DYNCON 1393 PROD.TYPE: COM ED: Mariamma
pp. 1–21 (col.fig.: NIL) PAGN: Umesh – SCAN: Monica

ELSEVIER

Journal of Economic Dynamics & Control
00 (2001) 000–000

JOURNAL OF Economic Dynamics & Control

www.elsevier.nl/locate/econbase

Robust portfolio selection using linear-matrix inequalities☆

O.L.V. Costa[a,*], A.C. Paiva[b]

[a]*Departamento de Engeneria de Telecomunicações e Controle, USP, 05508-900, São Paulo, SP, Brazil*
[b]*Departimento de Matemática, USP, 05315-970, São Paulo, SP, Brazil*

Received 14 December 1999; accepted 15 November 2000

Abstract

In this paper, we consider the problem of robust optimal portfolio selection for tracking error when the expected returns of the risky and risk-free assets as well as the covariance matrix of the risky assets are not exactly known. We assume that these parameters belong to a convex polytope defined by some known elements, which form the vertices of this polytope. We consider two problems: the first one is to find a portfolio of minimum worst case volatility of the tracking error with guaranteed fixed minimum target expected performance. The second one is to find a portfolio of maximum worst case target expected performance with guaranteed fixed maximum volatility of the tracking error. We show that these two problems are equivalent to solving linear-matrix inequalities (LMI) optimization problems, so that the powerful numerical packages nowadays available for this class of problems can be used. A numerical example in the São Paulo stock exchange (BOVESPA) is presented. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: C61; C63; G11

Keywords: Portfolio choice; Robustness; Linear-matrix inequalities

☆The authors would like to thank the helpful comments and suggestions made by the reviewers and editor. The research of O.L.V. Costa was supported in part by FAPESP (Research Council of the State of São Paulo), grant 97/04668-1, CNPq (Brazilian National Research Council) grant 305173/88-0, and PRONEX, grant 015/98.

* Corresponding author. Tel.: + 55-11-818-5771; fax: + 55-11-818-5718.
E-mail address: oswaldo@lac.usp.br (O.L.V. Costa).

PII: S0165-1889(00)00086-5

1. Introduction

The concept of mean-variance optimization, introduced by Markowitz (1959), is the most used and well-known tool for economic allocation of capital (Campbell et al., 1997; Elton and Gruber, 1995; Jorion, 1992). More recently, this concept has been extended to include tracking-error optimization (Roll, 1992). In this case the professional money manager is judged by total return performance relative to a pre-specified benchmark portfolio, usually a broadly diversified index of assets. The allocation decision problem is based on the difference between the manager's return and the benchmark return, the so-called tracking error. This means that tracking-error optimization problems could be posed in two forms: to find a portfolio with minimum tracking error volatility for a given target expected performance relative to the benchmark, or to find a portfolio with maximum expected performance relative to the benchmark for a given tracking error volatility.

As pointed out in Rustem et al. (2000), for the optimal mean-variance strategy to be useful the set of expected return of the component assets and the covariance matrix should be sufficiently precise. As shown by Black and Litterman (1991), small changes in expected returns can produce large changes in asset allocation decisions. In practice, this lack of robustness with respect to the inherent inaccuracy of the expected returns and covariance matrix estimates prevents the widespread use of mean-variance optimization by practitioners.

As an example, of the inherent inaccuracy in the calculation of the covariance matrix, consider the exponentially weighted moving average (EWMA) model for estimating the covariance matrix from historical data (which is the method most widely used by practitioners — see (*RiskMetrics Technical Document, 4th Edition,* 1996). Weights declining over time are applied to the set of data points with λ, the decay factor, between 0 and 1. A small λ implies a small number of data being used for the parameter calculations, leading to estimation errors. On the other hand, if λ is very close to one a large number of data would be taken into account, which could be undesirable, especially for non-stable economies.

Another situation that can be critical is when asset allocation is made for a long period of time with no frequent rebalancing. In this case, future changes of economic scenarios could lead to different expected returns and covariance matrices along this period of time. As a consequence, it is not unusual that mean-variance optimization generates asset allocation decisions that seem to be unacceptable or non-intuitive.

Linear matrix inequalities (LMI) applied to robust control and related problems have been extensively studied over the last years (Boyd et al., 1994). Due to the large number of fast and reliable computational techniques available for LMI optimization programming nowadays (Oliveira et al., n.d.), this approach has shown to be an important tool to derive numerical algorithms. In particular, algorithms using LMI optimization programming for obtaining control for uncertain systems have been recently presented in the literature (Boyd et al., 1994; Costa et al., 1997; Geromel et al., 1991). We refer the reader to these references for further details on the practical implementation of LMI algorithms.

The main goal of this paper is to use LMI to solve robust tracking-error optimization problems under the more realistic assumption that the expected returns and covariance matrix are not exactly known or cannot be confidentially estimated. We assume that these parameters are in a convex set defined by the convex combination of some known vertices. By robust tracking-error optimization problems we mean a robust generalization of the two tracking error problems mentioned above. The first one (problem MV) is to find a portfolio of minimum worst case tracking error volatility with respect to a benchmark, with guaranteed fixed target expected performance. The second one (problem MR) is to find a portfolio with maximum worst case expected performance with respect to a benchmark, with guaranteed fixed maximum tracking error volatility. We show that the solution to these problems are equivalent to the solution to LMI optimization problems.

The advantage of this approach is that, besides considering the inaccuracy of the parameter estimates, it also allows, as in Rustem et al. (2000), the possibility of including in the analysis possible future scenarios for the expected returns, risk-free interest rate, and covariance matrix of returns. This could minimize or reduce rebalancing and associated transaction costs in long-term portfolio management since possible future changes of scenario would be taken into account at the moment of the portfolio selection.

It is important to stress that the vertices of the polytope are assumed to be known and should be provided by the asset manager based on numerical estimations and investment analysis, and we shall not be particularly concerned here on how to derive them. Indeed we believe that this point would be interesting in itself and could be further developed in the future. Other approaches presented in the portfolio optimization literature which consider high-order moments of the distribution assets returns (David, 1997; David and Veronesi, 1999) will not be considered here either. As mentioned before, the main objective of this paper will be to cast the robust tracking-error optimization problems into LMI formulations so that the powerful numerical packages nowadays available for LMI optimization problems can be used. We believe that this technique represents a computational tool in the direction of overcoming the main limitations of standard mean-variance optimization mentioned above, with a special focus on real problems.

As examples of previous works on tracking error we can mention Roll (1992) who considers the problem of minimizing the volatility of the tracking error, and Rudolf et al. (1999), who used linear models for tracking error minimization. A paper very closely related to this one is Rustem et al. (2000), where the authors present a min–max strategy for robust portfolio strategies to multiple return and risk scenarios. As shown in Remark 3 below, the LMI problems presented in this paper could be written as min–max problems as defined in Rustem et al. (2000). Other related papers include Howe and Rustem (1997); Howe et al. (1996).

The paper is organized as follows: Section 2 presents the notation, basic results, and problem formulation that will be considered throughout the work. Section 3 presents the equivalence between the robust portfolio tracking-error optimization problems and the LMI optimization problems. In Section 4 we present some numerical examples in the São Paulo stock exchange (BOVESPA). The paper is concluded in Section 5 with some final conclusions and future works. We recall in appendix some basic facts from the LMI literature.

2. Preliminaries

We consider a financial model in which there are N risky assets represented by the random return vector A, with mean vector $\mu \in \mathbb{R}^N$, and covariance matrix Ω $(N \times N)$. Therefore A can be written as

$$A = \mu + \varepsilon, \tag{2.1}$$

where ε is a random vector with zero mean and covariance matrix Ω. We also consider a risk-free asset with return $r \in \mathbb{R}$. It is convenient to define the vector $\varphi \in \mathbb{R}^{N+1}$ as follows:

$$\varphi := \begin{pmatrix} \mu \\ r \end{pmatrix}.$$

A portfolio ω will be a vector belonging to a set Γ of the following form:

$$\Gamma = \left\{ \omega \in \mathbb{R}^N; F_0 + \sum_{i=1}^{N} \omega_i F_i \geq 0 \right\},$$

where F_i, $i = 0, \ldots, N$, are the given symmetric matrices. Notice that the set Γ corresponds to a LMI (see appendix). The components of the vector ω represent the weights on the risky assets A, that is, the ith entry ω_i of ω is the portfolio's proportion invested in asset i. The set Γ is suitable for representing constraints like the sum of the portfolio components is equal or less than 1, and no short sales are permitted, that is, constraints of the form

$$\omega' 1 \leq 1, \quad 0 \leq \omega_i, \ i = 1, \ldots, N,$$

where 1 represents the N vector formed by 1 in all positions, and $'$ denotes transpose. In this case the matrices F_i would be

$$F_0 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$$F_1 = \begin{pmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \dots, F_N = \begin{pmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We consider that $(1 - \omega' 1)$ is invested in a risk-free asset r. Therefore the return of the investor is

$$\omega' A + (1 - \omega' 1) r. \tag{2.2}$$

Let us denote by ω_{B} a fixed portfolio provided by the manager, called benchmark portfolio. As in (2.2), the return of the benchmark portfolio is

$$\omega_{\mathrm{B}}' A + (1 - \omega_{\mathrm{B}}' 1) r. \tag{2.3}$$

From (2.2) and (2.3) we have that the difference between the return of the investor's portfolio ω and the benchmark portfolio, defined as the tracking error $e(\omega)$, is

$$e(\omega) = (\omega - \omega_{\mathrm{B}})' A + (\omega_{\mathrm{B}} - \omega)' 1 r. \tag{2.4}$$

From (2.1) and (2.4) we have that the expected value of the tracking error $e(\omega)$, denoted by $\rho_\varphi(\omega)$, is given by

$$\rho_\varphi(\omega) = (\omega - \omega_{\mathrm{B}})' \mu + (\omega_{\mathrm{B}} - \omega)' 1 r = (\omega - \omega_{\mathrm{B}})' (\mu - 1 r) \tag{2.5}$$

and the variance (volatility), denoted by $\sigma_\Omega^2(\omega)$, by

$$\sigma_\Omega^2(\omega) = (\omega - \omega_{\mathrm{B}})' \Omega (\omega - \omega_{\mathrm{B}}). \tag{2.6}$$

For the case in which the expected returns μ and r, and covariance matrix Ω are perfectly known, two kinds of problem are usually considered in the literature. The first one is to minimize the volatility of the tracking error conditional on a target expected performance τ. By target expected performance we mean a real number τ provided by the investor which represents the minimum value that the expected value of the tracking error could achieve. More formally, the problem can be written as

$$\begin{aligned} \min_{\omega} \quad & \sigma^2_{\Omega}(\omega) \\ \text{s.t.} \quad & \rho_{\varphi}(\omega) \geq \tau, \\ & \omega \in \Gamma. \end{aligned} \tag{2.7}$$

The other problem is to maximize the expected value of the tracking error conditional on a maximum value ϑ^2 for the volatility of the tracking error. The value ϑ^2, provided by the manager, represents the maximum value that the volatility of the tracking error could achieve. Mathematically, this problem can be written as

$$\begin{aligned} \max_{\omega} \quad & \rho_{\varphi}(\omega) \\ \text{s.t.} \quad & \sigma^2_{\Omega}(\omega) \leq \vartheta^2, \\ & \omega \in \Gamma. \end{aligned} \tag{2.8}$$

The above problems can be solved for the case in which the expected returns μ and r, and covariance matrix Ω are perfectly known.

In this paper we shall assume that these quantities are not exactly known. Define the $(N+1)\times(N+1)$ matrix Φ as follows:

$$\Phi := \begin{pmatrix} \Omega & \mu \\ 0 & r \end{pmatrix}.$$

We will suppose that (see Eq. (A.4) for notation)

$$\Phi \in \mathrm{Con}\{\Phi_1, \dots, \Phi_n\},$$

where the elements

$$\Phi_i := \begin{pmatrix} \Omega_i & \mu_i \\ 0 & r_i \end{pmatrix}, \quad i = 1, \dots, n,$$

are assumed to be known. It will be also convenient to define

$$\varphi_i := \begin{pmatrix} \mu_i \\ r_i \end{pmatrix}, \quad i = 1, \dots, n$$

and

$$\mathscr{X} = \mathrm{Con}\{\Omega_1, \dots, \Omega_n\},$$

$$\mathscr{Y} = \mathrm{Con}\{\varphi_1, \dots, \varphi_n\}.$$

Consider $v \in \mathbb{R}^n$ where

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \tag{2.9}$$

The value v_i will denote the target-expected performance if the expected return vector is μ_i and the risk-free return is r_i, $i = 1, \ldots, n$. We shall call v the vector of target-expected performance. For a generic $\varphi \in \mathscr{Y}$, where

$$\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i, \qquad 0 \leq \lambda_i \leq 1, \quad i = 1, \ldots, n,$$

the target-expected performance $\tau(v, \mu, r)$ will be the corresponding convex combination as follows:

$$\tau(v, \varphi) := \sum_{i=1}^{n} \lambda_i v_i. \tag{2.10}$$

In this way the value of the target-expected performance $\tau(v, \varphi)$ is determined according to the expected return $\varphi \in \mathscr{Y}$ of the risky assets and risk-free return. Clearly $\tau(v, \varphi_i) = v_i$.

Similarly consider $\psi \in \mathbb{R}^n$, where

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \qquad \psi_i \geq 0, \quad i = 1, \ldots, n. \tag{2.11}$$

The value ψ_i will denote the maximum volatility of the tracking error if the covariance matrix of the risky assets is Ω_i, $i = 1, \ldots, n$. We shall call ψ the vector of maximum volatility. For a generic $\Omega \in \mathscr{X}$, where

$$\Omega = \sum_{i=1}^{n} \lambda_i \Omega_i, \qquad 0 \leq \lambda_i \leq 1, \quad i = 1, \ldots, n,$$

the maximum volatility $\vartheta(\psi, \Omega)$ of the tracking error will be the corresponding convex combination as follows:

$$\vartheta(\psi, \Omega) = \sum_{i=1}^{n} \lambda_i \psi_i. \tag{2.12}$$

In this way, the values of the maximum volatility of the tracking error $\vartheta(\psi, \Omega)$ is determined according to the covariance matrix $\Omega \in \mathscr{X}$. Clearly $\vartheta(\psi, \Omega_i) = \psi_i$.

We present the following definitions regarding the robustness properties we shall consider:

Definition 1. For a fixed $v \in \mathbb{R}^n$ as in (2.9) we say that a portfolio ω is robust with respect to the vector of target-expected performance v if

$$\rho_\varphi(\omega) \geq \tau(v, \varphi) \tag{2.13}$$

for every $\varphi \in \mathscr{Y}$, where $\rho_\varphi(\omega)$ and $\tau(v, \varphi)$ are given as in (2.5) and (2.10), respectively. Similarly, for $\psi \in \mathbb{R}^n$ as in (2.11), we say that a portfolio ω is robust with respect to the vector of maximum volatility ψ if

$$\sigma^2_\Omega(\omega) \leq \vartheta(\psi, \Omega) \tag{2.14}$$

for every $\Omega \in \mathscr{X}$, where $\sigma^2_\Omega(\omega)$ and $\vartheta(\psi, \Omega)$ are given by (2.6) and (2.12), respectively.

Two kinds of problem are considered, which can be seen as robust versions of problems (2.7) and (2.8), respectively.

Definition 2. The MVGRU problem (*minimum worst case variance*, *guaranteed return under uncertainty*): For a fixed $v \in \mathbb{R}^n$ as in (2.9), find a portfolio ω_v such that

(a) it is robust with respect to the vector of target-expected performance v,
(b) for any other portfolio ω robust with respect to the vector of target-expected performance v, we have

$$\max_{\Omega \in \mathscr{X}} \ \sigma^2_\Omega(\omega_v) \leq \max_{\Omega \in \mathscr{X}} \ \sigma^2_\Omega(\omega). \tag{2.15}$$

Therefore, in the MVGRU problem we want to find a portfolio ω_v such that the target-expected performance (2.13) is satisfied for every $\varphi \in \mathscr{Y}$ and if ω is another portfolio with this property then there exists $\Omega^\circ \in \mathscr{X}$ (in fact, $\Omega^\circ = \Omega_i$ for some $i \in \{1, \ldots, n\}$) such that the tracking-error volatility of the portfolio ω with respect to Ω° will be greater than the tracking-error volatility of the portfolio ω_v with respect to any $\Omega \in \mathscr{X}$ (Eq. (2.15)).

Definition 3. The MRGVU problem (*maximum worst case return*, *guaranteed variance under uncertainty*): For a fixed $\psi \in \mathbb{R}^n$ as in (2.11), find a portfolio ω_ψ such that

(a) it is robust with respect to the vector of maximum volatility ψ,
(b) for any other portfolio ω robust with respect to the vector of maximum volatility ψ, we have

$$\min_{\varphi \in \mathscr{Y}} \ \rho_\varphi(\omega) \leq \min_{\varphi \in \mathscr{Y}} \ \rho_\varphi(\omega_\psi). \tag{2.16}$$

Therefore, in the MRGVU problem we want to find a portfolio ω_ψ such that its tracking-error volatility satisfies (2.14) for all the values $\Omega \in \mathscr{X}$, and if ω is another portfolio with this property then there exists $\varphi^\circ \in \mathscr{Y}$ (in fact, $\varphi^\circ = \varphi_i$ for some $i \in \{1, \dots, n\}$) such that the expected tracking error of the portfolio ω with respect to φ° will be less than the expected-tracking error of the portfolio ω_ψ with respect to any $\varphi \in \mathscr{Y}$ (Eq. (2.16)).

Remark 1. It is worth noticing that by taking $\omega_{\mathrm{B}} = 0$ we obtain the traditional problem of portfolio optimization, which provides the "best" allocation based on given expected returns and covariance matrix (robust in our case).

3. LMI formulation

We shall show in this section that problems MVGRU and MRGVU can be formulated in terms of LMI optimization problems. To do this we shall use the Shur complement, presented in Proposition 3 of Appendix A. We start with the following Proposition:

Proposition 1. (a) *Portfolio ω is robust with respect to the vector of maximum volatility ψ, where $\psi \in \mathbb{R}^n$ is as in* (2.11) *if and only if* (b) $(\psi, \omega), \psi \in \mathbb{R}^n$, $\omega \in \Gamma$ *satisfies the following LMI*:

$$\begin{bmatrix} \psi_i & (\omega - \omega_{\mathrm{B}})'\Omega_i \\ \Omega_i(\omega - \omega_{\mathrm{B}}) & \Omega_i \end{bmatrix} \geq 0, \quad i = 1, \dots, n. \tag{3.1}$$

Proof. (a) $\Rightarrow$ (b): If portfolio ω is robust with respect to the vector of maximum volatility ψ then $\omega \in \Gamma$ and, by Definition 1 (see (2.6) and (2.14)), for every $\Omega \in \mathscr{X}$,

$$0 \leq (\omega - \omega_{\mathrm{B}})'\Omega(\omega - \omega_{\mathrm{B}}) \leq \vartheta(\psi, \Omega)$$

and in particular we have from (2.12) that

$$\psi_i - (\omega - \omega_{\mathrm{B}})'\Omega_i(\omega - \omega_{\mathrm{B}}) \geq 0, \quad i = 1, \dots, n.$$

The last inequality can be written as (see (A.1))

$$\psi_i - (\omega - \omega_{\mathrm{B}})'\Omega_i\Omega_i^\dagger\Omega_i(\omega - \omega_{\mathrm{B}}) \geq 0, \quad i = 1, \dots, n, \tag{3.2}$$

and from (A.1) again, for $i = 1, \dots, n$,

$$\begin{aligned}(\omega - \omega_{\mathrm{B}})'\Omega_i(I - \Omega_i^\dagger\Omega_i) &= (\omega - \omega_{\mathrm{B}})'\Omega_i - (\omega - \omega_{\mathrm{B}})'\Omega_i\Omega_i^\dagger\Omega_i \\ &= (\omega - \omega_{\mathrm{B}})'\Omega_i - (\omega - \omega_{\mathrm{B}})'\Omega_i = 0. \end{aligned} \tag{3.3}$$

From (3.2), (3.3) and Proposition 3 (see (A.2), (A.3)) we can conclude that (3.1) must be satisfied.

(b) ⇒ (a): If (3.1) is satisfied then from Proposition 3 again, and (A.1), (A.2), (A.3) we must have that

$$\begin{aligned}\psi_i &\geq (\omega-\omega_{\mathrm{B}})'\Omega_i\Omega_i^{\dagger}\Omega_i(\omega-\omega_{\mathrm{B}})\\ &=(\omega-\omega_{\mathrm{B}})'\Omega_i(\omega-\omega_{\mathrm{B}}),\quad i=1,\dots,n.\end{aligned}\tag{3.4}$$

For any $\lambda_i \geq 0$, $i=1,\dots,n, \sum_{i=1}^{n}\lambda_i=1$, and writing $\Omega=\sum_{i=1}^{n}\lambda_i\Omega_i$, we have from (2.12) and (3.4) that

$$\vartheta(\psi,\Omega)=\sum_{i=1}^{n}\lambda_i\psi_i \geq (\omega-\omega_{\mathrm{B}})'\left(\sum_{i=1}^{n}\lambda_i\Omega_i\right)(\omega-\omega_{\mathrm{B}})=(\omega-\omega_{\mathrm{B}})'\Omega(\omega-\omega_{\mathrm{B}})$$

and since it holds for every $\Omega\in\mathscr{X}$, we get (a). □

Proposition 2. (a) *Portfolio* ω *is robust with respect to the vector of target-expected performance* v, *for* $v\in\mathbb{R}^n$ *as in* (2.9) *if and only if* (b) $\omega\in\Gamma$ *satisfies the following linear inequalities*:

$$(\omega-\omega_{\mathrm{B}})'(\mu_i-r_i1)\geq v_i,\quad i=1,\dots,n.\tag{3.5}$$

Proof. (a) ⇒ (b): If portfolio ω is robust with respect to the minimum return v then from Definition 1 (see (2.5) and (2.13))

$$\rho_{\varphi}(\omega)=(\omega-\omega_{\mathrm{B}})'\mu-(\omega-\omega_{\mathrm{B}})'1r\geq\tau(v,\varphi),$$

for every $\varphi\in\mathscr{Y}$. In particular, from (2.10), we have that

$$(\omega-\omega_{\mathrm{B}})'\mu_i-(\omega-\omega_{\mathrm{B}})'1r_i\geq v_i,\quad i=1,\dots,n,$$

which is (3.5).

(b) ⇒ (a): If (3.5) is satisfied, then for any $\lambda_i\geq 0, i=1,\dots,n,\sum_{j=1}^{m}\lambda_j=1$ and writing $\varphi=\sum_{i=1}^{n}\lambda_i\varphi_i$, we have

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i((\omega-\omega_{\mathrm{B}})'\mu_i-(\omega-\omega_{\mathrm{B}})'1r_i)&=(\omega-\omega_{\mathrm{B}})'\mu-(\omega-\omega_{\mathrm{B}})'r\\ &\geq\sum_{i=1}^{n}\lambda_i v_i=\tau(v,\varphi)\end{aligned}$$

and since it holds for every $\varphi\in\mathscr{Y}$, we have (a). □

We now define the following LMI optimization problems.

Definition 4. LMI 1: For $v \in \mathbb{R}^n$ as in (2.9) fixed, find $(\hat{\alpha}, \hat{\omega})$ solution to the following LMI optimization

$$\min \quad \alpha$$

$$\text{s.t.} \quad \begin{bmatrix} \alpha & (\omega - \omega_{\mathrm{B}})'\Omega_i \\ \Omega_i(\omega - \omega_{\mathrm{B}}) & \Omega_i \end{bmatrix} \geq 0$$

$$(\omega - \omega_{\mathrm{B}})'(\mu_i - r_i 1) \geq v_i, \quad i = 1, \ldots, n$$

$$\omega \in \Gamma.$$

Definition 5. LMI 2: For $\psi \in \mathbb{R}^n$ is as in (2.11) fixed, find $(\hat{\beta}, \hat{\omega})$ solution to the following LMI optimization

$$\max \quad \beta$$

$$\text{s.t.} \quad \begin{bmatrix} \psi_i & (\omega - \omega_{\mathrm{B}})'\Omega_i \\ \Omega_i(\omega - \omega_{\mathrm{B}}) & \Omega_i \end{bmatrix} \geq 0,$$

$$(\omega - \omega_{\mathrm{B}})'(\mu_i - r_i 1) \geq \beta, \quad i = 1, \ldots, n$$

$$\omega \in \Gamma.$$

Remark 2. Notice that the restrictions of problem LMI 1 may be unfeasible, but problem LMI 2 has always the feasible solution $\omega = \omega_{\mathrm{B}}$, $\beta = 0$, provided that $\omega_{\mathrm{B}} \in \Gamma$.

From Propositions 1 and 2, the following results, linking problems LMI 1 and LMI 2 to problems MVGRU and MRGVU, respectively, are obtained.

Theorem 1. (a) *Problem LMI 1 has a solution* $(\hat{\alpha}, \hat{\omega})$ *if and only if* (b) *the MVGRU problem has a solution* ω_v. *Moreover, if* (a) *holds then* $\omega_v = \hat{\omega}$ *is a solution to the MVGRU problem and similarly if* (b) *holds then* $(\hat{\alpha}, \hat{\omega})$ *is a solution to LMI 1 where* $\hat{\omega} = \omega_v$ *and*

$$\hat{\alpha} = \max\{(\omega_v - \omega_{\mathrm{B}})'\Omega_i(\omega_v - \omega_{\mathrm{B}});\ i = 1, \ldots, n\}.$$

Proof. (a) $\Rightarrow$ (b): If $(\hat{\alpha}, \hat{\omega})$ is an optimal solution to problem LMI 1 then from Proposition 1 with $\psi = (\hat{\alpha} \cdots \hat{\alpha})$ and (2.14),

$$\sigma^2_{\Omega}(\hat{\omega}) \leq \hat{\alpha},$$

for every $\Omega \in \mathscr{X}$. From the optimality of $(\hat{\alpha},\hat{\omega})$ we must have

$$\hat{\alpha} = \max\{(\hat{\omega} - \omega_{\mathrm{B}})'\Omega_i(\hat{\omega} - \omega_{\mathrm{B}});\ i = 1, \dots, n\} = \max_{\Omega \in \mathscr{X}} \sigma^2_{\Omega}(\hat{\omega}). \tag{3.6}$$

From Proposition 2 the portfolio $\hat{\omega}$ is robust with respect to the vector of target-expected performance v. Consider another portfolio ω robust with respect to the vector of target expected performance v. Take

$$\alpha = \max\{(\omega - \omega_{\mathrm{B}})'\Omega_i(\omega - \omega_{\mathrm{B}});\ i = 1, \dots, n\} = \max_{\Omega \in \mathscr{X}} \sigma^2_{\Omega}(\omega). \tag{3.7}$$

From (3.7), portfolio ω is robust with respect to the vector of maximum volatility $(\alpha, \dots, \alpha)$. From Propositions 1 and 2, (α, ω) is a feasible solution to problem LMI 1 and from the optimality of $(\hat{\alpha}, \hat{\omega})$, we should have $\hat{\alpha} \leq \alpha$. From (3.6) and (3.7) we conclude that (2.15) holds with $\omega_v = \hat{\omega}$.

(b) $\Rightarrow$ (a): Following the same steps as above we have that if portfolio ω_v is a solution to the MVGRU problem then $(\hat{\alpha}, \omega_v)$ is a feasible solution to problem LMI 1, where

$$\hat{\alpha} = \max\{(\omega_v - \omega_{\mathrm{B}})'\Omega_i(\omega_v - \omega_{\mathrm{B}});\ i = 1, \dots, n\}.$$

For any other feasible solution (α, ω) to problem LMI 1 we must have, as seen above, that the portfolio ω is robust with respect to the vector of target-expected performance v and

$$\max_{\Omega \in \mathscr{X}} \sigma^2_{\Omega}(\omega) = \max\{(\omega - \omega_{\mathrm{B}})'\Omega_i(\omega - \omega_{\mathrm{B}});\ i = 1, \dots, n\}.$$

From (2.10) we have $\hat{\alpha} \leq \alpha$, and thus $(\hat{\alpha}, \omega_v)$ is optimal for LMI 1. $\square$

Theorem 2. (a) *Problem LMI 2 has a solution* $(\hat{\beta}, \hat{\omega})$ *if and only if* (b) *the problem has a solution* ω_ψ. *Moreover, if* (a) *holds then* $\omega_\psi = \hat{\omega}$ *is a solution to the MRGVU problem and similarly if* (b) *holds then* $(\hat{\beta}, \hat{\omega})$ *is a solution to LMI* 2 *where* $\hat{\omega} = \omega_\psi$ *and*

$$\hat{\beta} = \min\{(\omega_\psi - \omega_{\mathrm{B}})'(\mu_i - r_i 1);\ i = 1, \dots, n\}.$$

Proof. This proof follows the same steps as the previous theorem. (a) $\Rightarrow$ (b) If $(\hat{\beta}, \hat{\omega})$ is an optimal solution to problem LMI 2 then from Proposition 1 the portfolio $\hat{\omega}$ is robust with respect to the vector of maximum volatility ψ. From the optimality of $(\hat{\beta}, \hat{\omega})$ we must have

$$\begin{aligned}\hat{\beta} &= \min\{(\hat{\omega} - \omega_{\mathrm{B}})'(\mu_i - r_i 1);\ i = 1, \dots, n\} \\ &= \min_{\varphi \in \mathscr{Y}} \rho_\varphi(\hat{\omega}).\end{aligned} \tag{3.8}$$

Consider another portfolio ω robust with respect to the vector of maximum volatility ψ. From Proposition 1 we know that ω satisfies (3.1). Take

$$\begin{aligned}\beta &= \min\{(\omega-\omega_{\mathrm{B}})'(\mu_i - r_i 1);\ i=1,\dots,n\}\\ &= \min_{\varphi\in\mathscr{Y}} \rho_\varphi(\omega),\end{aligned} \tag{3.9}$$

so that (3.5) is satisfied. Therefore (β,ω) is feasible to problem LMI 2, and from the optimality of $(\hat\beta,\hat\omega)$, we must have $\hat\beta \geq \beta$. From (3.8) and (3.9) we conclude that

$$\min_{\varphi\in\mathscr{Y}} \rho_\varphi(\hat\omega) \geq \min_{\varphi\in\mathscr{Y}} \rho_\varphi(\omega)$$

and from (2.16) we see that $\hat\omega$ is a solution to the MRGVU problem.

(b) $\Rightarrow$ (a): If portfolio ω_ψ is a solution to the MRGVU problem then from Proposition 1, $(\hat\beta,\omega_\psi)$ is a feasible solution to problem LMI 2, where $\hat\beta = \min\{(\omega_\psi-\omega_{\mathrm{B}})'(\mu_i - r_i 1);\ i=1,\dots,n\}$. For any other feasible solution (β,ω) to problem LMI 2 we must have, as seen above, that the portfolio ω is robust with respect to the vector of maximum volatility ψ and

$$\begin{aligned}\beta &\leq \min\{(\omega-\omega_{\mathrm{B}})'(\mu_i - r_i 1);\ i=1,\dots,n\}\\ &= \min_{\varphi\in\mathscr{Y}} \rho_\varphi(\omega).\end{aligned}$$

From (2.16) we have $\hat\beta \geq \beta$, and thus $(\hat\beta,\omega_\psi)$ is optimal for LMI 2. $\square$

Remark 3. From (3.6), (3.7) and the proof of Theorem 1 we have that problem LMI 1 is equivalent to the following robust min–max portfolio problem as posed by Rustem et al. (2000):

$$\hat\alpha = \min_{\omega} \max_{i} \{(\omega-\omega_{\mathrm{B}})'\Omega_i(\omega-\omega_{\mathrm{B}});\ \omega\in\Lambda(v)\}$$

where

$$\Lambda(v) = \{\omega\in\Gamma;\ (\omega-\omega_{\mathrm{B}})'(\mu_i - r_i 1) \geq v_i,\ i=1,\dots,n\}.$$

Similarly, from (3.8), (3.9) and the proof of Theorem 2, we have that problem LMI 2 is equivalent to the following robust min–max portfolio problem as posed by Rustem et al. (2000):

$$\hat\beta = -\min_{\omega} \max_{i} \{-(\omega-\omega_{\mathrm{B}})'(\mu_i - r_i 1);\ \omega\in\Psi(\psi)\},$$

where

$$\Psi(\psi) = \left\{\omega\in\Gamma;\ \begin{bmatrix}\psi_i & (\omega-\omega_{\mathrm{B}})'\Omega_i\\ \Omega_i(\omega-\omega_{\mathrm{B}}) & \Omega_i\end{bmatrix} \geq 0,\ i=1,\dots,n\right\}.$$

Remark 4. Problems LMI 1 and LMI 2 with $n = 1$ are equivalent to the portfolio obtained from the efficient frontier.

4. Numerical example

In this section we provide a simple and illustrative example of the results obtained in the previous section. We assume that an investor wants to optimize his/her portfolio composed by 11 liquid stocks traded in the São Paulo stock exchange (BOVESPA), and the 1-day risk-free interest rate. The optimized portfolio is based on the previous daily observed returns of the 11 assets, as well as the present 1-day risk-free interest rate, from January 2, 1997 to June 17, 1999. This period is divided into two parts. The period from January 2, 1997 to November 2, 1998 is used for the volatility matrices calculations, while the period from November 3, 1998 to June 17, 1999 is used for the optimization and analysis of the methods. We take the benchmark portfolio to be $\omega_{\mathrm{B}} = (\frac{1}{11}, \ldots, \frac{1}{11})'$, that is, the benchmark return is just the mean of the daily returns of the 11 assets. In Fig. 1 we present the compounded daily returns of the

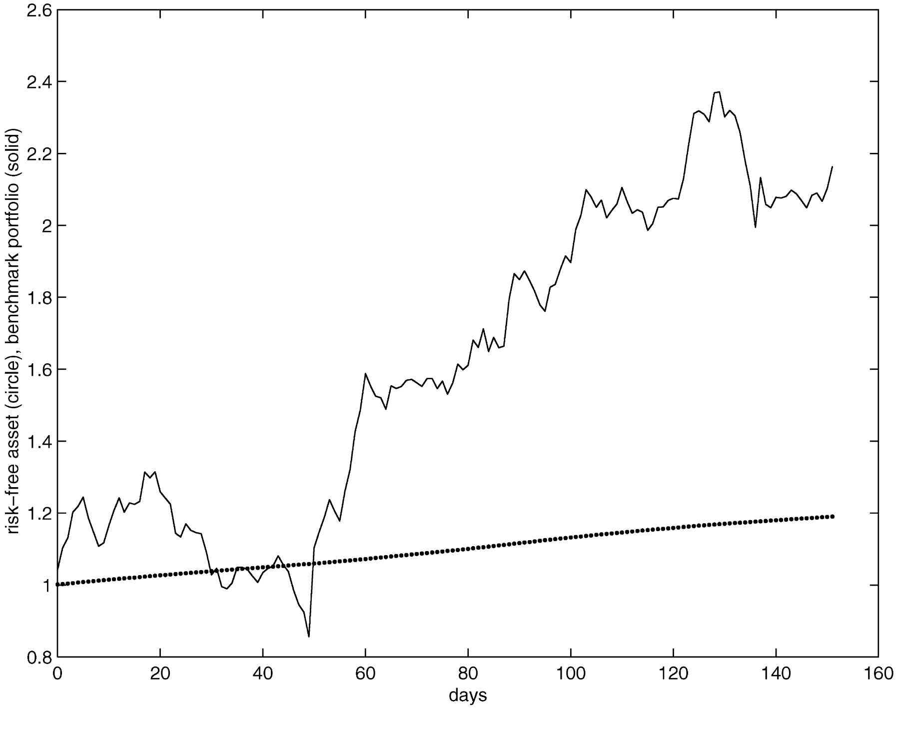

Fig. 1. Compounded daily return of benchmark portfolio (solid thin line) and 1-day risk-free interest rate (circle line) from November 3, 1998 to June 17, 1999.

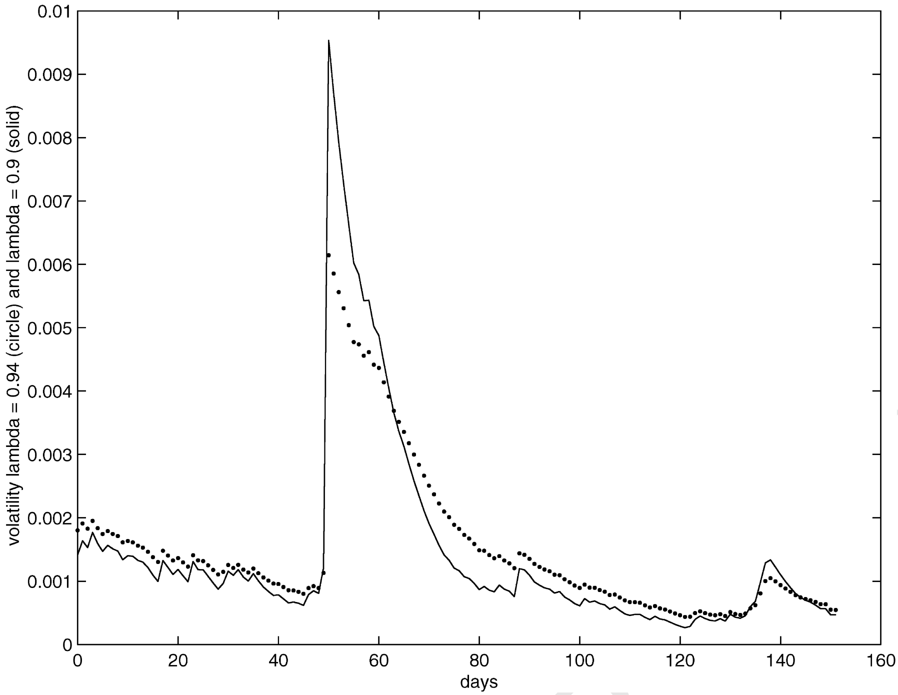

Fig. 2. Daily volatility of benchmark portfolio with decaying factor 0.94 (circle line) and 0.9 (solid line) from November 3, 1998 to June 17, 1999.

benchmark portfolio and the compounded risk-free 1-day interest rate from November 3, 1998 to June 17, 1999.

In Fig. 2 we present the volatility of the benchmark return over the same period, using two different ways of calculating the covariance matrices. The circle line corresponds to the covariance matrix using a decaying factor of 0.94, and the solid line corresponds to a decaying factor of 0.9, both using the EWMA methodology for computing covariance matrices, and the previous 100 daily historical returns for the calculations. Notice that the peak of volatility is related to the change in the exchange-rate policy which occurred in Brazil in January 1999, which led to a great Brazilian currency devaluation. With this devaluation, the prices of the Brazilian assets in dollar became cheaper, which caused an increase of their returns in the Brazilian currency.

Let us denote by $A(i)$ the random vector with the 11 returns of the assets observed at day i and $r(i)$ the risk-free return at day i. For the optimization problems we imposed short-selling constraints of the form $\omega \geq 0$, $0 \leq \omega' 1 \leq 1$. We considered the minimization problem as in problem LMI 1 and compared the portfolio returns obtained in two different ways at day i.

Portfolio 1: With $n = 1$, that is, 1 volatility matrix $\mathrm{Var}_1(i)$, 1 expected return vector $\mu_1(i)$ and 1 risk-free return $r(i)$ were considered. As noticed in Remark 4, this problem is equivalent to the portfolio obtained by the efficient frontier with short-selling constraints. $\mathrm{Var}_1(i)$ was calculated from the 100 previous returns and with a decaying factor of 0.94. The expected return $\mu_1(i)$ was obtained from the 3 previous daily assets returns with weights 0.7, 0.2 and 0.1, respectively. That is,

$$\mu_1(i) = 0.7A(i-1) + 0.2A(i-2) + 0.1A(i-3).$$

$v_1(i)$ was calculated as follows:

$$\omega'_{\mathrm{B}}\mu_1(i) + v_1(i) = \omega'_{\mathrm{B}}A(i-1) + r(i). \tag{4.1}$$

Therefore the constraint on the expected return of Portfolio 1 was to obtain the previous day benchmark portfolio return plus the current 1-day risk-free interest rate return (Eq. (4.1)). In our simulation, these optimization problems were performed from November 3, 1998 through June 17, 1999. The optimized portfolios $\omega(i)$ were updated everyday and the real return was obtained from the assets returns $A(i)$ (known at the end of the day i). That is, the real return of portfolio 1 at day i is

$$\omega(i)'A(i) + r(i)(1 - \omega(i)'1). \tag{4.2}$$

Portfolio 2: With $n = 4$ corresponding to 2 covariance matrices $\mathrm{Var}_1(i)$ and $\mathrm{Var}_2(i)$ and 2 return vectors $\mu_1(i)$ and $\mu_2(i)$. The values of $\mathrm{Var}_1(i)$, $\mu_1(i)$ and $v_1(i)$ were as in Portfolio 1. For $\mathrm{Var}_2(i)$ we considered the 100 previous returns and decaying factor of 0.9. For $\mu_2(i)$ we considered the 2 previous daily assets returns with weights 0.5 and 0.5. That is, at day i

$$\mu_2(i) = 0.5A(i-1) + 0.5A(i-2).$$

$v_2(i)$ was calculated as follows:

$$\omega'_{\mathrm{B}}\mu_2(i) + v_2(i) = \omega'_{\mathrm{B}}A(i-2) + r(i). \tag{4.3}$$

Therefore the second constraint on the return of Portfolio 2 was to obtain the benchmark return of two days ago plus the current risk-free 1-day interest rate return (Eq. (4.3)). In our simulation, these optimization problems were performed from November 3, 1998 through June 17, 1999. The optimized portfolios $\omega(i)$ were updated every 2 days and the real return obtained from the assets returns $A(i)$ (known at the end of the day i), that is, the real return of portfolio 2 at day i is as in Eq. (4.2). In our simulations, the problem LMI 1 was found to be unfeasible 4 times (out of 76 optimizations). In this situations the portfolio was kept unchanged.

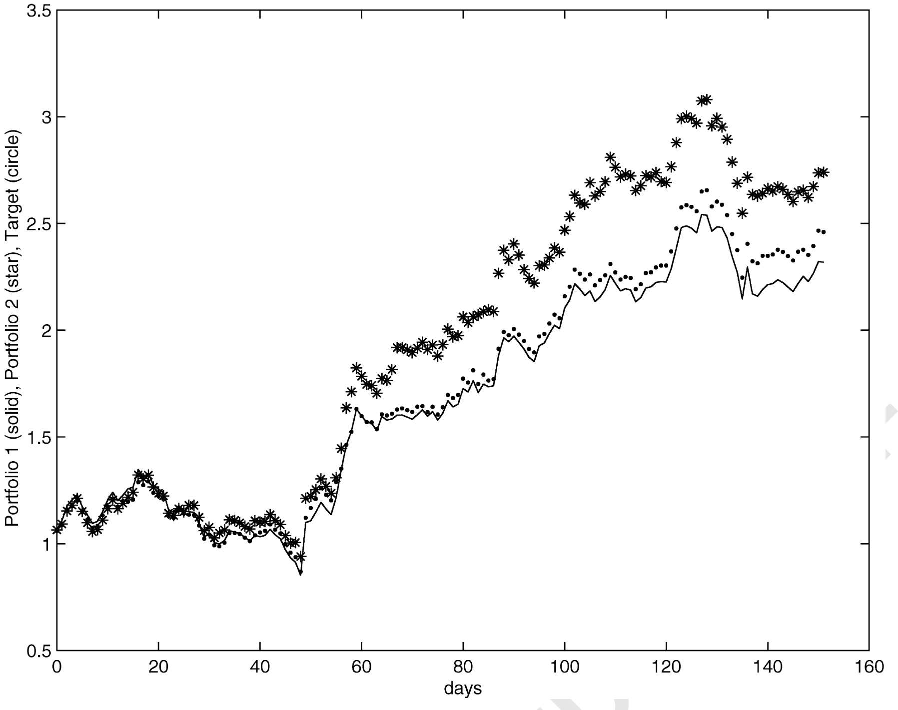

Fig. 3. Compounded daily returns of Portfolio 1 ($m = n = k = 1$, updated every day) – solid line, Portfolio 2 ($m = n = 2, k = 1$, updated every 2 days) – star line, and target portfolio as in (4.4) – circle line.

Fig. 3 presents the compounded daily returns of the portfolios obtained by Portfolios 1 and 2 as in Eq. (4.2), and the compounded daily returns of the target portfolio, obtained from the (single net) return at day i as

$$\omega_{\mathbf{B}}' A(i) + r(i). \tag{4.4}$$

The star line represents Portfolio 2, the solid line Portfolio 1, and the circle line the target portfolio.

Fig. 4 presents the difference between the compounded daily returns of the portfolios obtained by Portfolio 1 (solid line) and Portfolio 2 (star line) with respect to the compounded daily returns of the target portfolio. As mentioned before, the calculations were performed from November 3, 1998 through June 17, 1999.

From Figs. 3 and 4 we see that although Portfolio 2 was updated only every 2 days, its compounded return at the end of the period was 27.95% over the compounded return of the target portfolio, while the compounded return of Portfolio 1, updated every day, was about 14.13% below the compounded return of the target portfolio at the end of the period.

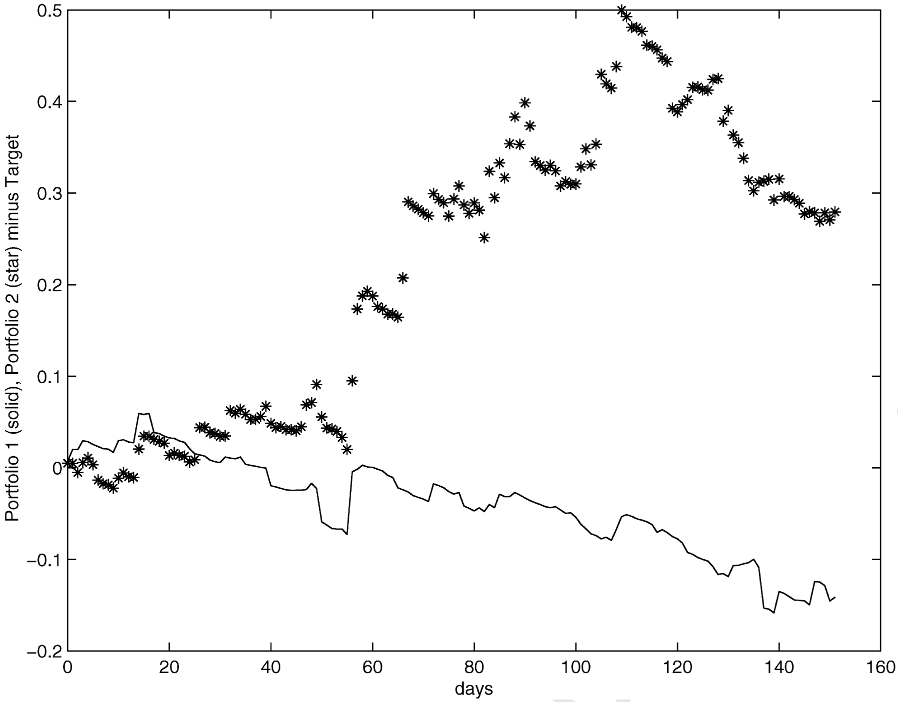

Fig. 4. Difference between compounded daily returns of Portfolio 1 – solid line, and Portfolio 2 – star line, with respect to the target portfolio.

An artificial and oversimplified example where one could intuitively grasp the reason for these results would be as follows. Consider a horizon time of two days, a financial market with one risky asset ($N = 1$) and returns $A(1)$ and $A(2)$ at days 1 and 2, respectively, and volatility 1 for the 2 days, one risk-free asset with return $r = \frac{1}{20}$ for the 2 days, $\omega_{\mathrm{B}} = 0$, and the target-expected performance $v = \frac{1}{40}$ for the 2 days. Therefore, the investor's expected returns would be $r + v = \frac{3}{40}$ at days 1 and 2. As in the numerical example above, consider Portfolio 1, updated at days 1 and 2 and obtained by the efficient frontier, and Portfolio 2, updated only at day 1 and obtained from problem LMI 1. For Portfolio 1, we consider expected returns for the risky asset to be $\mu_1(1) = \frac{1}{10}$ and $\mu_1(2) = \frac{2}{10}$ at days 1 and 2, respectively. Solving these 2 problems we obtain that the returns of Portfolio 1 will be at days 1 and 2 as follows:

$$\text{Day 1: } \tfrac{1}{2}A(1) + \tfrac{1}{2}r = \tfrac{1}{2}A(1) + \tfrac{1}{2} \times \tfrac{1}{20}$$

$$\text{Day 2: } \tfrac{1}{6}A(2) + \tfrac{5}{6}r = \tfrac{1}{6}A(2) + \tfrac{5}{6} \times \tfrac{1}{20}.$$

For Portfolio 2, we consider the 2 possible scenarios for the return of the risky asset at day 1 ($\frac{1}{10}$ and $\frac{2}{10}$), and since we want to make sure that the target expected performance will be achieved, the return of the Portfolio 2 must be for $i = 1, 2$

$$\tfrac{1}{2}A(i) + \tfrac{1}{2}r = \tfrac{1}{2}A(i) + \tfrac{1}{2} \times \tfrac{1}{20}.$$

It is easy to see that for any value of $A(2)$ between $[\frac{1}{10}, \frac{2}{10}]$ Portfolio 1 will have return at day 2 below $\frac{3}{40}$ while Portfolio 2 will have return at day 2 greater than $\frac{3}{40}$. Of course, the analysis is much more complicated in a more general case as the numerical example presented above. But we believe that this kind of argument, in which Portfolio 2 outperforms Portfolio 1 by taking into account possible future scenarios for the parameters, could intuitively explain the results obtained.

5. Conclusions

In this paper, we have considered the problem of optimal portfolio selection for tracking error when the return mean of the risky and risk-free assets as well as the covariance matrix of the risky assets belong to a convex polytope defined by some known elements which form the vertices of this polytope. We showed that the problems of finding a portfolio of minimum worst case volatility of the tracking error with guaranteed fixed minimum target-expected performance, or maximum worst case target-expected performance with guaranteed fixed maximum volatility of the tracking error, are equivalent to solving LMI optimization problems, so that the powerful numerical packages available for this class of problems can be used.

A numerical example in the São Paulo stock exchange (BOVESPA) was presented. We compared two portfolios, one, Portfolio 1, calculated as in the traditional efficient frontier way for minimum risk and fixed return gain, and updated every day. The other, Portfolio 2, as in problem LMI 1 of Section 3, with 2 covariance matrices and 2 expected returns, and updated every 2 days. The simulations were performed with real data collected from January 2, 1997 to June 17, 1999 with 11 liquid stocks assets traded in the São Paulo stock exchange (BOVESPA). The comparison results (Figs. 3 and 4) suggests that the use of LMI algorithms for more than one covariance matrix and expected returns can be useful in the optimization of portfolios, especially if it is desired to minimize or reduce re-balancing and associated transactions costs. At the moment, more comprehensive numerical tests of our methodology in the context of the Brazilian financial market are being performed.

As future works, we mention the possible extension of the LMI methodology for Value at Risk computation (including benchmark-VaR) and in multi-stage stochastic optimization problems. In this case, the main goal would be the robust long-term asset allocations without frequent re-balancing of the portfolio, bearing in mind applications in pension funds allocation problems. Another topic for future work is to consider norm bound uncertainties for the expected returns and covariance matrices (as in the international control literature) instead of polytope uncertainties as adopted here.

Appendix

A (non-strict) linear matrix inequality (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0,$$

where $x = (x_1, \ldots, x_m)$, $x_i \in R$, $i = 1, \ldots, m$ are the variables and the symmetric matrices F_i, $i = 0, \ldots, m$ are given. The inequality $\geq$ means that the symmetric matrix $F(x)$ must be positive semi-definite. A LMI optimization problem consists of finding a feasible x (that is, find x such that $F(x) \geq 0$) which minimizes (or maximizes) a convex function $c(x)$. The key feature of LMI optimization is that these problems are tractable both from theoretical and numerical point of view (e.g. Boyd et al. (1994) and Oliveira et al. (n.d.)).

A key result for converting non-linear convex inequalities into LMI formulation is the Schur complement. For a real matrix Q, we set Q' the transpose of Q. The generalized inverse of Q (or Moore–Penrose inverse of Q) is defined as the unique matrix $Q^{\dagger}$ such that (Saberi et al., 1995, p. 13)

$$\begin{aligned}
&\text{(a)} \quad QQ^{\dagger}Q = Q \\
&\text{(b)} \quad Q^{\dagger}QQ^{\dagger} = Q^{\dagger} \\
&\text{(c)} \quad (QQ^{\dagger})' = QQ^{\dagger} \quad \text{and} \\
&\text{(d)} \quad (Q^{\dagger}Q)' = Q^{\dagger}Q.
\end{aligned} \tag{A.1}$$

The Shur complement, presented next, establishes a link between LMI and nonlinear convex inequalities (Saberi et al., 1995, p. 13).

Proposition 3. *We have that*

$$\text{(a)} \quad \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0 \tag{A.2}$$

if and only if

$$\text{(b)} \quad R \geq 0, \quad S(I - R^{\dagger}R) = 0 \quad \textit{and} \quad Q - SR^{\dagger}S' \geq 0. \tag{A.3}$$

Notice that (a) in Proposition 3 is in the form of a LMI, and (b) is in the form of non-linear convex inequalities.

Finally we conclude this appendix introducing the following notation. Let $\mathbb{X}$ be a space of real vectors or matrices. For a collection of points $v_i \in \mathbb{X}$, $i = 1, \ldots, \kappa$, we shall define the convex polytope $\text{Con}\{v_1, \ldots v_\kappa\}$ as

$$\text{Con}\{v_1, \ldots, v_\kappa\} := \left\{ v \in \mathbb{X};\ v = \sum_{i=1}^{\kappa} \lambda_i v_i,\ \sum_{i=1}^{\kappa} \lambda_i = 1,\ \lambda_i \geq 0 \right\}. \tag{A.4}$$

The above formulation is used to characterize the robustness in the financial model.

References

Black, F., Litterman, R., 1991. Asset allocation: combining investor views with market equilibrium. Journal of Fixed Income 1, 7–18.

Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequalities and Control Theory. SIAM, Philadelphia, PA.

Campbell, J.Y., Lo, A.W., MacKinlay, A.C., 1997. The Econometrics of Financial Markets. Princenton University Press, Princeton, NJ.

Costa, O.L.V., do Val, J.B.R., Geromel, J.C., 1997. A convex programming approach to h_2-control of markovian jump linear systems. International Journal of Control 66, 557–579.

David, A., 1997. Fluctuating confidence in stock markets: implication for returns and volatility. Journal of Finance and Quantitative Analysis 32, 427–462.

David, A., Veronesi, P., 1999. Option prices with uncertain fundamentals: theory and evidence of the dynamics of implied volatilities. Federal Reserve Board Working Paper.

Elton, E.J., Gruber, M.J., 1995. Modern Portfolio Theory and Investment Analysis. Wiley, New York.

Geromel, J.C., Peres, P.L.D., Bernussou, J., 1991. On a convex parameter space method for linear control design of uncertain systems. SIAM Journal on Control and Optimization 29, 381–402.

Howe, M.A., Rustem, B., 1997. A robust hedging algorithm. Journal of Economic Dynamics and Control 21, 1065–1092.

Howe, M.A., Rustem, B., Selby, M.J.P., 1996. Multi-period minimax hedging strategies. European Journal of Operational Research 93, 185–204.

Jorion, P., 1992. Portfolio optimization in practice. Financial Analysis Journal 68–74.

Markowitz, H., 1959. Portfolio Selection: Efficient Diversification of Investments. Wiley, New York.

Oliveira, M.C., Faria, D.P., Geromel, J.C., (n.d.). Lmisol User's Guide. Available at http://www.dt.fee.unicamp.br/carvalho/#soft.

RiskMetrics Technical Document, 4th Edition, 1996. Available at http://www.riskmetrics.com.

Roll, R., 1992. A mean/variance analysis of tracking error. The Journal of Portfolio Management 18, 13–22.

Rudolf, M., Hans-Jurgen Wolter, Heinz Zimmermann, 1999. A linear model for tracking error minimization. The Journal of Banking and Finance 23, 85–103.

Rustem, B., Becker, R.G., Marty, W., 2000. Robust min-max portfolio strategies for rival forecast and risk scenarios. Journal of Economic Dynamics and Control 24, 1591–1621.

Saberi, A., Sannuti, P., Chen, B.M., 1995. H_2-Optimal Control. Prentice-Hall, London.