# Mixed $H_2/H_\infty$-Control of Discrete-Time Markovian Jump Linear Systems

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**Abstract** : In this paper we consider the mixed $H_2/H_\infty$-control problem for the class of discrete-time linear systems with parameters subject to Markovian jumps (MJLS). It is assumed that both the state variable and the jump variable are available to the controller. The transition probability matrix may not be exactly known, but belongs to an appropriate convex set. For this controlled discrete-time Markovian jump linear system, the problem of interest can be stated in the following way. Find a robust (with respect to the uncertainty on the transition Markov probability matrix) mean square stabilizing state and jump feedback controller that minimizes an upper bound for the $H_2$-norm, under the restriction that the $H_\infty$-norm is less than a pre-specified value $\delta$. The problem of the determination of the smallest $H_\infty$-norm is also addressed. We present an approximate version of these problems via LMI optimization.

**Keywords** : mixed $H_2/H_\infty$-control, Markovian jump systems, coupled algebraic Riccati equations, LMI optimization.

November 1996

This work was supported in part by CNPq (Brazilian National Research Council) and FAPESP (Research Council of the State of São Paulo).

# 1. Introduction

A great deal of attention has been given nowadays to a class of stochastic linear systems subject to abrupt variations, namely, Markovian jump linear systems (MJLS). This family of system is modeled by a set of linear systems, with the transitions between the models determined by a Markov chain taking values in a finite set. Due to a large number of applications in control engineering, several results on this field can be found in the current literature, regarding applications, stability conditions and optimal control problems (see, for instance, [1]-[11],[13]-[18],[21]-[28]).

The mixed $H_2/H_\infty$ and $H_\infty$ control problems for time-invariant discrete-time linear systems has been studied in the current literature usually using a state space approach, leading to non-standard algebraic Riccati equations and Lyapunov-like equations (see, for instance, [12], [19], [20]). The $H_2$ and $H_\infty$ control problems for MJLS have recently been analyzed in [5], [6], and [11]. For the $H_2$ control problem, a convex programming approach was applied in [5] and numerical algorithms developed. In this paper we study the mixed $H_2/H_\infty$-control and $H_\infty$ control problems of a discrete-time MJLS. We will assume that the transition probability matrix for the Markov chain is not exactly known, but belongs to an appropriate convex set. In this case a robust mean square (state and jump feedback) stabilizing controller is defined as a state-feedback controller, which also depends on the jump Markov variable, that stabilizes in the mean square sense the MJLS for every appropriate Markov transition probability matrix. This kind of concept was first introduced by Rami and El Ghaoui in [27] for continuous-time MJLS. Under these conditions, the mixed $H_2/H_\infty$ control problem of a MJLS can be formulated as follows: we are interested in finding a robust mean square stabilizing controller that minimizes an upper bound for the $H_2$-norm, under the restriction that the $H_\infty$-norm is less than a pre-specified value $\delta$. The problem of minimizing the $H_\infty$-norm is also addressed. We trace a parallel with the discrete-time linear system theory of $H_2/H_\infty$ and $H_\infty$ control to derive our results. When restricted to the case with no jumps, the equations presented here can be seen as dual to the ones derived in [12]. As in [12], we present an approximate version of the mixed $H_2/H_\infty$ and $H_\infty$ control problems of MJLS based on linear matrix inequalities (LMI) optimization.

The paper is organized in the following way. Section 2 presents the notation that will be used throughout the work. Section 3 deals with previous results derived for stability, $H_2$ and $H_\infty$-control of MJLS, as well as some other auxiliary results. Section 4 presents a sufficient condition for the existence of a mean square stabilizing controller that makes the $H_\infty$-norm of the MJLS less than a pre-specified value $\delta$. The condition is written in terms of the existence of a solution $P = (P_1,...,P_N)$ and $K = (K_1,...,K_N)$ for a set of coupled Lyapunov-like equations. This solution P leads to an upper bound for the $H_2$-norm of the MJLS, so that an approximation for the mixed $H_2/H_\infty$-control problem for the MJLS can be determined by minimizing this functional over the set of solutions P and K. The $H_\infty$-control problem can also be addressed through this Lyapunov-like equation. In section 5 we consider the case in which the transition probability matrix belongs to an appropriate convex set and, using the results of section 4, derive a LMI optimization problem that leads to an approximation for the mixed $H_2/H_\infty$ and $H_\infty$-control problems. Numerical examples are presented in section 6 and the paper is concluded in section 7 with some final comments.

# 2. Notation

We shall write $\mathbb{C}^n$ and $\mathbb{R}^n$ to denote the n-dimensional complex and real spaces respectively, and $\mathbb{M}(\mathbb{C}^n,\mathbb{C}^m)$ the normed linear space of all m by n complex matrices. For simplicity we set $\mathbb{M}(\mathbb{C}^n,\mathbb{C}^n) = \mathbb{M}(\mathbb{C}^n)$. We write $^*$ to indicate the adjoint operator and, for real matrices, $'$ will indicate transpose. $L \geq 0$ and $L>0$ will be used if a self-adjoint matrix is positive semi-definite or positive definite respectively and we write $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n);\ L = L^* \geq 0\}$. We denote by $\|\ .\ \|$ the standard norm in $\mathbb{C}^n$.

Let $\mathcal{H}^{m,n}$ be the linear space made up of all N-sequence of matrices $V = (V_1,...,V_N)$, $V_i \in \mathbb{M}(\mathbb{C}^m,\mathbb{C}^n)$. For $V \in \mathcal{H}^{m,n}$ we define the following norm $\|\ .\ \|_2$:

$$\|\ V\ \|_2 = \Big( \sum_{i=1}^{N} \mathrm{tr}\Big(V_i^* V_i\Big)\Big)^{1/2} \text{ (where tr(.) denotes the trace operator).}$$

It is easy to verify that $\mathcal{H}^{m,n}$ equipped with the norm $\|\ .\ \|_2$ is a complex Hilbert space with inner product given by:

$$< V;H > = \sum_{i=1}^{N} \mathrm{tr}\Big((V_i^* H_i)\Big).$$

We set $\mathcal{H}^{n,n} = \mathcal{H}^n$ and $\mathcal{H}^{n+} = \{V = (V_1,...,V_N) \in \mathcal{H}^n;\ V_i \in \mathbb{M}(\mathbb{C}^n)^+,\ i = 1,...,N\}$. For $H = (H_1,...,H_N)$ and $V = (V_1,...,V_N)$ in $\mathcal{H}^{n+}$ the notation $H \leq L$ ($H < L$) indicates that $H_i \leq L_i$ ($H_i < L_i$) for each i = 1,...,N.

For an increasing filtration $\{\mathcal{F}_k\}$ defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$, we set $\ell_2^r(\mathcal{F}_k)$ as the Hilbert space formed by the sequence of second order random variables $z = (z(0),z(1),...)$ with $z(k) \in \mathbb{R}^r$ and $\mathcal{F}_k$-adapted for each k=0,1,.., and such that

$$\|\ z\ \|_2^2 := \sum_{k=0}^{\infty} \|\ z(k)\ \|_2^2 < \infty \text{ where } \|\ z(k)\ \|_2^2 := E(\ \|\ z(k)\ \|^2)\ .$$

For any complex Banach space $\mathbb{Z}$ we denote by $\mathbb{B}(\mathbb{Z})$ the Banach space of all bounded linear operators of $\mathbb{Z}$ into $\mathbb{Z}$ with the uniform induced norm represented by $\|\ .\ \|$ and for $L \in \mathbb{B}(\mathbb{Z})$ we denote by $r_\sigma(L)$ the spectral radius of L.

Finally we conclude this section with the following well known result used in LMI's, which will be useful in the sequel.

**Remark 1** : If $R > 0$ then $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$ if and only if $Q \geq SR^{-1}S'$.

# 3. Auxiliary Results

### 3.1. Stability Results

Consider the following stochastic system on an appropriate probability space $(\Omega,\{\mathcal{F}_k\},\mathcal{F},P)$,

$$x(k+1) = \tilde{A}_{\theta(k)} x(k) \tag{1.a}$$

$$x(0) = x_0\ ,\ \theta(0) = \theta_0 \tag{1.b}$$

where $\{\theta(k);\ k=0,1,...\}$ is a discrete-time Markov chain with finite state space $\{1,...,N\}$ with transition probability matrix $\mathbb{P} = [\ p_{ij}\ ]$. We consider $\tilde{A} = (\tilde{A}_1,...,\tilde{A}_N) \in \mathcal{H}^n$ real, and $x_0$ a second order random variable in $\mathbb{R}^n$. We set $Q(k) = (Q_1(k),...,Q_N(k))$, where

$$Q_j(k) := E(\ x(k)\, x(k)'\, 1_{\{\ \theta(k) = j\ \}}\ ) \in \mathbb{M}(\mathbb{C}^n)^+ \tag{2}$$

and $1_{\{\ .\ \}}$ stands for the Dirac measure.

For $S = (S_1,...,S_N) \in \mathcal{H}^n$ we define the operator $\mathcal{T} \in \mathbb{B}(\mathcal{H}^n)$ as: $\mathcal{T}(S) = (\mathcal{T}_1(S),...,\mathcal{T}_N(S))$ where

$$\mathcal{T}_j(S) = \sum_{i=1}^{N} p_{ij}\tilde{A}_i\, S_i \tilde{A}_i'. \tag{3}$$

It is easy to verify that with the inner product as defined above we have $\mathcal{L} := \mathcal{T}^*$ given by:

$$\mathcal{L}_i(S) = \tilde{A}_i'\left( \sum_{j=1}^{N} p_{ij} S_j \right) \tilde{A}_i.$$

In particular, $r_\sigma(\mathcal{L}) = r_\sigma(\mathcal{T})$. The following result, shown in Proposition 3 of [7], provides a connection between (2) and (3):

**Proposition 1** : For every k = 0,1,2,..., $Q(k+1) = \mathcal{T}(Q(k))$.

We make the following definitions:

**Definition 1** : Model (1) is mean square stable (MSS) if $\| Q(k) \|_2^2 \to 0$ as $k \to \infty$ for any initial condition $x_0$ and initial distribution for $\theta_0$.

**Remark 2** : It can be shown that stability of each model (that is, $r_\sigma(\tilde{A}_i) < 1$ for i = 1,...,N) is neither necessary nor sufficient for MSS (see [16]). Moreover if (1) is MSS then $\| x(k) \| \to 0$ as $k \to \infty$ with prob. 1 (see [7]).

The next result has been proved in Theorems 1 and 2 of [7]:

**Proposition 2** : The following assertions are equivalent:

a) Model (1) is MSS

b) $r_\sigma(\mathcal{T}) < 1$.

c) $r_\sigma(\mathcal{L}) < 1$.

d) There exists $\alpha \in (0,1)$ and $a \in \mathbb{R}$, $a > 0$, such that for each k = 0,1,...,

$$E( \| x(k) \|^2) \leq a\,\alpha^k.$$

e) (coupled Lyapunov equations) given any $S = (S_1,...,S_N) > 0$ in $\mathcal{H}^{n+}$ there exists $P = (P_1,...,P_N) > 0$ in $\mathcal{H}^{n+}$ satisfying $P - \mathcal{T}(P) = S$ with $P = \sum_{k=0}^{\infty} \mathcal{T}^k(S)$.

f) (adjoint coupled Lyapunov equations) given any $S = (S_1,...,S_N) > 0$ in $\mathcal{H}^{n+}$ there exists $P = (P_1,...,P_N) > 0$ in $\mathcal{H}^{n+}$ satisfying $P - \mathcal{L}(P) = S$ with $P = \sum_{k=0}^{\infty} \mathcal{L}^k(S)$.

Moreover if $r_\sigma(\mathcal{T}) < 1$ then for any $S \in \mathcal{H}^n$ there exists a unique $P \in \mathcal{H}^n$ such that $P - \mathcal{T}(P) = S$. If $S \geq T \geq 0$ ($> 0$ respectively) and $P - \mathcal{T}(P) = S$, $L - \mathcal{T}(L) = T$ then $P \geq L \geq 0$ ($> 0$). These results also hold replacing $\mathcal{T}$ by $\mathcal{L}$.

We present now the definition of mean square stabilizability and detectability. Consider $A = (A_1,...,A_N) \in \mathcal{H}^n$, $B = (B_1,...,B_N) \in \mathcal{H}^{m,n}$ and $C = (C_1,...,C_N) \in \mathcal{H}^{n,p}$ real.

**Definition 2** : We say that (A,B) is mean square stabilizable if there exists $K = (K_1,...,K_N) \in \mathcal{H}^{n,m}$ such that model (1) is MSS with $\tilde{A}_i = A_i - B_i K_i$. In this case we say that K stabilizes (A,B) in the mean square sense and set $\mathbb{K} = \{ K \in \mathcal{H}^{n,m};$ K stabilizes (A,B) in the mean square sense$\}$. Similarly, we say that (C,A) is mean square detectable if there exists $H = (H_1,...,H_N) \in \mathcal{H}^{p,n}$ such that model (1) is MSS with $\tilde{A}_i = A_i - H_iC_i$, and we say that H stabilizes (C,A).

The next proposition follows from Proposition 6 in [9]. Consider $D = (D_1,...,D_N) \in \mathcal{H}^{m,p}$ such that $D_i'D_i > 0$ and set $\mathcal{E}_i(L) = \sum_{j=1}^{N} p_{ij}L_j$ , i = 1,...,N, for $L = (L_1,...,L_N)$.

**Proposition 3** : Suppose (C,A) is mean square detectable and $P = (P_1,...,P_N) \geq 0$, $K = (K_1,..., K_N) \in \mathcal{H}^{n,m}$ satisfy

$$- P_i + (A_i - B_i K_i)'\mathcal{E}_i(P)(A_i - B_i K_i) + (C_i - D_i K_i)'(C_i - D_i K_i) \leq 0. \tag{4}$$

Then $K = (K_1,...K_N) \in \mathbb{K}$.

### 3.2. The $H_2$-Norm

Consider again on $(\Omega,\{\mathcal{F}_k\},\mathcal{F},P)$, the following system $\mathcal{G}$

$$\mathcal{G} = \begin{cases} x(k+1) = \tilde{A}_{\theta(k)}x(k) + Jw(k) & (5.a) \\ x(0) = 0 \text{ , } \theta(0) = \theta_0 & (5.b) \\ z(k) = \tilde{C}_{\theta(k)}x(k) & (5.c) \end{cases}$$

where $\tilde{A} = (\tilde{A}_1,...,\tilde{A}_N) \in \mathcal{H}^n$, $\tilde{C} = (\tilde{C}_1,...,\tilde{C}_N) \in \mathcal{H}^{n,p}$ and $J \in \mathbb{M}(\mathbb{C}^r,\mathbb{C}^n)$, with $\tilde{A}$, $\tilde{C}$, J real and $JJ' > 0$.

Suppose that $r_\sigma(\mathcal{T}) < 1$ (that is, model (1) is MSS) and w = (w(0),...) is an impulse input. From Proposition 2.d) we have $z = (z(0),z(1),...) \in \ell_2^p(\mathcal{F}_k)$. The next definition is a generalization of the $H_2$-norm from discrete-time deterministic systems to the stochastic Markovian jump case:

**Definition 3** : We define the $H_2$-norm of system $\mathcal{G}$ as

$$\| \mathcal{G} \|_2^2 = \sum_{s=1}^{r} \sum_{j=1}^{N} \| z_{s,j} \|_2^2$$

where $z_{s,j}$ represents the output sequence (z(0),z(1),...) given by (5.c) when

a) the input sequence is given by w = (w(0),w(1),...), $w(0) = e_s$, $w(k) = 0$, $k > 0$, $e_s \in \mathbb{R}^r$ the unitary vector formed by 1 at the $s^{th}$ position and zero elsewhere, and

b) $\theta(0) = \theta(1) = j$.

For the deterministic case (N=1 and $p_{11}$=1) the above definition reduces to the usual $H_2$-norm. As in the deterministic case, we have that the $H_2$-norm as defined above can be calculated as the solution of the discrete-time coupled gramian of observability and controllability. For this, define $\mathcal{C} = (\tilde{C}_1'\tilde{C}_1,...,\tilde{C}_N'\tilde{C}_N) \in \mathcal{H}^{n+}$, $\mathcal{J} = (JJ',...,JJ') \in \mathcal{H}^{n+}$, and $L = (L_1,...,L_N) \in \mathcal{H}^{n+}$, $P = (P_1,...,P_N) \in \mathcal{H}^{n+}$ the unique solution of the equations (recall that $r_\sigma(\mathcal{T})=r_\sigma(\mathcal{L}) < 1$ and see Proposition 2)

$$L = \mathcal{L}(L) + \mathcal{C} \text{ (observability gramian)} \tag{6}$$

$$P = \mathcal{T}(P) + \mathcal{J} \text{ (controllability gramian).} \tag{7}$$

The next result was proved in [5] and represents a characterization of the $H_2$-norm in terms of the solution of the observability and controllability gramians.

**Proposition 4** : $\| \mathcal{G} \|_2^2 = \sum_{j=1}^{N} \mathrm{tr}(J' L_j J) = \sum_{j=1}^{N} \mathrm{tr}(\tilde{C}_j P_j \tilde{C}_j')$.

### 3.3. The $H_\infty$-Norm

Consider again system $\mathcal{G}$ as in (5) above with $w = (w(0),...) \in \ell_2^r(\mathcal{F}_k)$. The following result was proved in Proposition 2 of [6].

**Proposition 5** : $r_\sigma(\mathcal{T}) < 1$ if and only if $x = (0,x(1),...) \in \ell_2^n(\mathcal{F}_k)$ for every $w = (w(0),w(1),...) \in \ell_2^r(\mathcal{F}_k)$.

Suppose that $r_\sigma(\mathcal{T}) < 1$. From the above Proposition, $x = (0,x(1),...) \in \ell_2^n(\mathcal{F}_k)$ and thus $z = (0,z(1),...) \in \ell_2^p(\mathcal{F}_k)$. The $H_\infty$-norm of system $\mathcal{G}$ is defined as :

**Definition 4** : $\| \mathcal{G} \|_\infty := \sup_{\theta_0} \sup_{w \in \ell_2^r(\mathcal{F}_k)} \frac{\|z\|_2}{\|w\|_2}$ .

Again, for the deterministic case ($N = 1$ and $p_{11} = 1$), the above definition reduces to the usual $H_\infty$-norm.

# 4. Mixed $H_2/H_\infty$-Control Problem

Consider now a controlled version of system $\mathcal{G}$

$$\mathcal{G} = \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + Jw(k) & (8.a) \\ x(0) = 0 \, , \, \theta(0) = \theta_0 & (8.b) \\ z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k) & (8.c) \end{cases}$$

where $A = (A_1,...,A_N) \in \mathcal{H}^n$, $B = (B_1,...,B_N) \in \mathcal{H}^{m,n}$, $C = (C_1,...,C_N) \in \mathcal{H}^{n,p}$, $D = (D_1,...,D_N) \in \mathcal{H}^{m,p}$, $J \in \mathbb{M}(\mathbb{C}^r,\mathbb{C}^n)$ are real, and $C_i'D_i = 0$ for each $i = 1,...,N$.

For $K = (K_1,...,K_N)$ set $\mathcal{G}_K$ as system (8) with $u(k) = -K_{\theta(k)}x(k)$. We have the following result.

**Theorem 1** : Suppose (C,A) is mean square detectable and $\delta > 0$ fixed a real number. If there exists $P = (P_1,...,P_N) \geq 0$ and $K = (K_1,...,K_N) \in \mathcal{H}^{n,m}$ such that for each $i = 1,...,N$,

$$- P_i + (A_i - B_i K_i)'\mathcal{E}_i(P)(A_i - B_i K_i) + (C_i - D_i K_i)'(C_i - D_i K_i) + \frac{1}{\delta^2} P_i J J' P_i \leq 0. \qquad (9)$$

then $K = (K_1,...K_N) \in \mathbb{K}$ and

$$\| \mathcal{G}_K \|_\infty^2 \leq \delta^2(1 - \nu) \leq \delta^2$$

where $\nu \in (0, \frac{1}{\delta^2} \sum_{i=1}^{N} \mathrm{tr}(J'P_iJ))$. Moreover,

$$\| \mathcal{G}_K \|_2^2 \leq \sum_{i=1}^{N} \mathrm{tr}(J'P_iJ).$$

**Proof** : Comparing (4) and (9) it is immediate from Proposition (3) that $K \in \mathbb{K}$. Set $\tilde{A}_i = A_i - B_iK_i$ and $\tilde{C}_i = C_i - D_iK_i$. Recalling from Proposition 5 that, for any $w = (w(0),...) \in \ell_2^r(\mathcal{F}_k)$ we have $x = (0,x(1),...) \in \ell_2^n(\mathcal{F}_k)$, and that $x(k)$, $\theta(k)$ and $w(k)$ are $\mathcal{F}_k$-measurable, we get from (9) that

$$
\begin{aligned}
E(x(k+1)'P_{\theta(k+1)}x(k+1)) &= E(E(x(k+1)'P_{\theta(k+1)}x(k+1) \mid \mathcal{F}_k)) = E(x(k+1)'E(P_{\theta(k+1)} \mid \mathcal{F}_k)x(k+1)) \\
&= E((\tilde{A}_{\theta(k)}x(k) + Jw(k))'\mathcal{E}_{\theta(k)}(P)(\tilde{A}_{\theta(k)}x(k)+Jw(k))) \\
&\le E\Big(x(k)'\Big(P_{\theta(k)} - \tilde{C}'_{\theta(k)}\tilde{C}_{\theta(k)} - \frac{1}{\delta^2} P_{\theta(k)}JJ'P_{\theta(k)}\Big)x(k) + \\
&\quad w(k)'J'\mathcal{E}_{\theta(k)}(P)\tilde{A}_{\theta(k)}x(k) + x(k)'\tilde{A}_{\theta(k)}\mathcal{E}_{\theta(k)}(P)Jw(k) + w(k)'J'\mathcal{E}_{\theta(k)}Jw(k)\Big)
\end{aligned}
$$

so that,

$$
\| P^{1/2}_{\theta(k+1)}x(k+1) \|_2^2 - \| P^{1/2}_{\theta(k)}x(k) \|_2^2 + \| z(k) \|_2^2 \le - \tfrac{1}{\delta^2} \| J'P_{\theta(k)}x(k) \|_2^2 +
$$

$$
E(w(k)'J'\mathcal{E}_{\theta(k)}(P)\tilde{A}_{\theta(k)}x(k)) + E(x(k)'\tilde{A}'_{\theta(k)}\mathcal{E}_{\theta(k)}(P)Jw(k)) + \| \mathcal{E}^{1/2}_{\theta(k)}(P)Jw(k) \|_2^2 =
$$

$$
- \tfrac{1}{\delta^2} \| J'P_{\theta(k)}x(k) \|_2^2 + \tfrac{1}{\delta^2} \| J'P_{\theta(k+1)}x(k+1) \|_2^2 - \tfrac{1}{\delta^2} \| J'P_{\theta(k+1)}x(k+1) \|_2^2 +
$$

$$
2\, E(w(k)'J'\mathcal{E}_{\theta(k)}(P)(\tilde{A}_{\theta(k)}x(k) + Jw(k))) - E\Big(w(k)'J'\mathcal{E}_{\theta(k)}(P)Jx(k)\Big) .
$$

Thus,

$$
\| P^{1/2}_{\theta(k+1)}x(k+1) \|_2^2 - \| P^{1/2}_{\theta(k)}x(k) \|_2^2 - \tfrac{1}{\delta^2} \| J'P_{\theta(k+1)}x(k+1) \|_2^2 + \tfrac{1}{\delta^2} \| J'P_{\theta(k)}x(k) \|_2^2 + \| z(k) \|_2^2
$$

$$
\le
$$

$$
-\tfrac{1}{\delta^2} \| J'P_{\theta(k+1)}x(k+1) \|_2^2 + 2E(w(k)'J'P_{\theta(k+1)}x(k+1)) - \delta^2 \| w(k) \|_2^2 + E(w(k)'(\delta^2 I - J'\mathcal{E}_{\theta(k)}(P)J)w(k))
$$

$$
=
$$

$$
- \| \tfrac{1}{\delta}J'P_{\theta(k+1)}x(k+1) - \delta w(k) \|_2^2 + E(w(k)'(\delta^2 I - J'\mathcal{E}_{\theta(k)}(P)J)w(k)) \le E(w(k)'(\delta^2 I - J'\mathcal{E}_{\theta(k)}(P)J)w(k)).
$$

Taking the sum for $k = 0$ to $\infty$, and recalling that $x(0) = 0$, $\| x(k) \|_2 \to 0$ as $k \to \infty$, we get

$$
\| z \|_2^2 \le \delta^2 \sum_{k=0}^{\infty} E(w(k)'(I - \tfrac{1}{\delta^2} J'P_{\theta(k+1)}J)w(k)) \le \delta^2(1-\nu) \| w \|_2^2
$$

where $\nu \in (0, \frac{1}{\delta^2} \sum_{i=1}^{N} tr(J'P_iJ))$. Thus,

$$
\| \mathcal{G}_K \|_\infty = \sup_{\theta_0} \sup_{w \in \ell_2^r(\mathcal{F}_k)} \frac{\|z\|_2}{\|w\|_2} \le \delta(1 - \nu)^{1/2} < \delta.
$$

Finally notice from Proposition 4, $\| \mathcal{G}_K \|_2^2 = \sum_{i=1}^{N} tr(J'S_iJ)$, where

$$
S_i = \tilde{A}_i'\mathcal{E}_i(S)\tilde{A}_i + \tilde{C}_i'\tilde{C}_i .
$$

From (9) and some $V_i \ge 0$, $i = 1,\dots,N$,

$$
P_i = \tilde{A}_i'\mathcal{E}_i(P)\tilde{A}_i + \tilde{C}_i'\tilde{C}_i + \tfrac{1}{\delta^2} P_iJJ'P_i + V_i'V_i
$$

so that, from Proposition 2, $P_i \ge S_i$ for all $i = 1,\dots,N$. This implies that

$$\| \mathcal{G}_K \|_2^2 = \sum_{i=1}^{N} tr(J' S_i J) \leq \sum_{i=1}^{N} tr(J' P_i J)$$

completing the proof of the Theorem. □

The above Theorem suggests the following approximation for the mixed $H_2/H_\infty$-control problem: for $\delta > 0$ fixed, find $P = (P_1,...,P_N) \geq 0$ and $K = (K_1,...,K_N)$ such that

$$\min tr(\sum_{i=1}^{N} J' P_i J)$$

subject to (9). If we are interested in minimizing the $H_\infty$-norm, then $\delta$ becomes a variable of our problem, and we just have to replace $tr(\sum_{i=1}^{N} J' P_i J)$ above by $\delta^2$. For the case in which N=1, $p_{11}= 1$, equation (9) can be seen as dual to the one obtained in [12], Lemma 3.1.

# 5. Convex Approach

We will assume now that the transition probability matrix $\mathbb{P}$ is not exactly known, but belongs to a convex set $\mathbb{D} := \{\mathbb{P};\ \mathbb{P} = \sum_{t=1}^{q} \alpha_t \mathbb{P}^t,\ \alpha_t \geq 0,\ \sum_{t=1}^{q} \alpha_t = 1\}$, where $\mathbb{P}^t$, t = 1,...,q, are known transition probability matrices. We make the following definition.

**Definition 5** : We say that $K = (K_1,...,K_N) \in \mathcal{H}^{n,m}$ robustly stabilizes (A,B) in the mean square sense if system (1) with $\tilde{A}_i = A_i - B_i K_i$ is MSS for every $\mathbb{P} \in \mathbb{D}$, and we set $\mathbb{K}_r := \{ K \in \mathcal{H}^{n,m};$ K robustly stabilizes (A,B) in the mean square sense$\}$.

We want to solve the following mixed $H_2/H_\infty$ control problem: given $\delta > 0$, find $K \in \mathbb{K}_r$ which minimizes $\zeta$ subject to $\| \mathcal{G}_K \|_2 \leq \zeta$, $\| \mathcal{G}_K \|_\infty \leq \delta$, for every $\mathbb{P} \in \mathbb{D}$. Let us show now that an approximation for this problem can be obtained via a LMI optimization problem. Set $\Gamma_i^t = [\sqrt{p_{i1}^t}\, I \ ...\ \sqrt{p_{iN}^t}\, I] \in \mathbb{M}(\mathbb{C}^{Nn}, \mathbb{C}^n)$ for i = 1,...,N, t = 1,...,q, and define the following problem:

**Problem I** : Set $\mu = \delta^2$. Find $P = (P_1,...,P_N) > 0$, $Q = (Q_1,...,Q_N) > 0$, $L = (L_1,...,L_N) > 0$, $Y = (Y_1,...,Y_N)$ such that

$$\xi = \min tr(\sum_{i=1}^{N} J' P_i J)$$

subject to

$$\begin{bmatrix} Q_i & Q_i A_i' + Y_i' B_i' & Q_i C_i' & Y_i' D_i' & J \\ A_i Q_i + B_i Y_i & L_i & 0 & 0 & 0 \\ C_i Q_i & 0 & I & 0 & 0 \\ D_i Y_i & 0 & 0 & I & 0 \\ J' & 0 & 0 & 0 & \mu I \end{bmatrix} \geq 0,\ i = 1,...,N \qquad (10)$$

$$\begin{bmatrix} L_i & L_i \Gamma_i^t \\ \Gamma_i^{t\prime} L_i & diag\{Q_\kappa\} \end{bmatrix} \geq 0,\ i = 1,...,N,\ t = 1,...,q \qquad (11)$$

$$\begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0,\ i = 1,...,N \qquad (12)$$

where diag$\{Q_\kappa\}$ is the matrix in $\mathbb{M}(\mathbb{C}^{Nn})$ formed by $Q_1,...,Q_N$ in the diagonal, and zero elsewhere.

**Theorem 2** : Suppose Problem I has a solution P, Q, L and Y. Set $K = (K_1,...,K_N)$ as $K_i = - Y_i Q_i^{-1}$, $i = 1,...,N$ and $\xi = \sum_{i=1}^{N} tr(J'P_iJ)$. Then $K \in \mathbb{K}_r$ and $\| \mathcal{G}_K \|_2 \leq \xi^{\frac{1}{2}}$, $\| \mathcal{G}_K \|_\infty \leq \delta$, for every $\mathbb{P} \in \mathbb{D}$.

**Proof**: First of all notice that (10), (11) and (12) are equivalent to (see Remark 1)

$$Q_i \geq Q_i(A_i - B_i K_i)' L_i^{-1}(A_i - B_i K_i) Q_i + Q_i(C_i - D_i K_i)'(C_i - D_i K_i) Q_i + \mu^{-1} Q_i(Q_i^{-1}) JJ'(Q_i^{-1}) Q_i \tag{13}$$

$$L_i \geq L_i(\sum_{j=1}^{N} p_{ij}^t Q_j^{-1}) L_i, \; t = 1,...,q, \tag{14}$$

$$P_i \geq Q_i^{-1}. \tag{15}$$

Since we are minimizing $tr(\sum_{i=1}^{N} J'P_iJ)$ and $JJ' > 0$ by hypothesis, we must have from (15) that $P_i = Q_i^{-1}$. Consider any $\mathbb{P} \in \mathbb{D}$. Then by definition we have $p_{ij} = \sum_{t=1}^{q} \alpha_t p_{ij}^t$ for some $\alpha_t \geq 0$, $\sum_{t=1}^{q} \alpha_t = 1$. Thus from (14) we get

$$L_i^{-1} \geq (\sum_{j=1}^{N} p_{ij} Q_j^{-1}) = \mathcal{E}_i(P),$$

and from (13),

$$\begin{aligned} P_i = Q_i^{-1} &\geq (A_i - B_i K_i)' L_i^{-1}(A_i - B_i K_i) + (C_i - D_i K_i)'(C_i - D_i K_i) + \mu^{-1} P_i J J' P_i \\ &\geq (A_i - B_i K_i)' \mathcal{E}_i(P)(A_i - B_i K_i) + (C_i - D_i K_i)'(C_i - D_i K_i) + \mu^{-1} P_i J J' P_i. \end{aligned} \tag{16}$$

The desired result follows from (16), Proposition 2.f), and Theorem 1. □

**Remark 3** : If we desire to minimize the $H_\infty$-norm, then $\mu$ becomes a variable in problem 1 above, and we just have to replace the value function $tr(\sum_{i=1}^{N} J'P_iJ)$ by $\mu$. Inequalities (12) can be eliminated.

# 6. Numerical Examples

This example is adapted from [12] for the case in which we have two modes of operation, with transition probability matrix between the models given by $\mathbb{P}$. The matrices are:

$$A_1 = A_2 = \begin{bmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{bmatrix}, B_1 = \begin{bmatrix} 0.0013 \\ 0.0539 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.0013 \\ 0.1078 \end{bmatrix}, J = \begin{bmatrix} 1 & 0 \\ 1 & 0.1 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We consider the following cases:

a.1) $H_2/H_\infty$-control problem with $\delta = 80$, and transition probability matrix exactly known, given by:

$$\mathbb{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}.$$

For this case the obtained solution is $K_1$ = [1.36 4.43], $K_2$ = [1.6 4.64], and the optimal value function is $\xi$ = 728. The closed-loop system is mean square stable, with $r_\sigma(\mathcal{T})$ = 0.8624.

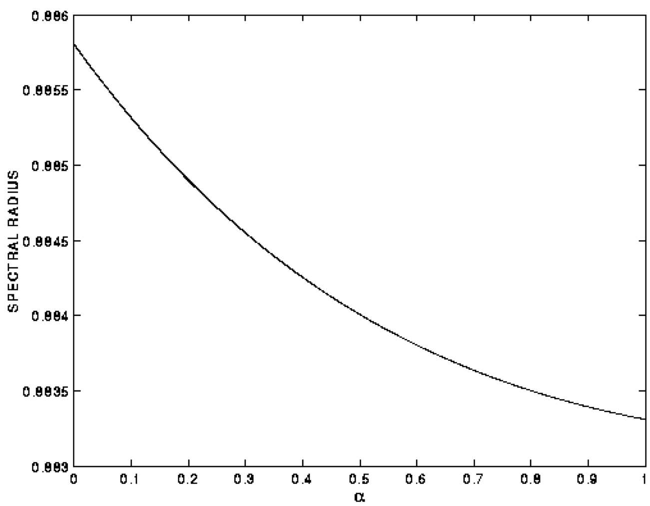
a.2) the same as above but with $\mathbb{P}$ belonging to $\mathbb{D}$, where $\mathbb{D}$ is defined through the transition probability matrices $\mathbb{P}^1$ and $\mathbb{P}^2$ defined below

$$\mathbb{P}^1 = \begin{bmatrix} 0.65 & 0.35 \\ 0.25 & 0.75 \end{bmatrix}, \mathbb{P}^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.15 & 0.85 \end{bmatrix}.$$

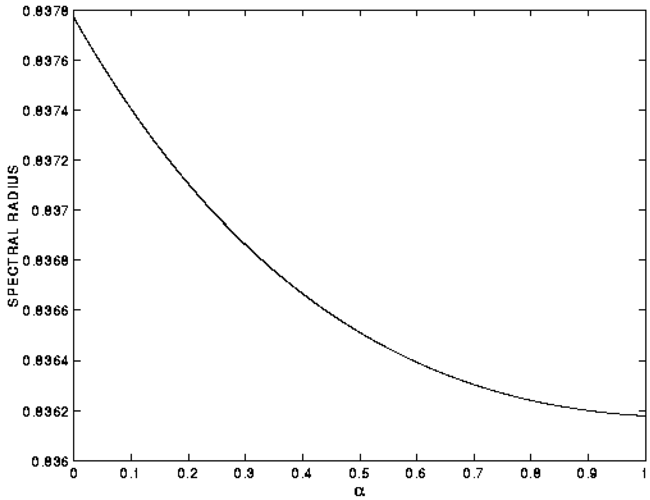
For this case the obtained solution is $K_1$ = [2.13 4.95], $K_2$ = [2.4 6], and the optimal value function is $\xi$ = 983. Figure 1.a shows $r_\sigma(\mathcal{T})$ for all elements of the convex set $\mathbb{D}$. This set is parametrized by $\alpha$, where $\mathbb{P}(\alpha) = \alpha\mathbb{P}^1 + (1-\alpha)\mathbb{P}^2$, $\alpha \in [0,1]$. Notice by the curve that the system is mean square stable for all elements of the convex set $\mathbb{D}$.

b.1) $H_\infty$-control problem with the same data as in a.1) above. The minimal value obtained for $\mu$ (= $\delta^2$) is 4369, with the controllers given by $K_1$ = [5.47 7.02], $K_2$ = [4.89 5.97]. For this case, $r_\sigma(\mathcal{T})$ = 0.8126.

b.2) the same as above but with $\mathbb{P} \in \mathbb{D}$, where $\mathbb{D}$ is defined as in a.2). The minimal value obtained for $\mu$ is 5042, with the controllers given by $K_1$ = [5.86 6.96], $K_2$ = [5.97 7.32]. Figure 1.b shows the spectral radius of $\mathcal{T}(.)$ as a function of $\alpha$, as in a.2). It can be seen from the curve that the closed-loop system is mean square stable for all elements of the convex set $\mathbb{D}$..




(a)




(b)

Figure 1: Spectral radii for $\mathbb{P} \in \mathbb{D}$ (parametrized by $\alpha$).

# 7. Conclusions

In this paper we have considered the problem of mixed $H_2/H_\infty$-control of discrete-time Markovian jump linear systems (MJLS). It has been assumed that both the state variable and the jump variable are available to the controller. The transition probability matrix may belong to an appropriate convex set. We are interested in finding a state and jump feedback controller that robustly stabilizes a MJLS in the mean square sense and minimizes an upper bound for the $H_2$ norm, under the restriction that the $H_\infty$-norm is less than a pre-specified value $\delta$. This kind of problem has been studied in the current literature for discrete-time deterministic linear systems, usually using a state space approach, leading to non-standard algebraic Riccati and Lyapunov-like equations. We have traced a parallel with the discrete-time linear system theory of $H_2/H_\infty$ and $H_\infty$ control to derive our results. An approximation for the problem has been proposed by minimizing a linear functional over the positive semi-definite solutions of a set of coupled Lyapunov-like equations. Furthermore it has been shown that this problem can be written in a convex programming formulation, leading to numerical algorithms. The $H_\infty$-control problem has also been addressed.

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