Y_n\}$ the Markov chain generated by the stochastic kernel G.

DEFINITION 4.1. *$\{Y_n\}$ is φ-irreducible if there exists a measure φ on $(E,\mathcal{E})$ such that, whenever $\varphi(A) > 0$, it follows that $P_x(\tau_A < \infty) > 0$ for all $x \in E$ where* $\tau_A = \inf\{n \geq 1 : Y_n \in A\}$. If $\{Y_n\}$ is φ-irreducible then $\psi = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \varphi G^n$ is a maximal irreducibility measure (see Proposition 4.2.2 in [14, p. 88]). In the following, we will use ψ to denote a maximal irreducibility measure for $\{Y_n\}$. Associated to a maximal irreducibility measure for $\{Y_n\}$, we define:

$$\mathcal{E}^+ \doteq \{A \in \mathcal{E} : \psi(A) > 0\}$$

DEFINITION 4.2. *$\{Y_n\}$ is called recurrent if it is ψ-irreducible and $\sum_{n=1}^{\infty} G^n(x, A) = \infty$ for all $x \in E$ and $A \in \mathcal{E}^+$.*

DEFINITION 4.3. *A set $C \in \mathcal{E}$ is called ν_a-petite if there exists a non-trivial measure ν_a on $(E,\mathcal{E})$ such that*

$$(\forall x \in C),\ (\forall B \in \mathcal{E}) \quad \sum_{n=1}^{\infty} a(n) G^n(x, B) \geq \nu_a(B)$$

where the positive sequence $\{a(n)\}$, satisfies $\sum_{n=0}^{\infty} a(n) = 1$.
Following similar ideas as in Proposition 14.1.1 of the book of Meyn and Tweedie [14, p. 333], we can now establish a sufficient condition for the existence and uniqueness of an invariant probability measure for the PDMP $\{x_t\}$ based on the associated Markov chain $\{Y_n\}$. Note however that, unlike in Proposition 14.1.1 in [14], it cannot be claimed that the invariant measure for the Markov chain $\{Y_n\}$ is finite, since $L(Y_n, E) < 1$.

THEOREM 4.4. *Suppose that $\{Y_n\}$ is recurrent and for a petite set $C \in \mathcal{E}^+$ we have:*

$$\sup_{x \in C} E_x[\sum_{n=1}^{\tau_C} L(Y_n, E)] < \infty \tag{4.1}$$

where $\tau_C = \inf\{n \geq 1 : Y_n \in C\}$.
Then there exists a unique invariant probability measure for the PDMP $\{x_t\}$.

Proof. Since $\{Y_n\}$ is recurrent then it admits a unique (up to constant multiples) σ-finite invariant measure π. From Proposition 10.1.2 in [14, p. 232], it follows that $\pi(C) < \infty$. Although here the function $L(.,E) < 1$, the rest of the proof is the same as in Proposition 14.1.1 in [14, p. 232] and we obtain the result. ☐

Condition given by equation (4.1) can be replaced by the following one, which may be easier to be checked, and is inspired from condition $(V3)$ of the book of Meyn and Tweedie [14, p. 337].

Condition ($\mathcal{D}$): For some set $C \in \mathcal{E}$, some constant $b < \infty$ and an extended real-valued function $V : E \to [0, \infty]$, we have

$$(\forall x \in E) \quad GV(x) - V(x) \leq -L(x, E) + bI_C(x).$$

This condition is of the same form as the drift or the Foster-Lyapunov criterion which is often used in the literature (see [13, 14] and references therein). However, we want to point out the fact that condition $(\mathcal{D})$ is weaker than conditions $(V2)$, $(V3)$ of the book of Meyn and Tweedie [14] due to the fact that $L(.,E) < 1$. Therefore, direct application of their results are not straightforward. Note in particular that, unlike Theorem 11.3.4 in [14, p. 265], condition $(\mathcal{D})$ does not guarantee that $E_x[\tau_C] \leq V(x) + bI_C(x)$ since $L(Y_n, E) < 1$ (see equation (4.2) in the proof of corollary 4.5). The next corollary gives the first way to ensure that there exists an invariant probability measure for $\{x_t\}$. It will be used in the next section to show that the pathologies found in the approach of Costa [2] and Davis (see section 34 in [4]) does not exist with our method.

COROLLARY 4.5. *Suppose that* $\{Y_n\}$ *is recurrent and that condition ($\mathcal{D}$) is satisfied for a function* V *bounded in a petite set* $C \in \mathcal{E}^+$. *Then there exists a unique invariant probability measure for the PDMP* $\{x_t\}$.

Proof. Applying Proposition 11.3.3 in [14, p. 264] with $Z_k = V(Y_k)$, $\varepsilon_k(x) = L(x, E)$ and $c = 1$, we obtain that

$$E_x[\sum_{n=1}^{\tau_C - 1} L(Y_n, E)] \leq \begin{cases} L(x, E) + GV(x), & x \in C \\ V(x), & \text{otherwise} \end{cases}.$$

Recall that for all $x \in C$, we have $GV(x) - V(x) \leq -L(x, E) + bI_C(x)$.
Therefore,

$$\begin{aligned} (\forall x \in E) \quad E_x[\sum_{n=1}^{\tau_C} L(Y_n, E)] &\leq E_x[\sum_{n=0}^{\tau_C - 1} L(Y_n, E)] + 1 \\ &\leq V(x) + 1 + bI_C(x) \end{aligned} \tag{4.2}$$

Then, $\sup_{x \in C} E_x[\sum_{n=1}^{\tau_C} L(Y_n, E)] < \infty$ and the result follows by using theorem 4.4. ☐

In the case where the stochastic kernel G is weak Feller, the next result gives a sufficient condition to check if $\{x_t\}$ has an invariant measure. Before presenting the result, we need the following definition of norm-like functions (see [14, p. 214]):

DEFINITION 4.6. *A function V is a norm-like function if it is a positive real-valued function such that $\lim_{n\to\infty} \inf_{x\notin K_n} V(x) = \infty$ where K_n is an increasing sequence of compact sets satisfying $\lim_{n\to\infty} \uparrow K_n = E$.*

THEOREM 4.7. *Suppose that the Markov chain $\{Y_n\}$ is weak Feller, $L(.,E)$ is a continuous function and that condition $(\mathcal{D})$ is satisfied with a compact set C and a positive norm-like function V which is finite at least at one $x \in E$. Then an invariant probability measure exists for the PDMP $\{x_t\}$.*

Proof. If there exists $x \in E$ such that $P_x(\tau_C < \infty) < 1$ then following the last part of the derivation of Theorem 12.3.3 in [14, p. 296] and using the fact that V is a norm-like function, there exists an invariant probability measure π for G. So, $\pi S(E) < \infty$ and using theorem 3.5 the result follows.
Now if $P_x(\tau_C < \infty) = 1$ for all $x \in E$, then using Proposition 9.1.1 in [14, p. 202], we have that

$$(\forall x \in E) \quad \sum_{n=0}^{\infty} G^n(x,C) = \infty.$$

Let us define the function $g(x) \in C_c(E)$ by:

$$g(x) \doteq \frac{d(x,O_1^c)}{d(x,O_1^c) + d(x,\overline{O}_2)}$$

where O_1, O_2 are open sets with compact closure for which $C \subset O_2 \subset \overline{O}_2 \subset O_1$ and $d(.,.)$ is as defined in section 2. Then, we have:

$$(\forall x \in E) \quad \sum_{n=0}^{\infty} G^n g(x) = \infty,$$

since $I_C(x) \leq g(x)$.
Therefore applying Theorem 5.1 in [10], there exists an invariant σ-finite measure π for the stochastic kernel G defined by:

$$(\forall f \in C_c(E)) \quad \int_E f(x)\pi(dx) = \text{bLIM}_N \frac{\sum_{n=0}^N \eta G^n f}{\sum_{n=0}^N \eta G^n g},$$

where η is a positive finite measure on $(E,\mathcal{E})$ and bLIM denotes Banach limit [18].
Using the hypothesis, we have

$$L(x,E) + GV(x) \leq V(x) + bg(x).$$

Consequently,

$$\frac{\sum_{n=0}^N \eta G^n L(E)}{\sum_{n=0}^N \eta G^n g} \leq b + \frac{V(x)}{\sum_{n=0}^N \eta G^n g}. \tag{4.3}$$

By hypothesis $L(.,E) \in C_b(E)$, so there exists a sequence $\{f_i\}$ in $C_c(E)$ such that for all $x \in E$, $\lim_{n\to\infty} \uparrow f_i(x) = L(x,E)$. Therefore, with equation (4.3) we obtain

$$(\forall i \in \mathbb{N}) \quad \int_E f_i(y)\pi(dy) \leq b.$$

Now using the monotone convergence theorem, it follows that $\int_E L(y,E)\pi(dy) \leq b$. So $\pi S(E) < \infty$ and with theorem 3.5 the result follows. ∎

REMARK 4.8. *In the proof of theorem 4.7, we only need that V is a norm-like function for the case in which $P_x(\tau_C < \infty) < 1$. Therefore if the condition $P_x(\tau_C < \infty) = 1$ for all $x \in E$ can be directly checked then the norm-like function assumption can be removed from theorem 4.7.*

Let us introduce the following weak assumptions that will be used in proposition 4.9 to show that the chain $\{Y_n\}$ is weak Feller:
(A1) The functions $t_* : E \to (0,\infty]$ is continuous.
(A2) For any x in E, $\lambda(\Phi(y,t))I_{\{t<t_*(y)\}} \to \lambda(\Phi(x,t))I_{\{t<t_*(x)\}}$ when $y \to x$, μ_{Leb} *a.s.* on $(0,t_*(x))$.
(A3) For any x in E, there exists a neighborhood $\mathcal{N}(x)$ of x and a function $B_x(.) : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ such that for all $y \in \mathcal{N}(x)$, $I_{\{t<t_*(y)\}}\lambda(\Phi(y,t)) \leq B_x(t)$, μ_{Leb} *a.s.* on $(0,t_*(x))$. If $\Lambda(x,t_*(x)) < \infty$ then $\int_0^{t_*(x)} B_x(s)ds < \infty$ otherwise $\int_0^t B_x(s)ds < \infty$ for all $t \in (0,t_*(x))$.
(A4) For any x in E, there exists a neighborhood $\mathcal{N}(x)$ of x and a constant $M_x > 0$ such that for all $y \in \mathcal{N}(x)$, $\lambda(\Phi(y,t))\exp\{-\Lambda(y,t)\} < M_x$.
(A5) For any open sets A in $\mathcal{E}$ and x in E, $Q(\Phi(y,t),A)I_{\{t<t_*(y)\}} \to Q(\Phi(x,t),A)I_{\{t<t_*(x)\}}$ when $y \to x$, μ_{Leb} *a.s.* on $(0,t_*(x))$.
(A6) For any open sets A in $\mathcal{E}$ and x in E, $Q(\Phi(y,t_*(y)),A) \to Q(\Phi(x,t_*(x)),A)$, when $y \to x$.

PROPOSITION 4.9. *(i) Suppose (A1)-(A6). Then the stochastic kernel G and U are weak Feller and $L(.,E) \in C_b(E)$.*

Proof. Let x and $t \in (0,t_*(x))$ be arbitrary fixed. Using assumptions *(A2)*, *(A3)* and the bounded convergence Theorem, we obtain that

$$\int_0^t \lambda(\Phi(y,s))I_{\{s<t_*(y)\}}ds \to \int_0^t \lambda(\Phi(x,s))I_{\{s<t_*(x)\}}ds \tag{4.4}$$

when $y \to x$. With *(A5)* it follows that for all open sets A in $\mathcal{E}$,

$$I_{\{t<t_*(y)\}}Q(\Phi(y,t),A)\lambda(\Phi(y,t))\exp -\{t+\Lambda(y,t)\}$$

tends to

$$I_{\{t<t_*(x)\}}Q(\Phi(x,t),A)\lambda(\Phi(x,t))\exp -\{t+\Lambda(x,t)\}$$

when $y \to x$, μ_{Leb} *a.s.* on $(0,t_*(x))$. Combining *(A4)* and the bounded convergence theorem, we have that

$$\int_0^{t_*(y)} Q(\Phi(y,t),A)\lambda(\Phi(y,t))\exp -\{t+\Lambda(y,t)\}dt$$

tends to

$$\int_0^{t_*(x)} Q(\Phi(x,t),A)\lambda(\Phi(x,t))\exp -\{t+\Lambda(x,t)\}dt$$

when $y \to x$.

If $\Lambda(x, t_*(x)) < \infty$ it is easy to deduce from *(A3)* that

$$\Lambda(y, t_*(y)) \to \Lambda(x, t_*(x))$$

when $y \to x$.

Now if $\Lambda(x, t_*(x)) = \infty$ and $t_*(x) < \infty$ then

$$(4.5)\ (\forall \varepsilon > 0), \quad (\exists \eta > 0) \quad \text{such that} \quad (\forall t \in [t_*(x) - \eta, t_*(x)]) \qquad \Lambda(x,t) > \frac{3\varepsilon}{2}.$$

Using *(A1)*, there exists a neighborhood $\mathcal{N}_1(x)$ of x such that:

$$(\forall y \in \mathcal{N}_1(x)) \qquad t_*(y) \in [t_*(x) - \frac{\eta}{2}, t_*(x) + \frac{\eta}{2}].$$

Therefore $t_*(x) - \eta < t_*(y)$.

So from equation (4.4), it follows that there exists a neighborhood $\mathcal{N}_2(x)$ of x such that:

$$(4.6) \qquad (\forall y \in \mathcal{N}_2(x)) \qquad |\Lambda(x, t_*(x) - \eta) - \Lambda(y, t_*(x) - \eta)| < \frac{\varepsilon}{2}.$$

Combining equations (4.5) and (4.6)

$$(\forall \varepsilon > 0), \quad (\forall y \in \mathcal{N}_1(x) \cap \mathcal{N}_2(x)), \qquad \Lambda(y, t_*(y)) > \varepsilon$$

since for all $x \in E$, $\Lambda(x, .)$ is increasing on $[0, t_*(x)]$.

The same result can be obtained for the case where $\Lambda(x, t_*(x)) = \infty$ and $t_*(x) = \infty$.

In conclusion

$$\Lambda(y, t_*(y)) \to \Lambda(x, t_*(x))$$

when $y \to x$.

Using *(A6)*, we obtain that for any open sets A in $\mathcal{E}$ and x in E, $K(y, A) \to K(x, A)$, when $y \to x$. Similarly, it can be shown that for any $f \in C_b(E)$, $Lf(y) \to Lf(x)$, when $y \to x$.

Consequently, it is straightforward with Proposition 6.1.1 in [14] to conclude that the stochastic kernel G is weak Feller.

Now using equation (3.12) and the result (12.7.7) in [6] it follows that for any positive function f in $C_b(E)$, Uf is a lower semicontinuous function. Since U is a stochastic kernel, Uf is continuous. Indeed, let $M_1 > 0$ be such that $f(x) < M_1$. Applying the previous result it follows that $U(M_1 - f)$ is lower semicontinuous. So Uf is upper semicontinuous and U is weak Feller. □

Therefore, the combination of theorem 4.7 and proposition 4.9 provides a second easy way of checking if there exists an invariant probability measure for $\{x_t\}$, as it will be shown in example 1 of the next section.

REMARK 4.10. *An interesting property of the PDMP's is that under the weak conditions of proposition 4.9 they generate weak Feller kernels G and U although the PDMP's themselves are generally not weak Feller Markov processes.* Indeed, it was shown in [4, example (27.5)] that the PDMP defined by $E = [0, 1[$, $\Phi(x,t) = x + t$, $\lambda(x) = 0$ and $Q(1, A) = I_A(0)$ is not a weak Feller process but it is easy to see that G and U are weak Feller kernels.

The results of this section have been developed considering the stochastic kernel G defined on $(E, \mathcal{E})$. It may be convenient in some applications to consider the extended state space $E \cup \Gamma^+$ instead of E, with the stochastic kernel G extended to include the points in Γ^+. Let us define the Kernel H on $(E, \mathcal{B}(E \cup \Gamma^+))$ (corresponding to an extension of G) by $(\forall x \in E \cup \Gamma^+)$, $(\forall A \in \mathcal{B}(E \cup \Gamma^+))$,

$$H(x, A) \doteq I_E(x) G(x, A \cap E) + I_{\Gamma^+}(x) Q(x, A \cap E). \tag{4.7}$$

It is easy to see that H is a stochastic kernel on $(E, \mathcal{B}(E \cup \Gamma^+))$. Moreover, it has the following important properties:
(i) if π is a σ-finite invariant measure for H then $\pi(\Gamma^+) = 0$.
(ii) a σ-finite measure is invariant for G if and only if it is invariant for H.
Therefore, all the results of the previous sections remain true if the kernel G is replaced by H.
It will be shown in the next section (see example 1), that it may be usefull to use H instead of G. Another possible illustration of the usefullness for introducing H is the following result:

COROLLARY 4.11. *Suppose* ***(A1)-(A6)*** *and that $E \cup \Gamma^+$ is compact. Then there exists an invariant probability measure for the PDMP $\{x_t\}$.*

Proof. As in proposition 4.9, it can be shown that H is a weak Feller kernel. Now using the fact that $E \cup \Gamma^+$ is compact and Theorem 3.1 in [9], it follows that H has an invariant probability measure. Therefore, using the above properties of H and theorem 3.5, we have the result. □

5. Examples. In this section we apply the results of section 3 and 4 to three examples to emphasize the relevance of our approach. In the first one, we generalize Proposition (34.46) in [4] and Proposition 8 in [2]. The last two examples are taken from Davis [4]. They show that the equivalence between $\mathbf{P}_1$ and $\mathbf{P}_2$ with $\overline{G}$ replacing G may fail if some conditions are not satisfied. Due to its generality we are able to show that this kind of "counter-examples" does not exist with our approach.

EXAMPLE 1. This example is based on the capacity expansion model (see [2, section 7] and [4, (34.45)]. Capacity expansion are general processes of adding facilities to meet a rising demand. For a complete description of these models, the reader is refered to [5]. The demand for some utility is modelled as a random point process, i.e., it increases by one unit at random times. This demand is met by consecutive construction of expansion projects. Each project meets K_i units of demand when completed where i corresponds the present level of excess demand. We assume that if there is an excess demand of at least p units then the construction of a new project is started at a rate r_i per unit of time and it is completed after a lead time of $\mathcal{L}_i$ units of time. If the excess demand is less than p then no construction takes place. New demand occurs with rate $\lambda_i(u)$ where u is the time spent by the current project. This problem can be modelled as a PDMP $\{x_t\}$ with state space

$$E \doteq \{\{p - K_p, \ldots, p-1\} \times \{0\}\} \cup \bigcup_{n=p}^{\infty} \{n, [0, \mathcal{L}_n[\} \tag{5.1}$$

where p and K_i are integers, $\mathcal{L}_i$ are strictly positive real numbers and $\mathbb{N}_p$ denotes the set of integers greater than or equal to p. The local characteristics are given by:

$$\lambda((i, u)) \doteq \lambda_i(u), \tag{5.2}$$

$$\Phi(t,(i,u)) = \begin{cases} (i,0) & \text{for } i \in \{p-K_p,\dots,p-1\}, \\ (i,u+r_it) & \text{for } i \geq p, \end{cases} \tag{5.3}$$

$$Q((i,u),\{(i+1,u)\}) = 1 \quad \text{and} \quad Q((i,\mathcal{L}_i),\{(i-K_i,0)\}) = 1. \tag{5.4}$$

Unlike [2, 4], we allow here that the rate r_i, K_i, $\mathcal{L}_i$ can vary according to the excess demand i and that $\lambda_i(u)$ can vary according to the excess demand and to the time spent to complete the projects.
Using equations (5.2)-(5.4), it is easy to show that for a bounded real-valued function $W(.)$ defined on $E \cup \Gamma^+$ we have for $i \geq p$ and $u \in [0,\mathcal{L}_i]$,

$$\begin{aligned} HW((i,u)) = &\int_0^{\frac{\mathcal{L}_i-u}{r_i}} \exp-\{s+\Lambda_i(u,s)\}\lambda_i(u+r_is)W((1+i,u+r_is))ds \\ &+\int_0^{\frac{\mathcal{L}_i-u}{r_i}} \exp-\{s+\Lambda_i(u,s)\}W((i,u+r_is))ds \\ &+\exp-\{\frac{\mathcal{L}_i-u}{r_i}+\Lambda_i(u,\frac{\mathcal{L}_i-u}{r_i})\}W((i-K_i,0)), \end{aligned} \tag{5.5}$$

where $\Lambda_i(u,s) = \displaystyle\int_0^s \lambda_i(u+r_iv)dv$.
Let us introduce the following assumptions:
(H1) There exists $r > 0$ such that for all i, $\quad r_i \geq r$.
(H2) For $i \geq p$, $i - K_i \geq p - K_p > 0$ and $\lim_{i\to\infty} i - K_i = \infty$.
(H3) There exists an integer i_0 such that for all $i \geq i_0$, $\dfrac{K_i}{\mathcal{L}_i} \leq \dfrac{K_{i+1}}{\mathcal{L}_{i+1}}$.
(H4) For $i \geq p$, $\lambda_i(u)$ is a continuous real-valued function on $[0,\mathcal{L}_i]$.
(H5) For $i \in \{p-K_p,\dots,p-1\}$, $\lambda_i > 0$.
(H6) $\limsup_{i\to\infty} \dfrac{\mathcal{L}_i}{K_ir_i} \max_{u\in[0,\mathcal{L}_i]} \lambda_i(u) < 1$.
We have the following result:

PROPOSITION 5.1. *Assume (H1)-(H6). Then there exists a unique invariant probability measure for the PDMP $\{x_t\}$.*

Proof. Let us introduce the real-valued function defined on $E \cup \Gamma^+$ by

$$V((i,u)) = i - \frac{K_i}{\mathcal{L}_i}u.$$

With *(H2)*, it follows that the function V is positive norm-like function. Moreover, using *(H1)*, *(H3)* and *(H6)*, we have that there exists $\varepsilon > 0$ and $k_0 \geq \max\{p,i_0\}$ such that

$$(\forall i \geq k_0), \quad (\varepsilon + \lambda_i(u))\frac{\mathcal{L}_i}{K_ir_i} \leq 1. \tag{5.6}$$

Using equations (5.5) and the previous definition of $V(.)$, we obtain that:

$$\begin{aligned} HV(i,u) - V(i,u) \leq &\int_0^{\frac{\mathcal{L}_i-u}{r_i}} \lambda_i(u+r_is)\exp-\{s+\Lambda_i(u,s)\}ds \\ &-\frac{r_iK_i}{\mathcal{L}_i}\left\{\int_0^{\frac{\mathcal{L}_i-u}{r_i}} s(1+\lambda_i(u+r_is))\exp-\{s+\Lambda_i(u,s)\}ds\right. \end{aligned}$$

$$+\frac{\mathcal{L}_i - u}{r_i} \exp -\{\frac{\mathcal{L}_i - u}{r_i} + \Lambda_i(u, \frac{\mathcal{L}_i - u}{r_i})\}\Big\}$$

$$\leq \int_0^{\frac{\mathcal{L}_i - u}{r_i}} (\lambda_i(u + r_i s) - \frac{r_i K_i}{\mathcal{L}_i}) \exp -\{s + \Lambda_i(u, s)\} ds$$

Using equation (5.6) and the fact that for (i, u) in $E \cup \Gamma^+$,

$$L((i, u), E) = \int_0^{\frac{\mathcal{L}_i - u}{r_i}} \exp -\{s + \Lambda_i(u, s)\} ds,$$

we obtain that that there exists a constant b and a compact set

$$C \doteq \{\{p - K_p, \ldots, p - 1\} \times \{0\}\} \cup \bigcup_{n=p}^{k_0} \{n, [0, \mathcal{L}_n]\}$$

such that

$$HV(i, u) - V(i, u) \leq -\varepsilon L((i, u), E)) + bI_C((i, u))$$

It is easy to check that assumptions *(A1)-(A6)* are satisfied for this process. So, $L(., E)$ is a continuous function and the stochastic kernel H is weak feller. Applying theorem 4.7, the result follows. □

REMARK 5.2. *The previous proposition generalizes the result of Costa [2] and Davis [4] since here K_i, r_i, $\mathcal{L}_i$ and λ_i (as function of i) may be unbounded or $\lambda_i(u)$ may depend on the time u spent to complete the project.*

EXAMPLE 2. (Example (34.28) in [4]) This example was designed in [4] to point out the fact that the PDMP $\{x_t\}$ may have an invariant probability measure but the associated Markov chain $\{Z_1, Z_2, \ldots\}$ may not. Indeed, the approach developed by Costa [2] suffers from the fact that if μ is an invariant probability measure for $\{x_t\}$ then the associated Markov chain $\{Z_1, Z_2, \ldots\}$ will have an invariant probability measure if the following boundness condition is satisfied: $\int_E \lambda(x)\mu(dx) < \infty$. Let us show however that in our case there exists a σ-finite invariant measure π for the stochastic kernel G as claimed in theorem 3.5 such that $\pi S(E) < \infty$.
In this case $E = \mathbb{Z}$ and for all $x \in E$, $\Phi(x, t) = x$, $Q(x, \{x+1\}) = Q(x, \{x-1\}) = \frac{1}{2}$ with $\lambda(x)$ satisfying the condition

$$\sum_{x=-\infty}^{\infty} \frac{1}{\lambda(x)} < \infty.$$

It was shown in [4] that $\{x_t\}$ has a unique invariant probability measure μ such that:

$$(\forall x \in \mathbb{Z}) \quad \mu(\{x\}) = \frac{1}{\lambda(x)} \sum_{y=-\infty}^{\infty} \frac{1}{\lambda(y)}.$$

Now it is easy to obtain that:

$$(5.7) \qquad (\forall x \in \mathbb{Z}) \quad G(x, \{y\}) = \begin{cases} \frac{1}{2}\frac{\lambda(x)}{1+\lambda(x)}, & \text{for } y = x+1 \text{ or } y = x-1 \\ \frac{1}{1+\lambda(x)}, & \text{for } y = x \\ 0, & \text{otherwise} \end{cases} .$$

It is easy to verify that for all $x \in \mathbb{Z}$ that the measure π defined by

$$(\forall x \in \mathbb{Z}) \quad \pi(\{x\}) = \frac{1+\lambda(x)}{\lambda(x)} \sum_{y=-\infty}^{\infty} \frac{1}{\lambda(y)}.$$

is invariant for G.
Indeed

$$\begin{aligned}\sum_{y=-\infty}^{\infty} G(y,\{x\})\pi(\{y\}) &= G(x-1,\{x\})\pi(\{x-1\}) + G(x,\{x\})\pi(\{x\}) \\ &\quad + G(x+1,\{x\})\pi(\{x+1\}) \\ &= \pi(\{x\}).\end{aligned}$$

Moreover, using the fact that $S(x,\mathbb{Z}) = \frac{1}{1+\lambda(x)}$, it is easy to prove that $\pi S(\mathbb{Z}) < \infty$. Therefore as claimed by theorem 3.5, if there exists a solution to $\mathbf{P}_1$ then there exists one to $\mathbf{P}_2$.

For this example the fact that $\{x_t\}$ has an invariant probability measure had been established by direct calculation (see [4]). We want now to illustrate the use of the weak conditions of theorem 4.5 for a suitable choice of function $V(.)$ and petite set C and recover this known result. Indeed, it is straightforward to prove that the Markov chain $\{Y_n\}$ associated to the kernel G is φ-irreducible for $\varphi(\{x\}) = \dfrac{1}{2^{|x|}}$. Then for all $x \in E$ we have $\psi(\{x\}) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \varphi G^n(\{x\}) > 0$. Therefore $\mathcal{E} = \mathcal{E}^+$.
Let us show that $\{x\}$ is a petite set for all $x \in \mathcal{E}$. Define

$$\nu(\{y\}) = \begin{cases} \frac{1}{1+\lambda(x)}, & \text{for } y = x \\ \frac{1}{2}\frac{\lambda(x)}{1+\lambda(x)}, & \text{for } y = x+1 \text{ or } y = x-1 \\ 0, & \text{otherwise} \end{cases}.$$

So, we have $(\forall B \in \mathcal{E})$, $G(x,B) \geq \nu(B)$ and the claim follows.
Now let us prove that $\{Y_n\}$ is recurrent. Define the function $V_1(i)$ for $i \in \mathbb{Z}$ by

$$\begin{aligned} V_1(i) &= 2\sum_{l=0}^{i}\sum_{j=l}^{\infty} \frac{1}{(j+1)^2}, \quad \text{for } i \geq 0; \\ V_1(i) &= 2\sum_{l=i}^{-1}\sum_{j=-\infty}^{l} \frac{1}{(|j|+1)^2}, \quad \text{for } i \leq -1. \end{aligned}$$

Then, for $i \geq 1$ and using equation (5.7), we have

$$\begin{aligned} GV_1(i) - V_1(i) &= \frac{\lambda(i)}{1+\lambda(i)}\{\frac{1}{2}V_1(i+1) + \frac{1}{2}V_1(i-1) - V_1(i)\} \\ &= \sum_{j=i+1}^{\infty} \frac{1}{(j+1)^2} - \sum_{j=i}^{\infty} \frac{1}{(j+1)^2} \quad = -\frac{1}{(i+1)^2}. \end{aligned}$$

Similarly, for $i \leq -2$, we obtain

$$GV_1(i) - V_1(i) = -\frac{1}{(|i|+1)^2}.$$

Therefore the norm-like function $V_1(.)$ satisfies condition $(\mathcal{D})$ in [14, p. 190] for $C = \{-1, 0\}$. Moreover, it is clear that this function is unbounded off petite sets and that $C = \{-1, 0\}$ is a petite since it is a finite set. Consequently, using Theorem 8.4.3 in [14, p. 191], we obtain that $\{Y_n\}$ is recurrent.
In order to show that condition $(\mathcal{D})$ is satisfied for the petite set $C = \{-1, 0\}$, let us introduce the function $V_2(i)$ for $i \in \mathbb{Z}$ by

$$V_2(i) = 2\sum_{l=0}^{i}\sum_{j=l}^{\infty}\frac{1}{\lambda(j)}, \quad \text{for } i \geq 0;$$

$$V_2(i) = 2\sum_{l=i}^{-1}\sum_{j=-\infty}^{l}\frac{1}{\lambda(j)}, \quad \text{for } i \leq -1.$$

Using equation (5.7), we obtain that

$$\begin{aligned} GV_2(i) - V_2(i) &= -\frac{1}{1+\lambda(i)} \\ &= -L(i, \mathbb{Z}) \end{aligned}$$

for $i \in \mathbb{Z} - C$.
Therefore the function $V_2(.)$ satisfies condition $(\mathcal{D})$ for a suitable constant b. From corollary 4.5, we conclude that there exists a unique invariant probability measure for the PDMP $\{x_t\}$.

Example 3. (Example (34.33) in [4]) This example shows that the inverse problem as described in the previous example may happen. Indeed, if there exists an invariant probability measure (labeled π) for the stochastic kernel $\overline{G}$ then an invariant probability measure for the PDMP $\{x_t\}$ will exist if the following condition is satisfied: $\displaystyle\int_E \int_0^{t_*(x)} \exp\{-\Lambda(x, s)\} ds \pi(dx) < \infty$. Davis studied the following case: $E = \mathbb{N}$, $\Phi(x, t) = x$, $Q(x, \{x+1\}) = p$, $Q(x, \{x-1\}) = q$, $Q(0, \{1\}) = p$, $Q(0, \{0\}) = q$ with $q = 1 - p$, $p < \frac{1}{2}$. $\lambda(x)$ is bounded. He has shown that there exists an invariant probability measure for $\{x_t\}$ if and only if

$$\sum_{y=0}^{\infty}\frac{1}{\lambda(y)}\left(\frac{p}{q}\right)^y < \infty. \tag{5.8}$$

Therefore, for the choice of $\lambda(x) = \left(\dfrac{p}{q}\right)^x$, $\{x_t\}$ has no invariant probability measure although the birth and death process generated by $\overline{G}$ has one. However we show that there exists an invariant probability measure for the stochastic kernel G if and only if there exists one for $\{x_t\}$ according to the fact that $\lambda(.)$ is bounded and remark 3.8. By direct calculations, it is easy to show that $(\forall x \in \mathbb{N})$ $G(x, \{y\}) = 0$ if $y \notin \{x-1, x, x+1\}$ and for $x \geq 1$ $G(x, \{x\}) = \dfrac{1}{1+\lambda(x)}$, $G(x, \{x+1\}) = \dfrac{p\lambda(x)}{1+\lambda(x)}$

$G(x,\{x-1\}) = \dfrac{q\lambda(x)}{1+\lambda(x)}$ and $G(0,\{0\}) = \dfrac{1+q\lambda(0)}{1+\lambda(0)}$, $G(0,\{1\}) = \dfrac{p\lambda(0)}{1+\lambda(0)}$.
If π is an invariant measure for $\{x_t\}$, it must satisfy the following equations: $(\forall x \geq 1)$

$$\pi(\{x\}) = G(x-1,\{x\})\pi(\{x-1\}) + G(x,\{x\})\pi(\{x\}) + G(x+1,\{x\})\pi(\{x+1\})$$

and

$$\pi(\{0\}) = G(0,\{0\})\pi(\{0\}) + G(1,\{0\})\pi(\{1\}).$$

Therefore,

$$(\forall x \geq 1) \quad pc(x-1) + qc(x+1) - c(x) = 0$$

and

$$-pc(0) + qc(1) = 0$$

with $c(x) \doteq \dfrac{\lambda(x)}{1+\lambda(x)}\pi(\{x\})$. The unique positive solution of the previous equations is

$$(\forall x \in \mathbb{N}) \quad c(x) = c(0)\left(\frac{p}{q}\right)^x$$

So, there exists an invariant probability measure for the stochastic kernel G if and only if

$$\sum_{y=0}^{\infty} \pi(\{y\}) < \infty,$$

that is, if and only if

$$\sum_{y=0}^{\infty} \frac{1}{\lambda(y)} \left(\frac{p}{q}\right)^y < \infty,$$

which is the same necessary and sufficient condition for the PDMP (see equation (5.8)) in agreement with theorem 3.5.

6. Conclusion. In this paper, we have shown that the existence of an invariant probability measure for a general PDMP is equivalent to the existence of a σ-finite invariant measure for the stochastic kernel G satisfying a boundedness condition or equivalently a Radon-Nikodym derivative. Here we generalize existing results of the literature [2, 4] since we do not require any additional assumptions to establish this equivalence. Moreover, we give sufficient conditions to ensure the existence of such σ-finite measure satisfying the boundedness condition. They are mainly based on a modified Foster-Lyapunov criterion. To emphasize the relevance of our results, we studied three examples and in particular, we are able to generalize the results obtained by Costa and Davis on the capacity expansion model.

REFERENCES

[1] J. Azema, M. Kaplan-Duflo, and D. Revuz, *Mesure invariante sur les classes récurrentes des processus de markov*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 8 (1967), pp. 157–181.

[2] O. Costa, *Stationary distributions for piecewise-deterministic Markov processes*, Journal Applied Probability, 27 (1990), pp. 60–73.

[3] M. Davis, *Piecewise-deterministic markov processes: A general class of non-diffusion stochastic models*, J.R. Statist. Soc. B, 46 (1984), pp. 353–388.

[4] ———, *Markov Models and Optimization*, Chapman and Hall, London, 1993.

[5] M. Davis, M. Dempster, S. Sethi, and D. Vermes, *Optimal capacity expansion under uncertainty*, Advances in Applied Probability, 19 (1987), pp. 156–176.

[6] J. Dieudonné, *Eléments d'Analyse, Tome 2*, Gauthier-Villard, Paris, 1974.

[7] S. Foguel, *The ergodic theory of positive operators on continuous functions*, Ann. Scuola Norm. Sup. Pisa, 27 (1973), pp. 19–51.

[8] O. Hernandez-Lerma and J.-B. Lasserre, *Existence and uniqueness of fixed points for Markov operators and Markov processes*, Proc. London Math. Soc, 76 (1998), pp. 711–736.

[9] J.-B. Lasserre, *Invariant probabilities for Markov chains on a metric space*, Statistics and Probability Letters, 34 (1997), pp. 259–266.

[10] M. Lin, *Conservative Markov processes on a topological space*, Israel Journal of Mathematics, 8 (1970), pp. 165–186.

[11] S. Meyn, *Ergodic theorems for discrete time stochastic systems using a stochastic Lyapunov function*, SIAM Journal of Control and Optimization, 27 (1989), pp. 1409–1439.

[12] S. Meyn and P. Caines, *Asymptotic behavior of stochastic systems possessing Markovian realizations*, SIAM Journal of Control and Optimization, 29 (1991), pp. 535–561.

[13] S. Meyn and R. Tweedie, *Criteria for stability of markovian processes III: Foster-Lyapunov criteria for continuous time processes, with examples*, Advances in Applied Probability, 25 (1993), pp. 518–548.

[14] ———, *Markov Chains and Stochastic Stability*, Springer-Verlag, Berlin, 1993.

[15] ———, *Stability of markovian processes II: continuous-time processes and sampled chains*, Advances in Applied Probability, 25 (1993), pp. 487–517.

[16] E. Nummelin, *General irreducible Markov chains and non-negative operators*, Cambridge University Press, Cambridge, 1984.

[17] D. Revuz, *Markov chains*, North Holland, Amsterdam, 1975.

[18] K. Yosida, *Functional Analysis*, Springer Verlag, Sixth edition, New york, 1980.