Constrained Quadratic Control of Discrete-Time Markovian Jump Linear Systems

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Abstract : In this paper we consider the quadratic optimal control problem of a discrete-time Markovian jump linear system, subject to constrains on the state and control variables. It is desired to find a state feedback controller, which may also depend on the jump variable, that minimizes a quadratic cost and satisfies some upper bounds on the norms of some random variables, related to the state and control variables of the system. The transition probability of the Markov chain and initial condition of the system may belong to appropriate convex sets. We obtain an approximation for the optimal solution of this problem in terms of LMI (linear matrices inequalities), so that convex programming can be used for numerical calculations. Examples are presented to illustrate the usefulness of the developed results.

Keywords : quadratic control, constrains, Markovian jump systems, LMI, convex programming.

February 1997; revised in December 1997

This work was supported in part by CNPq (Brazilian National Research Council) and FAPESP (Research Council of the State of São Paulo).

1. Introduction

Markovian jump linear systems (MJLS) comprise an important class of stochastic time-variant linear systems that have been in great evidence over the last years. This family of systems can model several problems where the structure of the plant is subject to random abrupt changes due to, for instance, failures or repairs, sudden environment changes, modification of the operating point of a non-linear system, etc. This large number of applications lead to a great interest on this field and several results, regarding applications, stability conditions and optimal control problems, can be found in the current literature (see, for instance, [1],[3]-[9],[11]-[14],[17]-[23]).

Convex programming applied to control problems have been extensively studied in the last years (see, for instance, [2],[4],[10],[15],[16],[22]). Due to the large number of fast and reliable computational techniques available for convex programming nowadays, this approach has showed to be an important tool to derive numerical algorithms for control problems. In particular algorithms using convex programming for obtaining constrained quadratic control of uncertain systems, H_2-guaranteed cost control for uncertain systems, H_∞-control problems, and mixed H_2/H_∞-control problems have been recently presented in the literature (cf. [2],[10],[15],[16]).

In this paper we study the constrained quadratic control problem of a discrete-time MJLS. Quadratic control problem for MJLS has been studied in the current literature (e.g. see [1],[3]-[5],[7],[8],[11]-[14],[18]-[21]) usually under the assumption that no constrains are imposed on the state and control variables. The transition probability matrix is also usually assumed to be known. In [22] the authors considers the case in which the transition matrix of the Markov chain of a continuous-time MJLS belongs to a convex set, and study the robust (with respect to the uncertainties on the transition matrix) state feedback stabilization problem of the system. We trace a parallel with the results presented in [4], [15] and [22] to derive an algorithm, written in terms of an LMI (linear matrix inequalities) optimization problem, that is capable of handling the quadratic control problem with restrictions on the state and control variables for MJLS, as well as uncertainties on the transition probability matrix $\mathbb{P}$. The initial state value is not assumed to be exactly known, only supposed to be in a convex set with probability one. To our knowledge, there is no other analytical or numerical way of handling this kind of problem in the literature.

The paper is organized in the following way. Section 2 presents the notation that will be used throughout the work. Section 3 deals with the appropriate notions of stability and stabilizability for MJLS, as well as some auxiliary results. In section 4 we show that the problem of constrained quadratic control for MJLS, when the initial state and transition probability matrix $\mathbb{P}$ belong to appropriate convex sets, can be stated in terms of an LMI optimization problem, so that convex programming can be used for obtaining an approximation of the optimal solution. Section 5 presents some numerical examples to illustrate the developed results. The paper is concluded in section 6 with some final comments.

2. Notation

We denote by $\mathbb{R}^n$ the n-dimensional real space, and set $\mathbb{M}(\mathbb{R}^n,\mathbb{R}^m)$ the normed linear space of all m by n real matrices. Whenever $m = n$ we write $\mathbb{M}(\mathbb{R}^n,\mathbb{R}^n) = \mathbb{M}(\mathbb{R}^n)$ for simplicity. The superscript $'$ will indicate transpose. $L \geq 0$ and $L > 0$ will be used if a self-adjoint matrix is positive semi-definite or positive definite respectively and we write

$\mathbb{M}(\mathbb{R}^n)^+ = \{ L \in \mathbb{M}(\mathbb{R}^n);\ L = L' \geq 0 \}$. We denote by $\|.\|$ either the induced norm in $\mathbb{M}(\mathbb{R}^n)$ or the standard norm in $\mathbb{R}^n$.

We define $\mathcal{H}^{m,n}$ the linear space made up of all N-sequence of matrices $V = (V_1, \ldots, V_N)$, $V_i \in \mathbb{M}(\mathbb{R}^m, \mathbb{R}^n)$, $i = 1, \ldots, N$. We set $\mathcal{H}^{n,n} = \mathcal{H}^n$ and $\mathcal{H}^{n+} = \{V = (V_1, \ldots, V_N) \in \mathcal{H}^n;\quad V_i \in \mathbb{M}(\mathbb{R}^n)^+, i = 1, \ldots, \mathrm{N}\}$. For $H = (H_1, \ldots, H_N)$ and $V = (V_1, \ldots, V_N)$ in $\mathcal{H}^{n+}$ the notation $H \leq L$ $(H < L)$ indicates that $H_i \leq L_i$ $(H_i < L_i)$ for each $i = 1, \ldots, N$.

We define ℓ_2^n as the Hilbert space formed by the sequence of second order random variables $z = (z(0), z(1), \ldots)$ with $z(k) \in \mathbb{R}^n$ for each $k = 0, 1, \ldots$ and such that

$$\|z\|_2^2 := \sum_{k=0}^{\infty} \|z(k)\|_2^2 < \infty,$$

where $\|z(k)\|_2^2 := E(\|z(k)\|^2)$.

For any set $\{H_1, \ldots, H_r\}$ of matrices (vectors) with the same dimensions, we write *conv*$\{H_1, \ldots, H_r\} := \{H; H = \sum_{\ell=1}^{r} \alpha_\ell H_\ell\ ,\ \alpha_\ell \geq 0, \sum_{\ell=1}^{r} \alpha_\ell = 1\}$.

Finally we conclude this section with the following remark which will be useful in the sequel.

Remark 1 : If $R > 0$ then $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$ if and only if $Q \geq S R^{-1} S'$ (see [2]).

3. Stability Results

Consider the following discrete-time MJLS on a probability space $(\Omega, \{\mathcal{F}_k\}, \mathcal{F}, P)$,

$$x(k+1) = \tilde{A}_{\theta(k)} x(k) \tag{1.a}$$

$$x(0) = x_0\ ,\ \theta(0) = \theta_0 \tag{1.b}$$

where $\{\theta(k); k = 0, 1, \ldots\}$ is a discrete-time Markov chain with finite state space $\{1, \ldots, N\}$ and transition probability matrix $\mathbb{P} = [p_{ij}]$, $\tilde{A} = (\tilde{A}_1, \ldots, \tilde{A}_\mathrm{N}) \in \mathcal{H}^n$, and x_0 a second order random variable in $\mathbb{R}^n$. For any $V = (V_1, \ldots, V_\mathrm{N}) \in \mathcal{H}^n$ we define $\mathcal{E}(V) = (\mathcal{E}_1(V), \ldots, \mathcal{E}_N(V))$ as

$$\mathcal{E}_i(V) := \sum_{j=1}^{N} p_{ij} V_j \in \mathbb{M}(\mathbb{R}^n). \tag{2}$$

We make the following definitions:

Definition 1 : Model (1) is mean square stable (MSS) if $E(\|x(k)\|^2) \to 0$ as $k \to \infty$ for any initial condition x_0 and initial distribution for θ_0.

Remark 2 : It can be shown that stability of each model is neither necessary nor sufficient for MSS. Moreover if (1) is MSS then $\|x(k)\| \to 0$ as $k \to \infty$ with prob. 1.

We present now the definition of mean square stabilizability and detectability. Consider $A=(A_1,\dots,A_N)\in\mathcal{H}^n$, $B=(B_1,\dots,B_N)\in\mathcal{H}^{m,n}$ and $C=(C_1,\dots,C_N)\in\mathcal{H}^{n,p}$.

Definition 2 : We say that (A,B) is mean square stabilizable if there exists $K=(K_1,\dots,K_{\mathrm{N}})\in\mathcal{H}^{\mathrm{n,m}}$ such that model (1) is MSS with $\widetilde{A}_i=A_i+B_iK_i$. In this case we say that K stabilizes (A,B) in the mean square sense and set $\mathbb{K}=\{K\in\mathcal{H}^{n,m}; K$ stabilizes (A,B) in the mean square sense$\}$. Similarly, we say that (C,A) is mean square detectable if there exists $H=(H_1,\dots,H_{\mathrm{N}})\in\mathcal{H}^{p,n}$ such that model (1) is MSS with $\widetilde{A}_i=A_i+H_iC_i$, and we say that H stabilizes (C,A) in the mean square sense.

The following auxiliary results will be required in the sequel. For the proofs, see [5],[6].

Proposition 1: Model (1) is MSS if and only if we can find $P=(P_1,\dots,P_N)>0$ such that

$$P_i-\widetilde{A}_i'\mathcal{E}_i(P)\widetilde{A}_i>0\ \ ,\ i=1,\dots,N. \tag{3}$$

Proposition 2: Suppose (C,A) is mean square detectable. If there exits $P=(P_1,\dots,P_N)\geq 0$ and $K=(K_1,\dots,K_N)\in\mathcal{H}^{n,m}$ such that for each $i=1,\dots,N$,

$$P_i-(A_i+B_iK_i)'\mathcal{E}_i(P)(A_i+B_iK_i)\geq\frac{1}{\delta^2}\,(C_i'C_i\ +\ K_i'D_i'D_iK_i) \tag{4}$$

for some $\delta>0$, then $K=(K_1,\dots,K_N)\in\mathbb{K}$.

4. Constrained Quadratic Control Problem

Consider now the following system $\mathcal{G}$ in $(\Omega,\{\mathcal{F}_k\},\mathcal{F},P)$,

$$\mathcal{G}=\begin{cases} x(k+1)=A_{\theta(k)}x(k)\ +\ B_{\theta(k)}u(k) & \text{(5.a)}\\ x(0)=x_0,\ \theta(0)=\theta_0 & \text{(5.b)}\\ z(k)=C_{\theta(k)}x(k)+D_{\theta(k)}u(k) & \text{(5.c)}\end{cases}$$

where $\theta_0\in\{1,\dots,N\}$, x_0 is a second order random variable belonging to *conv*$\{x_{0,1},\dots,x_{0,s}\}$ with probability 1, $A=(A_1,\dots,A_N)\in\mathcal{H}^n$, $B=(B_1,\dots,B_N)\in\mathcal{H}^{m,n}$, $C=(C_1,\dots,C_N)\in\mathcal{H}^{n,p}$, $D=(D_1,\dots,D_N)\in\mathcal{H}^{m,p}$ with $C_i'D_i=0$ for each $i=1,\dots,N$. The transition probability of the Markov chain $\mathbb{P}$ is not exactly known, but belongs to *conv*$\{\mathbb{P}_1,\dots,\mathbb{P}_\kappa\}$, where $\mathbb{P}_\ell=[p_{ij,\ell}]$, $\ell=1,\dots,\kappa$.

For $K=(K_1,\dots,K_N)\in\mathbb{K}$ set

$$J(K):=\|z\|_2=\sum_{k=0}^{\infty}E\Big(x(k)'C_{\theta(k)}'C_{\theta(k)}x(k)+u(k)'D_{\theta(k)}'D_{\theta(k)}u(k)\Big)$$

with $z=(z(0),\ldots)$ given by (5.c) when $u(k)=-K_{\theta(k)}x(k)$. We want to find the smallest $\delta>0$ such that

$$J(K)\leq\delta$$

subject to the restrictions

$$\|F_\iota x(k)+G_\iota u(k)\|\leq\rho_\iota \text{ with probability 1, for } k=0,1,\ldots,\ \iota=1,\ldots,t. \tag{6}$$

The motivation for restrictions in (6) is that many systems are subject to inequalities constrains on manipulated and controlled variables. For instance, in industrial processes, restrictions on control valves and process variables are quite often. In section 5 we present some examples to illustrate the use of these constrains.

We shall find an upper bound for the problem posed above in terms of a convex programming formulation. For this, set $\Gamma_{i,\ell}=[p_{i1,\ell}^{1/2}I\ldots p_{iN,\ell}^{1/2}I]\in\mathbb{M}(\mathbb{R}^{Nn},\mathbb{R}^n)$ for $i=1,\ldots,N$ and $\ell=1,\ldots,\kappa$, and define the following problem:

Problem I : find $\delta>0, Q=(Q_1,\ldots,Q_N)>0,\ Y=(Y_1,\ldots,Y_N)$ such that

$$min\,\delta$$

subject to,

$$\begin{bmatrix} 1 & x'_{0,\nu} \\ x_{0,\nu} & Q_i \end{bmatrix}\geq 0,\text{ for }\nu=1,\ldots,s,\ i=1,\ldots,N, \tag{7}$$

$$\begin{bmatrix} Q_i & (Q_iA'_i+Y'_iB'_i)\Gamma_{i,\ell} & Q_iC'_i & Y'_iD'_i \\ \Gamma'_{i,\ell}(A_iQ_i+B_iY_i) & diag\{Q_\epsilon\} & 0 & 0 \\ C_iQ_i & 0 & \delta I & 0 \\ D_iY_i & 0 & 0 & \delta I \end{bmatrix}\geq 0 \tag{8}$$

for $i=1,\ldots,N$, $\ell=1,\ldots,\kappa$,

$$\begin{bmatrix} Q_i & Q_iA'_i+Y'_iB'_i \\ A_iQ_i+B_iY_i & Q_j \end{bmatrix}\geq 0 \tag{9}$$

for $i=1,\ldots,N$ and j such that $p_{ij,\ell}>0$ for some $\ell=1,\ldots,\kappa$, and

$$\begin{bmatrix} \rho_\iota^2 I-(F_\iota Q_iF'_\iota+G_\iota Y_iF'_\iota+F_\iota Y'_iG'_\iota) & G_\iota Y_i \\ Y'_iG'_\iota & Q_i \end{bmatrix}\geq 0, \tag{10}$$

for $i=1,\ldots,N$, $\iota=1,\ldots,t$,
where $diag\{Q_\epsilon\}$ is the matrix in $\mathbb{M}(\mathbb{R}^{Nn})$ formed by $Q_1,\ldots,Q_N$ in the diagonal, and zero elsewhere.

The link between Problem I and the control problem established by equations (5), (6), and quadratic cost $J(K)$ is given by Theorem 1 below.

Theorem 1 : Suppose (C,A) is mean square detectable and there is a solution (δ, Q, Y) for Problem I. Define $K_j = Y_j Q_j^{-1}$, $j = 1, \ldots, N$, $K = (K_1, \ldots, K_N)$, and $u(k) = K_{\theta(k)} x(k)$ as in (5) above. Then $K \in \mathbb{K}$, (6) is satisfied, and

$$J(K) \leq \delta.$$

Proof: First of all notice that (8) is equivalent to (see Remark 1)

$$Q_i \geq (Q_i A_i' + Y_i' B_i')\Big(\sum_{j=1}^{N} p_{ij,\ell} Q_j^{-1}\Big)(A_i Q_i + B_i Y_i) + \frac{1}{\delta}(Q_i C_i' C_i Q_i + Y_i' D_i' D_i Y_i) \quad (11)$$

for $i = 1, \ldots, N$, $\ell = 1, \ldots, \kappa$. Set $P_i = Q_i^{-1}$, $i = 1, \ldots, N$, $P = (P_1, \ldots, P_N)$. Since $\mathbb{P} = \sum_{\ell=1}^{\kappa} \alpha_\ell \mathbb{P}_\ell$ for some $\alpha_\ell \geq 0$, $\sum_{\ell=1}^{\kappa} \alpha_\ell = 1$, we get that

$$\mathcal{E}_i(P) = \sum_{\ell=1}^{\kappa} \alpha_\ell \Big(\sum_{j=1}^{N} p_{ij\ell} P_j \Big)$$

and thus from (11),

$$P_i \geq (A_i + B_i K_i)' \mathcal{E}_i(P)(A_i + B_i K_i) + \frac{1}{\delta}(C_i' C_i + K_i' D_i' D_i K_i). \quad (12)$$

The assumption that (C,A) is mean square detectable and Proposition 2 imply that K stabilizes (A,B) (see (4)). Recalling that, with probability 1, $x_0 \in conv\{x_{0,1}, \ldots, x_{0,s}\}$, we get from (7) that for $\omega \in \Omega$, $x_0(\omega) = \sum_{\nu=1}^{s} \alpha_\nu x_{0,\nu}$, $\alpha_\nu \geq 0$, $\sum_{\nu=1}^{s} \alpha_\nu = 1$, $\theta_0(\omega) = i$,

$$\begin{bmatrix} 1 & x_0'(\omega) \\ x_0(\omega) & Q_{\theta_0(\omega)} \end{bmatrix} = \sum_{\nu=1}^{s} \alpha_\nu \begin{bmatrix} 1 & x_{0,\nu}' \\ x_{0,\nu} & Q_i \end{bmatrix} \geq 0$$

and from Remark 1, $1 \geq x_0'(\omega) Q_{\theta_0(\omega)}^{-1} x_0(\omega)$, that is, with probability 1,

$$x_0' P_{\theta_0} x_0 \leq 1. \quad (13)$$

Pre and pos multiplying equation (12) by $x(k)'$ and $x(k)$ respectively, and recalling that

$$x(k+1) = (A_{\theta(k)} + B_{\theta(k)} K_{\theta(k)}) x(k),$$

we get that

$$\begin{aligned} x(k)' P_{\theta(k)} x(k) \geq\; & x(k)'(A_{\theta(k)} + B_{\theta(k)} K_{\theta(k)})' \mathcal{E}_{\theta(k)}(P)(A_{\theta(k)} + B_{\theta(k)} K_{\theta(k)}) x(k) \; + \\ & \frac{1}{\delta} x(k)'(C_{\theta(k)}' C_{\theta(k)} + K_{\theta(k)}' D_{\theta(k)}' D_{\theta(k)} K_{\theta(k)}) x(k) \\ =\; & x(k+1)' \mathcal{E}_{\theta(k)}(P) x(k+1) + \\ & \frac{1}{\delta} x(k)'(C_{\theta(k)}' C_{\theta(k)} + K_{\theta(k)}' D_{\theta(k)}' D_{\theta(k)} K_{\theta(k)}) x(k). \end{aligned} \quad (14)$$

Note that $x(k+1)$ is $\mathcal{F}_k$-measurable, and thus

$$E(x(k+1)'P_{\theta(k+1)}x(k+1) \mid \mathcal{F}_k) = x(k+1)'\mathcal{E}_{\theta(k)}(P)x(k+1)$$

so that (14) leads to

$$x(k)'P_{\theta(k)}x(k) \;\geq\; E(x(k+1)'P_{\theta(k+1)}x(k+1) \mid \mathcal{F}_k) + \frac{1}{\delta}x(k)'(C'_{\theta(k)}C_{\theta(k)} + K'_{\theta(k)}D'_{\theta(k)}D_{\theta(k)}K_{\theta(k)})x(k).$$

Taking the expected value from both sides, we get from (14) that

$$\begin{aligned}\left\|P^{1/2}_{\theta(k)}x(k)\right\|_2^2 &= E(x(k)'P_{\theta(k)}x(k)) \geq E(x(k+1)'P_{\theta(k+1)}x(k+1)) + \frac{1}{\delta}\|z(k)\|_2^2 \\ &= \left\|P^{1/2}_{\theta(k+1)}x(k+1)\right\|_2^2 + \frac{1}{\delta}\|z(k)\|_2^2. \end{aligned} \tag{15}$$

Summing up (15) from $k=0$ to ∞, and recalling that $K \in \mathbb{K}$ so that, from MSS of (5), $\left\|P^{1/2}_{\theta(k)}x(k)\right\|_2^2 \to 0$ as $k \to \infty$, we obtain that

$$J(K) = \|z\|_2^2 = \sum_{k=0}^{\infty}\|z(k)\|_2^2 \leq \delta\left\|P^{1/2}_{\theta(0)}x(0)\right\|_2^2 = \delta E(x_0'P_{\theta_0}x_0) \leq \delta,$$

where the last inequality follows from (13). From (9) it follows that (see Remark 1)

$$Q_i \geq Q_i(A_i + B_iK_i)'Q_j^{-1}(A_i + B_iK_i)Q_i$$

and thus

$$Q_i^{-1} \geq (A_i + B_iK_i)'Q_j^{-1}(A_i + B_iK_i)$$

for all $i, j = 1, \ldots, N$, such that $p_{ij,\ell} > 0$ for some $\ell \in \{1, \ldots, \kappa\}$ (that is, the above equation holds for all j that can be reached from i in one step for some transition probability $\mathbb{P}_\ell$). Pre and pos multiplying the above equation by $x(k)'$ and $x(k)$ respectively, we obtain that

$$x(k)'Q^{-1}_{\theta(k)}x(k) \geq x(k+1)'Q^{-1}_{\theta(k+1)}x(k+1)$$

and from equation (13), we get that with probability 1,

$$1 \geq x_0'P_{\theta_0}x_0 \geq x(k)'P_{\theta(k)}x(k) \geq x(k+1)'P_{\theta(k+1)}x(k+1) \tag{16}$$

for all $k = 0, 1, \ldots$. From Remark 1, we have that (10) is equivalent to

$$\rho_\iota^2 I - (F_\iota Q_i F_\iota' + G_\iota Y_i F_\iota' + F_\iota Y_i' G_\iota') - G_\iota Y_i Q_i^{-1} Y_i' G_\iota' \geq 0,\; i = 1, \ldots, N,\; \iota = 1, \ldots, t$$

which is equivalent to

$$\rho_\iota^2 I \geq (F_\iota Q_i^{\frac{1}{2}} + G_\iota Y_i Q_i^{-\frac{1}{2}})(F_\iota Q_i^{\frac{1}{2}} + G_\iota Y_i Q_i^{-\frac{1}{2}})',\; i = 1, \ldots, N,\; \iota = 1, \ldots, t$$

and thus

$$\rho_\iota^2 \geq \left\| F_\iota Q_i^{\frac{1}{2}} + G_\iota Y_i Q_i^{-\frac{1}{2}} \right\|^2, \; i = 1, \ldots, N, \; \iota = 1, \ldots, t. \tag{17}$$

From (16) and (17) we get that with probability 1,

$$\begin{aligned}
&\|F_\iota x(k) + G_\iota u(k)\|^2 = \left\| (F_\iota + G_\iota Y_{\theta(k)} Q_{\theta(k)}^{-1}) x(k) \right\|^2 = \\
&\left\| (F_\iota Q_{\theta(k)}^{\frac{1}{2}} + G_\iota Y_{\theta(k)} Q_{\theta(k)}^{-\frac{1}{2}}) Q_{\theta(k)}^{-\frac{1}{2}} x(k) \right\|^2 \leq \left\| F_\iota Q_{\theta(k)}^{\frac{1}{2}} + G_\iota Y_{\theta(k)} Q_{\theta(k)}^{-\frac{1}{2}} \right\|^2 \left\| Q_{\theta(k)}^{-\frac{1}{2}} x(k) \right\|^2 \\
&\leq \rho_\iota^2 x(k)' Q_{\theta(k)}^{-1} x(k) = \rho_\iota^2 x(k)' P_{\theta(k)} x(k) \leq \rho_\iota^2
\end{aligned}$$

for any $k = 0, 1, \ldots,$ and $\iota = 1, \ldots, t$, completing the proof of the Theorem. □

Remark 3 : For the particular case with only one mode of operation ($N=1$), equation (9) is always satisfied from equation (8) and thus could be removed. Indeed in this case equation (12) with $\mathcal{E}_1(P) = P_1$ leads to $P_1 \geq (A_1 + B_1 K_1)' P_1 (A_1 + B_1 K_1)$ which is equivalent to (9) from Remark 1. For $N > 1$ however we cannot guarantee that equation (8) will imply equation (9) and we need to keep it as a restrictions for our LMI optimization problem.

We have assumed in Theorem 1 that (C, A) is mean square detectable. Since $\mathbb{P}$ is not exactly known, we need a condition to check this, which is provided below. Define the convex set

$\mathcal{V} = \{ \begin{bmatrix} Z_i & R_i \\ R_i' & V_i \end{bmatrix} \geq 0, \; Z_i > 0, \; P_i > 0, \; i = 1, \ldots, N,$ such that $Z_i \geq \sum_{j=1}^{N} p_{ij\ell} P_j, \; \ell = 1, \ldots, \kappa, i = 1, \ldots, N$ and $A_i' Z_i A_i + C_i' R_i' A_i + A_i' R_i C_i + C_i' V_i C_i - P_i < 0, i = 1, \ldots, N\}$.

Proposition 3: Suppose that $\mathcal{V} \neq \emptyset$. Then (C, A) is mean square detectable.

Proof: Let $\begin{bmatrix} Z_i & R_i \\ R_i' & V_i \end{bmatrix} \geq 0, \; P_i > 0, \; i = 1, \ldots, N,$ belong to $\mathcal{V}$. Then from Remark 1, $V_i \geq R_i' Z_i^{-1} R_i$. For some $\alpha_\ell \geq 0$, $\sum_{\ell=1}^{\kappa} \alpha_\ell = 1$, $\mathbb{P} = \sum_{\ell=1}^{\kappa} \alpha_\ell \mathbb{P}_\ell$, and thus $Z_i \geq \sum_{\ell=1}^{\kappa} \alpha_\ell \Big(\sum_{j=1}^{N} p_{ij\ell} P_j \Big) = \mathcal{E}_i(P) > 0, i = 1, \ldots, N$. Therefore

$$\begin{aligned}
&(A_i + (Z_i^{-1} R_i) C_i)' \mathcal{E}_i(P) (A_i + (Z_i^{-1} R_i) C_i) - P_i \leq \\
&(A_i + (Z_i^{-1} R_i) C_i)' Z_i (A_i + (Z_i^{-1} R_i) C_i) - P_i = \\
&A_i' Z_i A_i + C_i' R_i' A_i + A_i' R_i C_i + C_i' R_i' Z_i^{-1} R_i C_i - P_i \leq \\
&A_i' Z_i A_i + C_i' R_i' A_i + A_i' R_i C_i + C_i' V_i C_i - P_i < 0
\end{aligned}$$

for $i = 1, \ldots, N$. From Proposition 1 and equation (3), we conclude that $H = (Z_1^{-1} R_1, \ldots, Z_N^{-1} R_N)$ stabilizes (C, A) in the mean square sense. □

5. Numerical Examples

In this section we present two examples, based on [1] and [23], to show the usefulness of the developed results. In both examples we have that the linear systems represent linearization of non-linear plants, and thus the state and control variables are variations around operating points. For further details, the reader is refereed to [1] and [23].

Example 1: The first example considers a MJLS with three operation modes describing an economic system, adapted from [1], which relates government expenditure (the control variable u) and national income (the state variable x). See [1] and references therein for more details. The operation modes account for the general economic situation ("normal", "boom" or "slump"), which is assumed as non-deterministic, changing from one mode to another according to a Markov chain $\theta(k)$. The parameters for this system are presented in Table 1.

	operation modes		
parameters	$i = 1$ (normal)	$i = 2$ (boom)	$i = 3$ (slump)
A_i	$\begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -4.3 & 4.5 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix}$
B_i	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
C_i	$\begin{bmatrix} 1.5477 & -1.0976 \\ -1.0976 & 1.9145 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 3.1212 & -0.5082 \\ -0.5082 & 2.7824 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.8385 & -1.2728 \\ -1.2728 & 1.6971 \\ 0 & 0 \end{bmatrix}$
D_i	$\begin{bmatrix} 0 \\ 0 \\ 1.6125 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1.0794 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1.0540 \end{bmatrix}$

Table 1: Parameters for Example 1.

The nominal transition probability matrix that relates the three operation modes is given by

$$\mathbb{P}_{nom} = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{bmatrix}.$$

Three design cases are considered:

a) Unconstrained (that is, no limits on the government expenditure) quadratic control without uncertainties. For this case, the transition probability matrix is assumed to be exactly known, and such that $\mathbb{P} = \mathbb{P}_{nom}$.

b) Constrained quadratic control without uncertainties. For this case we assume that the government expenditure is bounded as $||u|| < 1$. As on the first case, $\mathbb{P} = \mathbb{P}_{nom}$.

c) Constrained quadratic control with uncertainties. As on the second case, $||u|| < 1$. The transition probability matrix is not exactly known, but is such that

$$\mathbb{P} \in conv\left\{ \begin{bmatrix} 0.55 & 0.23 & 0.22 \\ 0.36 & 0.35 & 0.29 \\ 0.32 & 0.16 & 0.52 \end{bmatrix}, \begin{bmatrix} 0.79 & 0.11 & 0.10 \\ 0.27 & 0.53 & 0.20 \\ 0.23 & 0.07 & 0.70 \end{bmatrix} \right\}.$$

For all three cases the initial condition $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is assumed to be exactly known. The designed controllers, along with the value of δ are presented in Table 2. Montecarlo simulations (using $\mathbb{P}_{nom}$) of the closed-loop system were performed for a total of 1000 possible realizations of the Markov chain. Figure 1 presents the mean value of the state (only the second component of the state vector) for the simulated realizations. Note that although case (b) has a smaller cost than case (c), the differences on the expected state value are almost negligible, to the extent of not being discernible in the figure. As would be expected, performance of the unconstrained case is better than the constrained ones. The mean system input is represented as a solid line on the plots of Figure 2. The dashed line on the graphics is an envelope of the obtained inputs for all simulations, reflecting thus eventual worst cases. As can be seen from the figure, although the expected input sequence does not violate the restrictions, an unfavorable evolution of the Markov chain might take the input well out the range of admissible values (i. e. $\|u(k)\| < 1$), which does not happen in cases (b) and (c). Case (c) has also the advantage of guaranteeing mean square stability and non violation of the restrictions for any transition probability matrix in the convex set considered.

case	δ	controller	restrictions	uncertainties
(a)	23.95	$F_1 = [2.3172 \quad -2.3317]$ $F_2 = [4.1684 \quad -3.7131]$ $F_3 = [-5.1657 \quad 5.7933]$		
(b)	114.86	$F_1 = [2.3107 \quad -2.0425]$ $F_2 = [3.4189 \quad -2.6547]$ $F_3 = [-4.5374 \quad 5.3507]$	$\|u(k)\| < 1$	
(c)	130.27	$F_1 = [2.3039 \quad -2.0359]$ $F_2 = [3.3753 \quad -2.6051]$ $F_3 = [-4.5418 \quad 5.3554]$	$\|u(k)\| < 1$	$\mathbb{P}$ not exactly known

Table 2: Controllers for Example 1.

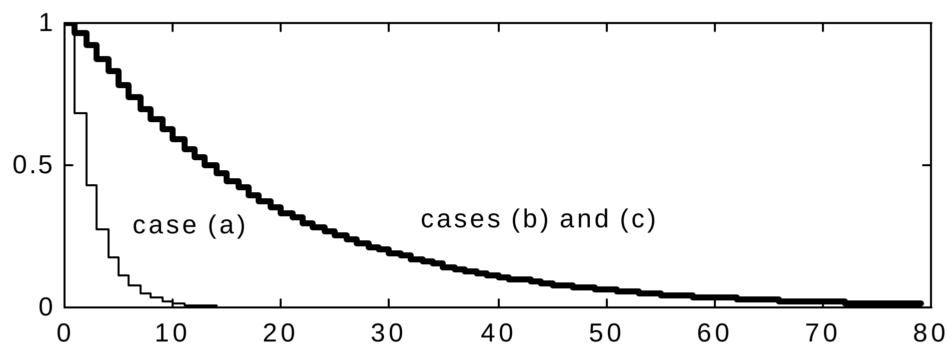

Figure 1: State for example 1.

As seen in figure 1, the performance of the controllers for cases b) and c) were deteriorated by the constrains on the control variable u. This is due to the fact that the violation of the control constrain is more likely to happen when the state variables are further away from the origin (see figure 2), and we have used a fixed conservative controller from beginning to end, no matter how close to the origin the system was getting to. A possible way of improving it would be to solve the LMI optimization problem for different initial conditions closer to the origin, and switch from one controller to another as the system gets closer and closer to it. By doing this we would get less conservative controllers, and the performance should be improved.

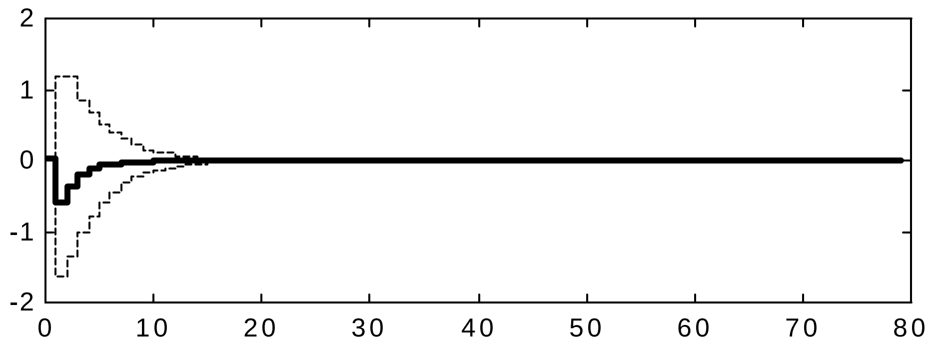

Figure 2: Input for example 1.

Example 2: This example, taken from [23], considers the dynamics of the steam boiler of a solar-powered central receiver. In this case the state variable represents the steam temperature and the manipulated variable the inlet flow, controlled by a valve. In this plant, movable mirrors reflect and focus sunlight on a boiler. The atmospheric conditions (clear or obstructed sky) play an important role in system dynamics, and are essentially unpredictable. [23] suggests a two-operation-mode Markovian model for the steam temperature regulation system of the plant, as presented in Table 3, with nominal transition probability matrix given by

$$\mathbb{P}_{nom} = \begin{bmatrix} 0.9767 & 0.0233 \\ 0.0435 & 0.9565 \end{bmatrix}.$$

	operation modes	
parameters	$i = 1$ (clear sky)	$i = 2$ (dense cloud)
A_i	0.8353	0.9646
B_i	0.0915	0.0982
C_i	$\begin{bmatrix} 0.1884 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.1884 \\ 0 \end{bmatrix}$
D_i	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Table 3: Parameters for Example 2.

As before, three cases are considered.

a) Unconstrained (that is, no constrains on the valve position) quadratic control without uncertainties, with $\mathbb{P} = \mathbb{P}_{nom}$.

b) Constrained (that is, limitations on the valve position) quadratic control without uncertainties, with $\|u\| < 0.015$ and $\mathbb{P} = \mathbb{P}_{nom}$.

c) Constrained quadratic control with uncertainties, with $\|u\| < 0.015$ and

$$\mathbb{P} \in conv\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

In all cases the initial condition $x_0 = 1$ is assumed to be exactly known The obtained δ´s and controllers are presented in Table 4.

case	δ	controller	restrictions	uncertainties
(a)	0.3603	$F_1 = [\ -0.0103\,]$ $F_2 = [\ -0.0331\,]$		
(b)	0.4220	$F_1 = [\ -0.0125\,]$ $F_2 = [\ -0.0150\,]$	$\|u(k)\| < 0.015$	
(c)	0.4935	$F_1 = [\ -5.33\times10^{-4}\,]$ $F_2 = [\ -0.0150\,]$	$\|u(k)\| < 0.015$	$\mathbb{P}$ not exactly known

Table 2: Controllers for Example 1.

200 possible realizations (using $\mathbb{P}_{nom}$) for the Markov chain were considered for the simulations presented in Figures 3 and 4. The figures are similar to those of Example 1. As before, it should be noted that in the unconstrained case, unfavorable realizations of the Markov chain could cause the system to violate the restrictions, which does not happen in cases (b) and (c). Case (c) of this example also shows the viability of considering uncertainties in matrix $\mathbb{P}$. Even for a very large uncertainty there is no significant loss of performance when compared with case (b). This is a typical application for a constrained controller, with the input given by a control valve position.

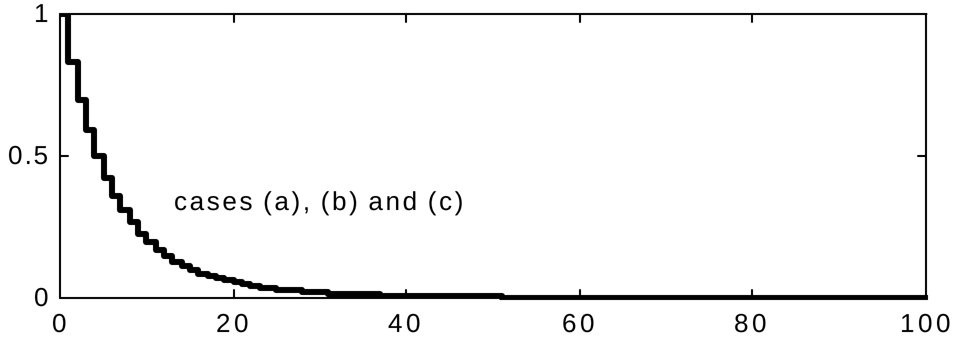

Figure 3: State for Example 2.

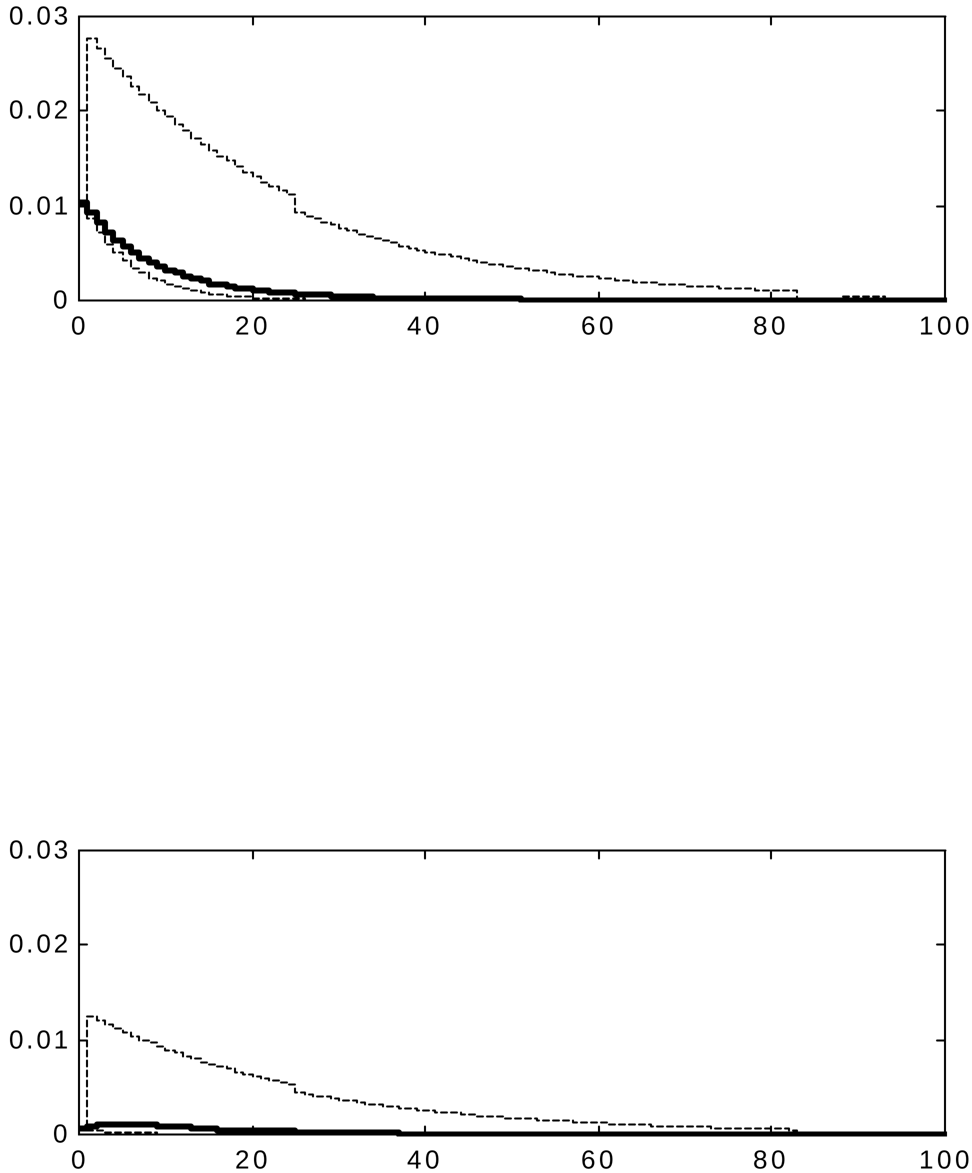

Figure 4: Input for Example 2.

6. Conclusions

In this paper we have considered the quadratic control problem of discrete-time MJLS with constrains on the norm of the state and control variables, and uncertainties on the transition probability matrix $\mathbb{P}$. The initial state is assumed to be in an appropriate convex set with probability 1. Both the state variable and the jump variable are assumed to be available to the controller. Tracing a parallel with the current literature for discrete-time deterministic linear systems we derived a formulation for the problem in terms of a convex optimization involving LMI. The solution of this problem leads to a state feedback mean square stabilizing controller that satisfies the constrains and makes the quadratic cost less than a value δ, whatever the initial value and transition probability matrix inside the appropriate convex sets are. Finally numerical examples were presented to show the usefulness of the proposed technique.

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